Probab. Theory Relat. Fields (2010) 147:303–332
DOI 10.1007/s00440-009-0209-7

Convex bodies and norms associated to convex measures

Sergey G. Bobkov

Received: 9 May 2008 / Revised: 6 February 2009 / Published online: 4 March 2009

Abstract Isotropy-like properties are considered for finite measures with heavy tails. As a basic tool, we extend K. Ball's relationship between convex bodies and finite logarithmically concave measures to a larger class of distributions, satisfying convexity conditions of the Brunn–Minkowski type.

Keywords Convex measures · Isotropic convex bodies · Floating bodies

Mathematics Subject Classification (2000) Primary 60xx; Secondary 46xx

1 Introduction

A symmetric convex body K in $\mathbf{R}^n$ is called isotropic or to be in isotropic position, if for some constant $L_K > 0$,

$$\frac{1}{\mathrm{vol}_n(K)^{1+\frac{2}{n}}} \int_K \langle x, \theta\rangle^2 dx = L_K^2, \tag{1.1}$$

for all (unit vectors) $\theta \in S^{n-1}$, where $\mathrm{vol}_n(K)$ stands for the n-dimensional volume. Intuitively, this means that K is more/less round and not dilated in any direction.

Supported in part by the NSF grant DMS-0706866.

S. G. Bobkov (✉)
School of Mathematics, University of Minnesota,
Vincent Hall 228, 206 Church St SE,
Minneapolis, MN 55455, USA
e-mail: bobkov@math.umn.edu

The quantity L_K is referred to as an isotropic constant of K. In general, for any symmetric convex body K, there is a linear invertible map $T : \mathbf{R}^n \to \mathbf{R}^n$, such that the image $T(K)$ represents an isotropic body. So, often, the condition (1.1) has the matter of normalization, only. Although the existence of the isotropic position [that is, of the map T or the body $T(K)$] is a rather obvious algebraic fact, it is of a large importance in many geometric problems, where it is essential that all linear functionals behave over K in a similar manner. We refer the reader to the pioneering work [34], where the isotropic position is discussed in various aspects; cf. also [7,23].

For a start, let us recall a theorem due to Hensley [28]: If K is isotropic, then all its central sections have approximately equal size in the sense that

$$\frac{\mathrm{vol}_{n-1}(K \cap H_1)}{\mathrm{vol}_{n-1}(K \cap H_2)} \leq C, \tag{1.2}$$

for all hyperplanes H_1, H_2 (passing through the origin). Moreover, Hensley proved (1.2) with an optimal dimension free constant $C = \sqrt{6}$. A similar property was also shown to hold for all subspaces of $\mathbf{R}^n$ of a fixed codimension (with some constants depending on codimensions). The inequality (1.2) was later rediscovered by Milman, as follows from [14], where this result was applied to obtain dimension free maximal inequalities.

As a natural extension, one may wonder, if an analogue of (1.2) continues to hold for classes of finite symmetric measures on $\mathbf{R}^n$, rather than for convex bodies, only. (When speaking about measures, bodies, or functions, the symmetry assumption will always be meant with respect to the origin.) Given a finite Borel measure μ on $\mathbf{R}^n$ with a (nice) density f and a linear subspace H in $\mathbf{R}^n$ of dimension $n-1$, introduce μ-perimeter of the half-space with boundary H,

$$\mu^+(H) = \int_H f(x)dx,$$

where dx stands for the Lebesgue measure on H. In particular, when μ is the Lebesgue measure on $\mathbf{R}^n$ restricted to a convex body K, $\mu^+(H)$ represents the $(n-1)$-dimensional volume of $K \cap H$, i.e., the size of the corresponding section of K. A more general intriguing question is therefore whether or not after some linear transformation the measure μ shares the property

$$\frac{\mu^+(H_1)}{\mu^+(H_2)} \leq C \tag{1.3}$$

similarly to (1.2). For example, if μ is spherically invariant, then it trivially satisfies (1.3) with $C = 1$, and no linear transformation is needed.

Since (1.3) requires that μ must be in a sense “round” like in the convex body case, it is natural to assume that the integral

$$\sigma^2 = \int \langle x, \theta \rangle^2 d\mu(x) \tag{1.4}$$

as a function of θ is constant on the unit sphere. This property, extending the definition (1.1) of the isotropy to the measure case, is typically used when μ is log-concave, i.e, when it has a log-concave density f. As was shown in [7,28], although it was not stated there explicitly, if a symmetric finite log-concave measure μ on $\mathbf{R}^n$ is isotropic in the sense of (1.4), it satisfies the desired relation (1.3), and with the same constant C as in the convex body case. This is actually reduced to the statement that, for any symmetric, log-concave probability density f on the real line,

$$\frac{1}{6} \le 2f(0)^2 \int_{-\infty}^{+\infty} |x|^2 f(x)dx \le 1.$$

Some further extensions of Hensley's theorem to non-symmetric bodies, considered in [9,20], remain also to hold for non-symmetric log-concave measures.

