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GAUSSIAN CONCENTRATION FOR A CLASS OF SPHERICALLY INVARIANT MEASURES

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Concentration and logarithmic Sobolev inequalities are derived for a class of multidimensional probability distributions, including spherically invariant log-concave measures. Bibliography: 17 *titles.*

1. Introduction

Let μ be a spherically invariant probability measure on the Euclidean space $\mathbf{R}^n$ ($n \geqslant 2$) with density

$$\frac{d\mu(x)}{dx} = \rho(|x|), \quad x \in \mathbf{R}^n,$$

where $\rho = \rho(r)$ is a nonnegative function on the positive half-axis $r > 0$.

In particular, all linear functionals $x \to \langle x, \theta \rangle$ have distributions under μ that only depend on the Euclidean norm $|\theta|$. For the sake of normalization, we assume that

$$\int x_1^2 \, d\mu(x) = 1, \tag{1.1}$$

which may also be written as

$$\int |x|^2 \, d\mu(x) = n.$$

We introduce the concentration function

$$\alpha_\mu(h) \,=\, \sup \Big[1 - \mu(A^h) \Big], \quad h > 0,$$

where the supremum is taken over all measurable sets A in $\mathbf{R}^n$ with $\mu(A) \geqslant 1/2$, and denote by

$$A^h = \{x \in \mathbf{R}^n : \exists\, y \in A,\, |x - y| < h\}$$

an open h-neighborhood of A with respect to the Euclidean distance.

It is known that, if ρ is log-concave, i.e., for all $x, y > 0$ and $t \in (0,1)$

$$\rho(tx + (1-t)y) \geqslant \rho(x)^t \rho(y)^{1-t},$$

then the concentration function associated to μ admits an exponentially decaying bound

$$\alpha_\mu(h) \leqslant e^{-ch}, \tag{1.2}$$

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where $c > 0$ is an absolute constant. Moreover [**1**], in the class of all smooth functions f on $\mathbf{R}^n$ with μ-mean zero, we have

$$\lambda_1 \int f^2 \, d\mu \leqslant \int |\nabla f|^2 \, d\mu \tag{1.3}$$

with an optimal constant (spectral gap) $\lambda_1 = \lambda_1(\mu)$ satisfying

$$c \leqslant \lambda_1 \leqslant 1.$$

The fact that the concentration function of μ has at worst an exponential decay and may be controlled in terms of λ_1 is an observation due to Gromov and Milman [**2**] and Borovkov and Utev [**3**]. More precisely (cf. [**4**]), in the presence of the Poincaré type inequality (1.3), one always has

$$1 - \mu(A^h) \leqslant C\, e^{-2\sqrt{\lambda_1}\, h}, \quad \mu(A) \geqslant 1/2,$$

with some absolute constant C (for example, $C = 18$). So, if λ_1 is separated from zero by an absolute constant, we arrive at the concentration inequality (1.2) with some absolute constant $c > 0$.

In general, the bound (1.2) cannot be improved on the basis of (1.3). However, in many interesting examples, especially when the dimension n is large, the concentration function α_μ decays on a large interval like in the Gaussian case. The goal of this paper is the following refinement of (1.2).

Theorem 1.1. *If a log-concave function ρ defines a probability measure μ on $\mathbf{R}^n$ satisfying the normalization condition* (1.1), *then*

$$\alpha_\mu(h) \leqslant e^{-ch^2}, \quad 0 \leqslant h \leqslant \sqrt{n}, \tag{1.4}$$

where $c > 0$ is an absolute constant.

In the functional language, the inequality (1.4) is equivalent to the property that for any Lipschitz $f : \mathbf{R}^n \to \mathbf{R}$ with

$$\|f\|_{\mathrm{Lip}} \leqslant 1$$

and the mean

$$\int f \, d\mu = 0$$

we have

$$\mu\{x \in \mathbf{R}^n : |f(x)| \geqslant h\} \leqslant 2e^{-ch^2}, \quad 0 \leqslant h \leqslant \sqrt{n},$$

up to some absolute constant $c > 0$.

Recall that a measure μ on $\mathbf{R}^n$ is *log-concave* if for all $t \in (0,1)$ it satisfies the Brunn–Minkowski type inequality

$$\mu(tA + (1-t)B) \geqslant \mu(A)^t \mu(B)^{1-t}$$

in the class of all nonempty Borel sets A and B in $\mathbf{R}^n$. If μ is absolutely continuous, the log-concavity of μ is equivalent to the log-concavity of its density (Prékopa's theorem, cf. [**5, 6**]). Using this characterization, it is readily seen that a spherically invariant measure μ on $\mathbf{R}^n$ is log-concave if and only if the function ρ is log-concave and nonincreasing. Thus, Theorem 1.1 involves the class of all spherically invariant, log-concave probability measures on $\mathbf{R}^n$ that are absolutely continuous with respect to the Lebesgue measure.