However, when trying to obtain (1.3) via (1.4) for more general classes of measures, a main difficulty is that (1.4) requires finiteness of the second moment $\int |x|^2 d\mu(x)$. And even if it is finite, the previous argument does not work [while, the example of the spherically invariant measures shows that any integrability assumption is irrelevant for assertions such as the Hensley-type property (1.3)]. This inspires to look for other isotropy-like conditions that would be still appropriate for measures with heavy tails.

In this note we consider such problems for general convex measures. Following Borell [12,13], a finite measure μ on $\mathbf{R}^n$ is called convex, if

$$\mu(tA + sB) \ge \min\{\mu(A), \mu(B)\} \tag{1.5}$$

for all non-empty Borel sets A and B in $\mathbf{R}^n$ and $t, s > 0$, such that $t + s = 1$, with the usual understanding of the Minkowski sum $tA + sB = \{tx + sy : x \in A, y \in B\}$. The definition also makes sense, when a measure is finite at least on compact subsets of the space. If μ is absolutely continuous with respect to Lebesgue measure (we also say that it is full-dimensional), the inequality (1.5) is equivalent to the property that μ is concentrated on some open convex set $\Omega \subset \mathbf{R}^n$, where it has a density f satisfying

$$f(tx + sy) \ge \left(tf(x)^{-1/n} + sf(y)^{-1/n}\right)^{-n} \tag{1.6}$$

for all $x, y \in \Omega$ and $t, s > 0$ such that $t + s = 1$. That is, the density should be of the form $f = V^{-n}$ for some positive convex function V on Ω. This is part of Borell's characterization theorem ([12], Theorem 1.1; [13], Theorem 3.2). For definiteness, we define f to be zero outside Ω.

Note f must be continuous on Ω. However, in contrast with the log-concave case, the tail function $\mu\{|x| > r\}$ may decrease to zero as $r \to +\infty$ as slow, as we wish.

As we will see, Hensley's theorem admits the following generalization.

Theorem 1.1 *Let μ be a full-dimensional symmetric finite convex measure on $\mathbf{R}^n$. There is a linear invertible map $T : \mathbf{R}^n \to \mathbf{R}^n$, such that the image $\nu = T(\mu)$ satisfies*

$$\nu^+(H_1) \le \sqrt{6}\nu^+(H_2), \tag{1.7}$$

for all hyperplanes H_1, H_2 in $\mathbf{R}^n$ passing through the origin.

The image $\nu = T(\mu)$ is defined to be the measure $\nu(B) = \mu\{x \in \mathbf{R}^n : T(x) \in B\}$ on Borel subsets B of $\mathbf{R}^n$.

It turns out that the inequality (1.7) may be reduced to the convex body case by virtue of a remarkable correspondence, discovered by K. Ball in his study of logarithmically concave functions. Namely, given a non-negative, symmetric function f on $\mathbf{R}^n$, put

$$\|x\| = \left(\int_0^{+\infty} f(rx)\, dr^p \right)^{-1/p}, \quad x \in \mathbf{R}^n. \tag{1.8}$$

A principal result of [7] is that, if f is log-concave and integrable, with $f(0) > 0$, then $\|\cdot\|$ represents a norm on $\mathbf{R}^n$ for any $p > 0$. Therefore, we obtain a family of symmetric convex bodies in $\mathbf{R}^n$, parameterized by p, and they indeed may be used to reduce various problems about log-concave measures to the ones about convex bodies (such as the slicing problem, for example). Here we prove that, if f is not log-concave, still there is a somewhat weaker property:

Theorem 1.2 *Let μ be a finite symmetric, full-dimensional convex measure on $\mathbf{R}^n$ with density f. Whenever $0 < p \le n-1$, the equality (1.8) defines a norm.*

The particular case $p = n - 1$ together with Hensley's theorem yields (1.7). More precisely, we obtain that, for any finite symmetric, full-dimensional convex measure μ on $\mathbf{R}^n$, there exists a unique symmetric convex body K, such that

$$\mu^+(H) = \mathrm{vol}_{n-1}(K \cap H),$$

for any linear subspace H of $\mathbf{R}^n$ of dimension $n-1$. And this body represents the unit ball with respect to the norm (1.8). Hence, for a linear map T in Theorem 1.1 one may take the one, which puts in the isotropic position the convex body K (and then $\nu = T(\mu)$ itself may be viewed as an isotropic measure).

The interval of possible values of p in Theorem 1.2 may be enlarged and related to the convexity properties according to Borell's hierarchy of convex measures. This we discuss in the next section, where the proofs are also included. Some applications, including Theorem 1.1 and its generalization for subspaces of a fixed codimension, are considered in Sect. 3. In Sect. 4, we derive a lower bound on the size of the slices of the measure, $\mu^+(H)$, in terms of the size of the slices of the associated convex body. In Sects. 5 and 6 we examine another possible approach to the isotropy in terms of the so-called floating bodies and surfaces of convex measures (in fact with arguments which are also used in the proof of Theorem 1.2). Finally, in Sect. 7 we conclude with remarks on the duality and Santaló-type inequalities for densities of convex measures.

2 Extension of K. Ball's theorem

The description (1.6) represents a particular case of the characterization of the so-called κ-concave measures, given by Borell [12,13]. We say that a finite Borel measure μ

on $\mathbf{R}^n$ is κ-concave, where $-\infty \le \kappa \le +\infty$, if it satisfies

$$\mu(tA + sB) \ge (t\mu(A)^\kappa + s\mu(B)^\kappa)^{1/\kappa} \tag{2.1}$$

for all non-empty Borel sets A and B in $\mathbf{R}^n$ and $t, s > 0$ with $t + s = 1$. If μ is not concentrated at a point, then necessarily $\kappa \le 1$, and moreover, $\kappa \le 1/n$ in the absolutely continuous case. More precisely, in that case (2.1) is equivalent to the property that μ is concentrated on some open convex set $\Omega \subset \mathbf{R}^n$, where it has a density f, satisfying, for all $x, y \in \Omega$ and $t, s > 0$, $t + s = 1$,

$$f(tx + sy) \ge (tf(x)^{\kappa_n} + sf(y)^{\kappa_n})^{1/\kappa_n}, \tag{2.2}$$

where $\kappa_n = \kappa/(1 - n\kappa)$. Following Caplin and Nalebuff [17], one may say that f is κ_n-concave (cf. also [5] for a special discussion of such functions).

One may start with an arbitrary κ_n-concave function f on Ω, and then it will serve as a density of a σ-finite measure μ, concentrated on Ω and satisfying the Brunn–Minkowski-type inequality (2.1), cf. Lemma 2.3 below. In this sense, μ will be κ-concave; for example, the Lebesgue measure itself is $\frac{1}{n}$-concave. However, all statements below are restricted to the family of finite κ-concave measures.

Note the inequality (2.1) becomes stronger, as κ increases, so for $\kappa = -\infty$ we obtain the largest class of (convex) measures, described by the Brunn–Minkowski-type inequality (1.5). If $\kappa = 0$, (2.1)–(2.2) describe log-concave measures, while the case $\kappa = 1/n$ is only possible, when, up to a factor, μ is the Lebesgue measure, restricted to Ω (provided that it is full-dimensional).

Theorem 2.1 *Let μ be a finite, symmetric, full-dimensional κ-concave measure on $\mathbf{R}^n$, $-\infty \le \kappa \le 0$, with density f as above. The equality*

$$\|x\| = \left(\int_0^{+\infty} f(rx)\, dr^p \right)^{-1/p} \tag{2.3}$$

defines a norm on $\mathbf{R}^n$, whenever $0 < p \le n - 1 - \frac{1}{\kappa}$.

If K is a symmetric convex body in $\mathbf{R}^n$ with the inner norm

$$\|x\|_K = \min\{\lambda \ge 0 : x \in \lambda K\},$$

and μ is the Lebesgue measure on $\mathbf{R}^n$, restricted to K (thus with density $f = 1_K$, the indicator function of K), the above definition leads to the equal norms $\|x\| = \|x\|_K$ regardless of $p > 0$. Hence, general convex bodies do not create special bodies.

When a finite symmetric measure μ or its density f are log-concave ($\kappa = 0$), Theorem 2.1 corresponds to the result of Ball [7]. In that case, f decays exponentially fast at infinity, so the integral in the theorem is finite for any $p > 0$.