On the other hand, since we do not require that ρ be nonincreasing, the inequality (1.4) is consistent with and may be viewed as an extension of the concentration phenomenon on the unit sphere

$$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}.$$

Indeed, replacing (1.1) with the condition

$$\int |x|^2 \, d\mu(x) = 1,$$

(1.4) should be changed to

$$\alpha_\mu(h) \leqslant e^{-cnh^2}, \quad 0 \leqslant h \leqslant 1.$$

In particular, this may be applied to the uniform distribution $\mu = \sigma_{n-1}$ on S^{n-1} (since this measure is a "limit point" for the class of all absolutely continuous, spherically invariant probability measures μ with log-concave ρ). Moreover, in this case, the assumption $0 \leqslant h \leqslant 1$ may be removed by noting that $\alpha_\mu(h) = 0$, as long as $h > 1$. Therefore, for any set $A \subset S^{n-1}$ with $\sigma_{n-1}(A) \geqslant 1/2$ we have

$$1 - \sigma_{n-1}(A^h) \leqslant e^{-cnh^2}, \quad h \geqslant 0.$$

This is one of the forms of the concentration property of the sphere, which is usually obtained as a consequence of the isoperimetric theorem of P. Lévy (cf. [**7, 8**]).

The concentration inequality of Theorem 1.1 will be obtained by involving logarithmic Sobolev inequalities, which we consider for the restriction of μ to the ball of radius of order $\sqrt{n}$ (cf. Sections 4 and 5). We discuss a concentration property of the distribution of the Euclidean norm under μ separately in Sections 2 and 3.

2. Log-Concave Measures of Order p

Using spherical coordinates, we can regard any spherically invariant probability measure μ on $\mathbf{R}^n \setminus \{0\}$ as the image of the product measure $\nu \otimes \sigma_{n-1}$ on $(0, +\infty) \times S^{n-1}$ under the map $(r, \theta) \to r\theta$. The probability measure ν is characterized as the distribution of the Euclidean norm $|X|$, where X is a random vector in $\mathbf{R}^n$ distributed according to μ. Moreover, if μ is absolutely continuous and has density

$$\frac{d\mu(x)}{dx} = \rho(|x|),$$

the measure ν is absolutely continuous on $(0, +\infty)$ and has density

$$\frac{d\nu(r)}{dr} = n\omega_n \, r^{n-1} \rho(r), \quad r > 0,$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. If a function ρ is log-concave, we obtain a special class of one-dimensional log-concave probability distributions on $(0, +\infty)$ with a number of remarkable properties.

Definition 2.1. We say that a random variable $\xi > 0$ has *log-concave distribution of order* $p \geqslant 1$ if it has a density of the form

$$q(r) = r^{p-1} \rho(r)$$

for some log-concave function ρ on $(0, +\infty)$.

For example, the standard Gamma-distribution with p degrees of freedom, which has the density

$$q(r) = \frac{1}{\Gamma(p)}\, r^{p-1}\, e^{-r}$$

is log-concave of order p, as long as $p \geqslant 1$.

In many relations about general log-concave distributions of order p, the Gamma-distribution plays an extremal role. In particular, the following assertion holds.

Proposition 2.2. *If $\xi > 0$ has log-concave distribution of order $p \geqslant 1$, then*

$$\mathrm{Var}\,(\xi) \leqslant \frac{1}{p}\,(\mathbf{E}\xi)^2. \tag{2.1}$$

Here, $\mathrm{Var}\,(\xi) = \mathbf{E}\xi^2 - (\mathbf{E}\xi)^2$ denotes the variance of ξ. Equality in (2.1) is attained for the standard Gamma-distribution with p degrees of freedom.

Thus, the meaning of the parameter p is that it allows one to control the concentration of distribution of ξ around its mean.

When p is integer, the inequality (2.1) is easily derived from the reverse Lyapunov inequality due to Barlow, Marshall, and Proshan [**9**], while the general case may be obtained from a result of Borell [**10**]. For more details we refer to [**1**], where the concentration aspect of log-concave distributions of a large order is emphasized and applied to study the K-L-S conjecture for the class of log-concave spherically symmetric measures.

3. Gaussian Concentration in Terms of Order p

For log-concave distributions on the real line the bounds on the variance, such as (2.1), can automatically be sharpened in the form of exponential bounds for large deviations. As a variant, under the assumption that ξ has a log-concave distribution, (2.1) yields, for example,

$$\mathbf{P}\{|\xi - \mathbf{E}\xi| \geqslant h\,\mathbf{E}\xi\} \leqslant 3\,e^{-\frac{1}{4}h\sqrt{p}}, \quad h > 0. \tag{3.1}$$

However, the order of log-concavity provides an additional information. In particular, in the interval $0 \leqslant h \leqslant 1$, the expression $h\sqrt{p}$ on the right-hand side of (3.1) may be replaced with $h^2 p$. An observation of this kind was first made by Klartag [**11**] in his study of the central limit theorem for isotropic convex bodies (see Remark 3.4 below for more details).

Using an alternative approach, let us explain how one can derive Gaussian deviations of ξ from its mean $\mathbf{E}\xi$ on the basis of the same bound on the variance (2.1). Thus, let ξ have a log-concave distribution of order p with density

$$q(r) = r^{p-1}\rho(r),$$

where $\rho(r)$ is a log-concave function in $r > 0$. Necessarily, ρ is Lebesgue integrable. We temporarily assume that

$$\int_0^{+\infty} e^{tr}\rho(r)\,dr < +\infty$$

for all $t \in \mathbf{R}$ and consider positive random variables ξ_t with densities of the form

$$q_t(r) = \frac{r^{p-1} e^{tr} \rho(r)}{\int_0^{+\infty} r^{p-1} e^{tr} \rho(r)\, dr}, \qquad r > 0,$$

where $t \in \mathbf{R}$ is a parameter. Since all ξ_t also have log-concave distributions of order p, we have, by Proposition 2.2,

$$\operatorname{Var}(\xi_t) \leqslant \frac{1}{p}\, (\mathbf{E}\xi_t)^2.$$

In terms of ξ, this is the same as

$$\mathbf{E}\,\xi^2 e^{t\xi}\, \mathbf{E}\, e^{t\xi} - \big(\mathbf{E}\,\xi e^{t\xi}\big)^2 \leqslant \frac{1}{p}\, (\mathbf{E}\,\xi e^{t\xi})^2. \tag{3.2}$$

Introduce the convex function

$$u(t) = \log \mathbf{E}\, e^{t\xi},$$

and set

$$v(t) = u'(t) = \frac{\mathbf{E}\,\xi e^{t\xi}}{\mathbf{E}\, e^{t\xi}}.$$

This function is strictly positive and increasing on the whole real line. Moreover, (3.2) can be recognized as a differential inequality

$$v'(t) = \frac{\mathbf{E}\,\xi^2 e^{t\xi}\, \mathbf{E}\, e^{t\xi} - \big(\mathbf{E}\,\xi e^{t\xi}\big)^2}{\big(\mathbf{E}\, e^{t\xi}\big)^2} \leqslant \frac{1}{p}\, v(t)^2.$$

It follows that

$$0 < \Big(-\frac{1}{v(t)}\Big)' \leqslant \frac{1}{p}$$

for all $t \in \mathbf{R}$, so

$$\left|\frac{1}{v(t)} - \frac{1}{v(0)}\right| \leqslant \frac{|t|}{p}. \tag{3.3}$$

Now, for the sake of definiteness, we set $\mathbf{E}\xi = 1$, so that $v(0) = 1$. We put

$$u_0(t) = \log \mathbf{E}\, \exp\{t(\xi - \mathbf{E}\xi)\} = u(t) - t$$

and

$$v_0(t) = u_0'(t) = v(t) - 1.$$

Then

$$u_0(t) \geqslant 0, \quad u_0(0) = v_0(0) = 0, \quad v_0(t) > -1$$

for all $t \in \mathbf{R}$, and (3.3) gives

$$\frac{|v_0(t)|}{1 + |v_0(t)|} \leqslant \frac{|v_0(t)|}{1 + v_0(t)} \leqslant \frac{|t|}{p}. \tag{3.4}$$

Let

$$|t| \leqslant \frac{p}{2}.$$

Then, necessarily

$$|v_0(t)| \leqslant 1,$$

and, by (3.4) once more,

$$|v_0(t)| \leqslant 2\, \frac{|v_0(t)|}{1 + v_0(t)} \leqslant \frac{2\,|t|}{p}, \quad |t| \leqslant \frac{p}{2}.$$

Integrating from 0 to t, we get

$$|u_0(t)| \leqslant \left| \int_0^t |v_0(s)|\, ds \right| \leqslant \frac{t^2}{p}.$$

Thus, if $\mathbf{E}\xi = 1$ and $|t| \leqslant \frac{p}{2}$, we have

$$\mathbf{E}\, \exp\{t(\xi - \mathbf{E}\xi)\} \leqslant e^{t^2/p}. \tag{3.5}$$

At this step, the assumption that $\mathbf{E}e^{t\xi}$ is finite for all real t can be weakened to the requirement that it is finite for $t \leqslant \frac{p}{2}$. In fact, a similar argument shows that $\mathbf{E}e^{t\xi}$ is finite, as long as $t < p$. To remove the assumption $\mathbf{E}\xi = 1$, we just apply (3.5) to $\xi/\mathbf{E}\xi$. As a result, we obtain the following assertion.

Proposition 3.1. *If $\xi > 0$ has log-concave distribution of order $p \geqslant 1$, then the exponential moment $\mathbf{E}\, e^{t\xi}$ is finite, whenever $t\mathbf{E}\xi < p$. Moreover, for $|t| \leqslant \dfrac{p}{2\,\mathbf{E}\xi}$*

$$\mathbf{E}\, \exp\{t(\xi - \mathbf{E}\xi)\} \leqslant \exp\left\{\frac{t^2}{p}\,(\mathbf{E}\xi)^2\right\}. \tag{3.6}$$

The example of Gamma-distributions with p degrees of freedom shows that the region $t\mathbf{E}\xi < p$ cannot be enlarged in terms of $\mathbf{E}\xi$ and p.

Applying the Chebyshev inequality, from (3.6) we find that for all $h > 0$ and $0 \leqslant t \leqslant p/2$

$$\mathbf{P}\{\xi - \mathbf{E}\xi \geqslant h\mathbf{E}\xi\} \ \leqslant\ \exp\left\{\frac{t^2}{p}\,(\mathbf{E}\xi)^2 - th\mathbf{E}\xi\right\},$$

which for $t = ph/(2\mathbf{E}\xi)$ yields

$$\mathbf{P}\{\xi - \mathbf{E}\xi \geqslant h\mathbf{E}\xi\} \ \leqslant\ e^{-\frac{ph^2}{4}}$$

provided that $h \leqslant 1$. A similar bound holds also for left deviations. One can summarize.

Corollary 3.2. *If $\xi > 0$ has log-concave distribution of order $p \geqslant 1$, then for all $0 \leqslant h \leqslant 1$*

$$\mathbf{P}\{\xi - \mathbf{E}\xi \geqslant h\mathbf{E}\xi\} \ \leqslant\ e^{-\frac{ph^2}{4}}, \tag{3.7}$$

$$\mathbf{P}\{|\xi - \mathbf{E}\xi| \geqslant h\,\mathbf{E}\xi\} \leqslant 2\,e^{-\frac{ph^2}{4}}. \tag{3.8}$$

Taking, for example, $h = 1$, we obtain in (3.7)

$$\mathbf{P}\{\xi \geqslant 2\,\mathbf{E}\xi\} \leqslant e^{-\frac{1}{4}p}.$$

By the log-concavity of the function

$$h \to \mathbf{P}\{\xi \geqslant h\mathbf{E}\xi\},$$

the above inequality immediately implies that

$$\mathbf{P}\{\xi \geqslant h\mathbf{E}\xi\} \leqslant e^{-\frac{ph}{8}}, \quad h \geqslant 2. \tag{3.9}$$

In order to involve the values $h \geqslant 1$ in (3.8), in which case the bound should be exponential (and not Gaussian – like in (3.9)), one can use a general dilation type inequality of Lovász–Simonovits [**12**] (cf. also [**13**])

$$1 - \nu(hB) \leqslant (1 - \nu(B))^{-(h+1)/2}, \quad h \geqslant 1, \tag{3.10}$$

where ν may be an arbitrary log-concave measure on $\mathbf{R}^n$ and B is a Euclidean ball with center at the origin. Given $h > 0$, we apply (3.10) in dimension one to the distribution ν of the random variable $\xi - \mathbf{E}\xi$, by taking h/a in place of the parameter h, where $a = \min(1, h)$. Then, by (3.8) and (3.10),

$$\begin{aligned} \mathbf{P}\{|\xi - \mathbf{E}\xi| \geqslant h\,\mathbf{E}\xi\} &= \mathbf{P}\Big\{|\xi - \mathbf{E}\xi| \geqslant \frac{h}{a}\,a\,\mathbf{E}\xi\Big\} \\ &\leqslant \mathbf{P}\{|\xi - \mathbf{E}\xi| \geqslant a\,\mathbf{E}\xi\}^{-\frac{1}{2}\,(\frac{h}{a}+1)} \\ &\leqslant 2\,e^{-\frac{a^2}{8}\,(\frac{h}{a}+1)p} \;\leqslant\; 2\,e^{-\frac{aph}{8}}. \end{aligned}$$

Therefore, we obtain the following assertion.

Corollary 3.3. *If $\xi > 0$ has log-concave distribution of order $p \geqslant 1$, then for all $h \geqslant 0$*

$$\mathbf{P}\{|\xi - \mathbf{E}\xi| \geqslant h\,\mathbf{E}\xi\} \leqslant 2\,e^{-\frac{p}{8}\,\min(h,h^2)}.$$

Remark 3.4. Klartag considered deviations of ξ from the mode $t_p = t_p(\xi)$, i.e, from the point of maximum of the density $q(r) = r^{p-1}\rho(r)$ of ξ. This point is well defined, whenever ρ is C^2-smooth on $(0, +\infty)$ and $p > 1$, and satisfies the equation

$$(\log\rho)'(r) = \frac{\rho'(r)}{\rho(r)} = -\frac{p-1}{r}.$$

As is shown in [**11**, Lemma 4.4], if $p \geqslant 2$ is integer, then for $0 \leqslant h \leqslant 1$

$$\mathbf{P}\{|\xi - t_p(\xi)| \geqslant h\,t_p(\xi)\} \leqslant C\,e^{-cph^2} \tag{3.11}$$

with some absolute constants $C > 1$ and $0 < c < 1$.

From (3.8) and (3.11) it readily follows that with some absolute $c > 0$

$$\Big(1 - \frac{c}{\sqrt{p}}\Big)\,\mathbf{E}\,\xi \leqslant t_p(\xi) \leqslant \Big(1 + \frac{c}{\sqrt{p}}\Big)\,\mathbf{E}\,\xi.$$

So, the deviation inequalities (3.8) and (3.11) are in essence equivalent (and we believe that the assumption that p is integer may be removed).

4. Logarithmic Sobolev Inequalities

According to Corollary 3.2, if a random variable ξ has log-concave distribution of a large order p, then its distribution is nearly supported on the interval $0 < r < 2\mathbf{E}\xi$ and has Gaussian rate of concentration around the mean $\mathbf{E}\xi$. It is, therefore, not surprising that, on long intervals, such distributions satisfy analytic inequalities, such as logarithmic Sobolev inequalities.

Introduce the entropy functional

$$\mathrm{Ent}_\mu(g) = \int g\log g\,d\mu - \int g\,d\mu\ \log\int g\,d\mu.$$

It is well defined for any measurable $g \geqslant 0$ on an abstract probability space (M, μ) and, in general,

$$0 \leqslant \mathrm{Ent}_\mu(g) \leqslant +\infty.$$

Like in case of the variance,

$$\mathrm{Ent}_\mu(g) = 0$$

if and only if g is a constant μ-almost everywhere. It should also be clear that $\mathrm{Ent}_\mu(g)$ is finite if and only if

$$\int g \log(1+g)\, d\mu < +\infty.$$

It is well known [**14**] that a log-concave probability measure μ on $\mathbf{R}^n$ satisfies the logarithmic Sobolev inequality

$$\alpha\ \mathrm{Ent}_\mu(f^2) \leqslant 2 \int |\nabla f|^2\, d\mu \tag{4.1}$$

with some positive constant $\alpha > 0$, where f is an arbitrary smooth function on $\mathbf{R}^n$, if and only if for some $t > 0$

$$\int e^{t|x|^2}\, d\mu(x) < +\infty.$$

The optimal value $\alpha = \alpha(\mu)$ in (4.1) is called the *logarithmic Sobolev constant* of μ. It may be related to the Orlicz norm, generated by the Young function

$$\psi_2(r) = e^{r^2} - 1.$$

In general, given a random variable η on (M,μ) such that $\mathbf{E}\, e^{t\eta^2}$ is finite for some $t > 0$, the norm $c = \|\eta\|_{\psi_2}$ is defined as the minimal nonnegative number, such that

$$\mathbf{E}\, \psi_2(|\eta|/c) \leqslant 1.$$

As a basic tool, we use a lower bound on the logarithmic Sobolev constant for one-dimensional log-concave distributions, obtained in [**14**, Proposition 4.4].

Lemma 4.1. *Given a random variable ξ with log-concave distribution μ,*

$$\frac{1}{24\, \|\xi - \mathbf{E}\xi\|_{\psi_2}^2} \leqslant \alpha(\mu) \leqslant \frac{8}{3\, \|\xi - \mathbf{E}\xi\|_{\psi_2}^2}.$$

The inequality on the right-hand side is optimal and is attained for Gaussian measures, while the constant $1/24$ on the left-hand side is apparently far from being optimal.

Our next task is to estimate from above the norm $\|\xi - \mathbf{E}\xi\|_{\psi_2}$ under the assumptions that $\xi > 0$ has compactly supported log-concave distribution of a given order $p \geqslant 1$. So, given $r_0 \geqslant 1$, suppose additionally that with probability one

$$\xi \leqslant (1 + r_0)\, \mathbf{E}\xi. \tag{4.2}$$

To simplify formulas (without loss of generality), assume that $\mathbf{E}\xi = 1$. By Proposition 3.1, the random variable $\eta = \xi - \mathbf{E}\xi$ satisfies

$$\mathbf{E}\, e^{t\eta} \leqslant e^{\frac{1}{p}t^2}, \quad |t| \leqslant \frac{p}{2}.$$

By (4.2), we also have $|\eta| \leqslant r_0$, so

$$\mathbf{E}\, e^{t\eta} \leqslant e^{r_0|t|} \leqslant e^{\frac{2r_0}{p}t^2}, \quad |t| \geqslant \frac{p}{2}.$$

The two estimates give

$$\mathbf{E}\, e^{t\eta} \leqslant e^{\frac{2r_0}{p}t^2}$$

without any constraint on t. Integrating this inequality over t with respect to the Gaussian measure on $\mathbf{R}$ with mean zero and variance

$$\sigma^2 > \frac{p}{4r_0},$$

we get

$$\mathbf{E}\, e^{\sigma^2\eta^2/2} \leqslant \frac{1}{\sqrt{1 - \frac{4r_0}{p}\,\sigma^2}}.$$

Here, the right-hand side is equal to 2 for

$$\sigma^2 = \frac{3}{16}\,\frac{p}{r_0},$$

which means that

$$\|\eta\|_{\psi_2}^2 \leqslant \frac{2}{\sigma^2} = \frac{32}{3}\,\frac{r_0}{p}.$$

Moreover, if we drop the condition $\mathbf{E}\xi = 1$, this estimate should be changed to

$$\|\eta\|_{\psi_2}^2 \leqslant \frac{32}{3}\,\frac{r_0}{p}\,(\mathbf{E}\xi)^2.$$

Hence, by Lemma 4.1, we arrive at the following assertion.

Lemma 4.2. *If $\xi > 0$ has log-concave distribution ν of order $p \geqslant 1$ and satisfies the condition* (4.2)*, then*

$$\alpha(\nu) \geqslant \frac{c}{r_0}\,\frac{p}{(\mathbf{E}\xi)^2}$$

with some absolute constant $c > 0$.

One may take, for example,

$$c = \frac{3}{32}\cdot\frac{1}{24} = \frac{1}{256}.$$

We use Lemma 4.2 to derive a logarithmic Sobolev inequality for compactly supported spherically symmetric probability measures μ on $\mathbf{R}^n$, $n \geqslant 2$, with density

$$\frac{d\mu(x)}{dx} = \rho(|x|),$$

such that $\rho = \rho(r)$ is log-concave. This is assumed in the next statement.

Proposition 4.3. *Let X be a random vector in $\mathbf{R}^n$ with distribution μ, and let $r_0 \geqslant 1$. If with probability one*

$$|X| \leqslant (1 + r_0)\,\mathbf{E}\,|X|, \tag{4.3}$$

then

$$\alpha(\mu) \geqslant \frac{c}{r_0^2}\,\frac{n}{(\mathbf{E}\,|X|)^2}, \tag{4.4}$$

where $c > 0$ is an absolute constant.

Proof. We need to derive the logarithmic Sobolev inequality (4.1) for the measure μ with a constant, given on the right-hand side of (4.4). So, take a bounded, positive, smooth function f on $\mathbf{R}^n$.

The distribution ν of the random variable $\xi = |X|$ is log-concave of order $p = n$ and, by the hypothesis (4.3), it has the supporting interval

$$0 \leqslant r \leqslant (1 + r_0)\,\mathbf{E}\xi. \tag{4.5}$$

So, one may apply Lemma 4.2: For any smooth function g on $(0,+\infty)$

$$\mathrm{Ent}_\nu(g^2) \leqslant A \int (g')^2 \, d\nu, \quad A = \frac{Cr_0}{n} \, (\mathbf{E}\,|X|)^2,$$

where C is an absolute constant. In particular, for $g(r) = f(r\theta)$ with a fixed $\theta \in S^{n-1}$, and using $|g'(r)| \leqslant |\nabla f(r\theta)|$, we get

$$\int f(r\theta)^2 \log f(r\theta)^2 \, d\nu(r) \leqslant \int f(r\theta)^2 d\nu(r) \, \log \int f(r\theta)^2 \, d\nu(r) + A \int |\nabla f(r\theta)|^2 \, d\nu(r),$$

i.e., in terms of the function on the unit sphere

$$u(\theta) = \Big(\int f(r\theta)^2 d\nu(r) \Big)^{1/2},$$

we have

$$\int f(r\theta)^2 \log f(r\theta)^2 \, d\nu(r) \leqslant u(\theta)^2 \log u(\theta)^2 + A \int |\nabla f(r\theta)|^2 \, d\nu(r).$$

Integrating this inequality with respect to σ_{n-1} and recalling that the map $(r,\theta) \to r\theta$ pushes forward $\nu \otimes \sigma_{n-1}$ into μ, we get

$$\int f^2 \log f^2 \, d\mu \leqslant \int u^2 \, \log u^2 \, d\sigma_{n-1} + A \int |\nabla f|^2 \, d\mu. \tag{4.6}$$

Now, it is known that the logarithmic Sobolev constant for the uniform measure on S^{n-1} is given by $\alpha(\sigma_{n-1}) = n-1$ (cf. [**15**]), i.e., there is a logarithmic Sobolev inequality

$$\int u^2 \, \log u^2 \, d\sigma_{n-1} \leqslant \int u^2 \, d\sigma_{n-1} \, \log \int u^2 \, d\sigma_{n-1} + \frac{2}{n-1} \int |\nabla u|^2 \, d\sigma_{n-1}.$$

Since

$$\int u^2 \, d\sigma_{n-1} = \int f^2 \, d\mu,$$

together with (4.6) it gives

$$\mathrm{Ent}_\mu(f^2) \leqslant \frac{2}{n-1} \int |\nabla u|^2 \, d\sigma_{n-1} + A \int |\nabla f|^2 \, d\mu. \tag{4.7}$$

In order to estimate the first integral, note that, by the very definition of u, for all $\theta, \widetilde{\theta} \in S^{n-1}$

$$\Big\langle \nabla u(\theta), \widetilde{\theta} \Big\rangle = \frac{2}{u(\theta)} \int r f(r\theta) \Big\langle \nabla f(r\theta), \widetilde{\theta} \Big\rangle \, d\nu(r).$$

Applying the Cauchy–Bunyakovski inequality together with the assumption (4.5), we get

$$\begin{aligned}\Big\langle \nabla u(\theta), \widetilde{\theta} \Big\rangle^2 &\leqslant \frac{4}{u(\theta)^2} \int r^2 f(r\theta)^2 \, d\nu(r) \int \Big\langle \nabla f(r\theta), \widetilde{\theta} \Big\rangle^2 \, d\nu(r) \\ &\leqslant 4\,(1+r_0)^2 \, (\mathbf{E}\,|X|)^2 \int |\nabla f(r\theta)|^2 \, d\nu(r).\end{aligned}$$

Taking the supremum over all $\widetilde{\theta} \in S^{n-1}$, we find

$$|\nabla u(\theta)|^2 \leqslant 4\,(1+r_0)^2 \, (\mathbf{E}\,|X|)^2 \int |\nabla f(r\theta)|^2 \, d\nu(r).$$

It remains to integrate this inequality with respect to σ_{n-1}, which gives

$$\int |\nabla u|^2 \, d\sigma_{n-1} \leqslant 4\,(1+r_0)^2 \, (\mathbf{E}\,|X|)^2 \int |\nabla f|^2 \, d\mu$$

and, by (4.7),

$$\mathrm{Ent}_\mu(f^2) \leqslant \left[\frac{8}{n-1}\,(1+r_0)^2\,(\mathbf{E}\,|X|)^2 + A\right] \int |\nabla f|^2\,d\mu.$$

Here, the factor in front of the integral does not exceed

$$\left[\frac{8}{n-1}\,(1+r_0)^2 + \frac{Cr_0}{n}\right](\mathbf{E}\,|X|)^2 \leqslant \frac{C' r_0^2}{n}\,(\mathbf{E}\,|X|)^2$$

with some absolute constant C'.

Finally, the assumption $f > 0$ in (4.1) may easily be weakened to $f \geqslant 0$ and then removed at all by virtue of the general bound $|\nabla |f|\,| \leqslant |\nabla f|$. Proposition 4.3 is proved. $\square$

Remark 4.4. In dimension $n = 1$, Proposition 4.3 is no longer true since the support of μ may have two (connected) components. For example, the uniform distribution μ on $[-2,-1] \cup [1,2]$ is "spherically" symmetric and corresponds to the log-concave function

$$\rho(r) = \frac{1}{2}\,1_{[1,2]}(r).$$

But, in this case,

$$\lambda_1(\mu) = \alpha(\mu) = 0.$$

5. Proof of Theorem 1.1

First, we isolate one important particular case or consequence from Proposition 4.3.

As before, let μ be a spherically symmetric probability measure on $\mathbf{R}^n$, $n \geqslant 2$, with density

$$\frac{d\mu(x)}{dx} = \rho(|x|),$$

where ρ is a log-concave function on $(0,+\infty)$. For normalization, we assume like in Theorem 1.1 that

$$\int |x|^2\,d\mu(x) = n. \tag{5.1}$$

We do not pose the condition that μ is compactly supported. Instead, we consider the restrictions of μ to the balls of an appropriate radius.

Proposition 5.1. *Given $a \geqslant 2$, the normalized restriction μ_a of μ to the ball $|x| < a\sqrt{n}$ satisfies the logarithmic Sobolev inequality*

$$\mathrm{Ent}_{\mu_a}(f^2) \leqslant Ca^2 \int |\nabla f|^2\,d\mu_a \tag{5.2}$$

with some absolute constant C.

As we know from Corollary 3.2 (cf. (3.9)), the measure μ is essentially supported on the balls $B_a = B(0, a\sqrt{n})$ in the sense that

$$\mu(B_a) \geqslant 1 - e^{-an/8}. \tag{5.3}$$

Proof of Proposition 5.1. Let X be a random vector in $\mathbf{R}^n$ distributed according to μ_a. By construction, $|X| \leqslant a\sqrt{n}$. Hence the condition (4.3) of Proposition 4.3 is satisfied, as long as

$$\mathbf{E}\,|X| \geqslant \frac{a\sqrt{n}}{1+r_0}.$$

The random variable $\xi(x) = |x|$ on the probability space $(\mathbf{R}^n, \mu)$ has log-concave distribution (of order n) with the second moment

$$\mathbf{E}\,\xi^2 = n$$

according to the assumption (5.1). Since

$$\mathbf{E}\,|X| = \frac{1}{\mu(B_a)} \int\limits_{B_a} \xi\, d\mu,$$

it suffices to choose $r_0 \geqslant 1$ such that

$$\int\limits_{B_a} \xi\, d\mu \geqslant \frac{a\sqrt{n}}{1 + r_0}. \tag{5.4}$$

By the Chebyshev inequality, for the complement $\widetilde{B}_a = \mathbf{R}^n \setminus B_a$ we have

$$\mu(\widetilde{B}_a) \leqslant \mu\left\{|x| \geqslant 2\sqrt{n}\,\right\} \leqslant \frac{1}{4n}\,\mathbf{E}\,\xi^2 \; = \; \frac{1}{4}.$$

Hence, by the Cauchy–Bunyakovski inequality,

$$\int\limits_{\widetilde{B}_a} \xi\, d\mu \;\leqslant\; \sqrt{\mathbf{E}\,\xi^2}\sqrt{\mathbf{E}\,1_{\widetilde{B}_a}} \;\leqslant\; \frac{1}{2}\,\sqrt{\mathbf{E}\,\xi^2}.$$

Since ξ is positive and has log-concave distribution, there is a Khinchin type bound

$$\mathbf{E}\,\xi^2 \leqslant 2\,(\mathbf{E}\,\xi)^2$$

which is an inequality due to Karlin, Proshan, and Barlow [**16**]. Hence

$$\int\limits_{\widetilde{B}_a} \xi\, d\mu \;\leqslant\; \frac{1}{2}\,\sqrt{2\,(\mathbf{E}\,\xi)^2} \;=\; \frac{1}{\sqrt{2}}\,\mathbf{E}\,\xi.$$

It follows that

$$\int\limits_{B_a} \xi\, d\mu \;\geqslant\; \Big(1 - \frac{1}{\sqrt{2}}\Big)\,\mathbf{E}\,\xi \;\geqslant\; \Big(1 - \frac{1}{\sqrt{2}}\Big)\,\frac{1}{\sqrt{2}}\,\sqrt{\mathbf{E}\,\xi^2} \;=\; \frac{\sqrt{2} - 1}{2}\,\sqrt{n}.$$

Therefore, (5.4) is valid, as long as $r_0 \geqslant 2a\,(1 + \sqrt{2}) - 1$. One may take, for example, $r_0 = 5a$.

Thus, Proposition 5.1 is implied by the bound (4.4) of Proposition 4.3. □

Now, let us turn to the application of the logarithmic Sobolev inequality (5.2) about the measures μ_a to the concentration property of μ. There is a standard Herbst argument leading to the Gaussian concentration, but we prefer to follow an infimum-convolution approach of [**17**]. Namely, it is shown there (cf. Theorem 2.1) that (5.2) implies the inequality

$$\int e^{Q_t f}\, d\mu_a \int e^{-f}\, d\mu_a \leqslant 1, \tag{5.5}$$

where $t = Ca^2$, f is an arbitrary bounded measurable function on $\mathbf{R}^n$, and

$$Q_t f(x) \;=\; \inf\left[\, f(y) + \frac{|x - y|^2}{2t}\,\right], \quad x \in \mathbf{R}^n,$$

where the infimum is taken over all $y \in \mathbf{R}^n$, is the classical infimum-convolution operator. One may also involve in (5.5) unbounded f, possibly taking the values $\pm\infty$.

Given a nonempty measurable set A in $\mathbf{R}^n$, we apply (5.5) to the function f taking the value zero on A and $+\infty$ on $\mathbf{R}^n \setminus A$. Since for all $x \in \mathbf{R}^n$

$$Q_t f(x) \, = \, \frac{1}{2t}\, d(A,x)^2 \, = \, \frac{1}{2t} \inf_{y\in A} |x-y|^2,$$

we get

$$\int \exp\left\{\frac{1}{2Ca^2}\, d(A,x)^2\right\} d\mu_a(x) \, \leqslant \, \frac{1}{\mu_a(A)}.$$

Hence, by the Chebyshev inequality, for all $h \geqslant 0$

$$1 - \mu_a(A^h) \leqslant \frac{1}{\mu_a(A)}\, e^{-h^2/(2Ca^2)}.$$

In terms of μ, this is the same as

$$\mu(B_a) - \mu(A^h \cap B_a) \, \leqslant \, \frac{\mu(B_a)^2}{\mu(A\cap B_a)}\, e^{-h^2/(2Ca^2)},$$

which, by the concentration bound (5.3), implies

$$1 - \mu(A^h) \, \leqslant \, \frac{1}{\mu(A\cap B_a)}\, e^{-h^2/(2Ca^2)} + e^{-an/8}. \tag{5.6}$$

To further simplify, we take, for example, $a = 16$ and assume that

$$\mu(A) \geqslant e^{-n}.$$

Since

$$\mu(B_a) \geqslant 1 - e^{-2n},$$

we have

$$\mu(A \cap B_a) \geqslant \frac{1}{2}\, \mu(A),$$

and (5.6) yields

$$1 - \mu(A^h) \, \leqslant \, \frac{2}{\mu(A)}\, e^{-ch^2} + e^{-n} \tag{5.7}$$

with some absolute constant $c > 0$. Here, choosing a smaller c (for example, replacing it with $c/2$), one can replace the factor 2 with 1. In addition, (5.7) is fulfilled automatically in the case

$$\mu(A) \leqslant e^{-n}.$$

Therefore,

$$1 - \mu(A^h) \leqslant \frac{1}{\mu(A)}\, e^{-ch^2} + e^{-n}.$$

In particular, if $h \leqslant \sqrt{n}$, the last term e^{-n} may be ignored, and we get the following assertion.

Proposition 5.2. *Under the normalization condition* (5.1), *for any nonempty measurable set A in $\mathbf{R}^n$*

$$1 - \mu(A^h) \leqslant \frac{1}{\mu(A)}\, e^{-ch^2}, \quad 0 \leqslant h \leqslant \sqrt{n},$$

with some absolute constant $c > 0$.

If additionally $\mu(A) \geqslant 1/2$, we obtain the statement of Theorem 1.1 (by choosing a smaller $c > 0$, if necessary).

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References

1. S. G. Bobkov, "Spectral gap and concentration for some spherically symmetric probability measures," *Lect. Notes Math.* **1807** (2003), 37–43.
2. M. Gromov, V. D. Milman, "A topological application of the isoperimetric inequality," *Am. J. Math.* **105**, 843–854 (1983).
3. A. A. Borovkov, S. A. Utev, "On an inequality and related characterization of the normal distribution," *Probab. Theory Appl.* **28**, 209–218 (1983).
4. S. G. Bobkov, "Remarks on Gromov–Milman's inequality" [in Russian], *Vestn. Syktyvkar Univ. Ser. 1* **3**, 15–22 (1999).
5. A. Prékopa, "Logarithmic concave measures with application to stochastic programming." *Acta Sci. Math. (Szeged)* **32**, 301–316 (1971).
6. C. Borell, "Convex measures on locally convex spaces," *Ark. Math.* **12**, 239–252 (1974).
7. V. D. Milman, G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces*, Lect. Notes Math. **1200**, Springer, Berlin (1986).
8. M. Ledoux, *The Concentration of Measure Phenomenon*, Am. Math. Soc., Providence, RI (2001).
9. R. E. Barlow, A. W. Marshall, F. Proshan, "Properties of probability distributions with monotone hazard rate," *Ann. Math. Stat.* **34**, 375–389 (1963).
10. C. Borell, "Complements of Lyapunov's inequality," *Math. Ann.* **205**, 323–331 (1973).
11. B. Klartag, "A central limit theorem for convex sets," *Invent. Math.* **168**, No. 1, 91–131 (2007).
12. L. Lovász, M. Simonovits, "Random walks in a convex body and an improved volume algorithm," *Random Struct. Algorithms* **4**, No. 3, 359–412 (1993).
13. O. Guédon, "Kahane–Khinchine type inequalities for negative exponent," *Mathematika*, **46**, No. 1, 165–173 (1999).
14. S. G. Bobkov, "Isoperimetric and analytic inequalities for log-concave probability distributions," *Ann. Probab.* **27**, No. 4, 1903–1921 (1999).
15. C. E. Mueller, F. B. Weissler, "Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere," *J. Funct. Anal.* **48**, 252–283 (1992).
16. S. Karlin, F. Proshan, R. E. Barlow, "Moment inequalities of Polya frequency functions," *Pacific J. Math.* **11**, 1023–1033 (1961).
17. S. G. Bobkov, I. Gentil, M. Ledoux, "Hypercontractivity of Hamilton-Jacobi equations," *J. Math. Pures Appl.* **80**, No. 7, 669–696 (2001).

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