If μ is κ-concave with $\kappa < 0$, then for some constant C, depending on the measure, its density satisfies, for all $x \in \mathbf{R}^n$ (cf. [11]),

$$f(x) \le \frac{C}{1 + |x|^{n+\alpha}}, \quad \alpha = -\frac{1}{\kappa}. \tag{2.4}$$

Therefore, for any $x \neq 0$, the integral $\int_0^{+\infty} f(rx)\,dr^p$ is finite, as long as

$$p < n - \frac{1}{\kappa}. \tag{2.5}$$

It would be rather interesting to clarify whether or not this condition provides a more precise restriction for the functional (2.3) to be a norm. In particular, one may think this is true for general symmetric convex measures in the important case $p = n$, when (2.3) becomes

$$\|x\| = \left(\int_0^{+\infty} f(rx)\,dr^n \right)^{-1/n}.$$

By Theorem 2.1, it is a norm, if $\kappa \ge -1$. Anyway, by the polar representation of the Lebesgue integral, the set $K = \{x \in \mathbf{R}^n : \|x\| \le 1\}$ has a finite volume, $\mu(\mathbf{R}^n)$.

Example Let μ denote the generalized Cauchy distribution on $\mathbf{R}^n$ with density

$$f(x) = \frac{c}{(1 + |x|^2)^{\frac{n+d}{2}}}, \quad x \in \mathbf{R}^n,$$

where $d > 0$ is a parameter and $c = c_{n,d}$ is a normalizing constant. (The standard Cauchy distribution corresponds to $d = 1$.) As easy to check, this measure is κ-concave with optimal value $\kappa = -1/d$. In this case, all norms (2.3) are proportional to the Euclidean norm in $\mathbf{R}^n$, provided that they are finite. The latter is equivalent to the requirement $p < n + d$, which is exactly the condition (2.5).

Remark 2.2 In dimension one, according to Borell's characterization (2.2), a probability measure μ on the real line $\mathbf{R}$ with density f is κ-concave, if and only if the function f is $\frac{\kappa}{1-\kappa}$-concave. In particular, μ is convex if and only if $1/f$ is convex on the supporting interval, say (a, b), of the measure μ. This characterization may equivalently be stated in terms of the associated function

$$I(t) = f(F^{-1}(t)), \quad 0 < t < 1,$$

where $F^{-1} : (a, b) \to (0, 1)$ is the inverse to the distribution function $F(x) = \mu(a, x), a < x < b$. Namely (cf. [11], Lemma 2.2), μ is κ-concave, if and only if the function $I^{1/(1-\kappa)}$ is concave on (0,1). This immediately follows from the general identity

$$\kappa\,(I^{1/(1-\kappa)})'(F(x)) = (f(x)^{\kappa/(1-\kappa)})', \quad \kappa \neq 0.$$

Similarly, μ is convex, if and only if the function $\log I$ is concave on (0,1).

Therefore, in order to describe an arbitrary non-degenerate convex probability measure μ on the line with median at zero (for definiteness), one may start with an arbitrary positive log-concave function I on (0,1) and then put $F^{-1}(t) = \int_{1/2}^{t} \frac{ds}{I(s)}$. The latter shows, in particular, that the tail function of the measure may decay at infinity as slow as we wish. More precisely, for any function $\varepsilon = \varepsilon(r)$ such that $\varepsilon(r) \downarrow 0$, as $r \to +\infty$, there exists a convex symmetric probability measure μ, satisfying

$$\mu\{|x| > r\} > \varepsilon(r), \quad \text{for all } r \text{ large enough.}$$

Equivalently, $F^{-1}(t) > R(t)$ for all t close to 1, where R is any prescribed increasing function on $[1/2, 1)$. To see this, first note that R may be assumed to be smooth (otherwise, one may rescale the coordinates and apply convolution) and to satisfy $R(t) \uparrow +\infty$ for $t \uparrow 1$. Secondly, any smooth function S on $[1/2, 1)$ is majorized by the convex function $\varphi(t) = S(1/2) + \int_{1/2}^{t} \max_{1/2 \leq u \leq s} S'(u)\, ds$. Take such a convex function for $S(t) = \log(1 + 2R'(t))$. Then (by comparing derivatives), we readily get $F^{-1}(t) = \int_{1/2}^{t} e^{\varphi(s)}\, ds \geq 2(R(t) - R(1/2))$. But $2(R(t) - R(1/2)) > R(t)$ for all t close to 1, and then $F^{-1}(t) > R(t)$. Finally, extending φ to (0,1/2] by symmetry about 1/2, we obtain F^{-1}, which corresponds to a convex probability measure with the associated log-concave function $I(t) = e^{-\varphi(t)}$.

Now, let us turn to the proof of Theorem 2.1. We need in the dimension $n = 1$ Borell–Brascamp–Lieb's functional form for the Brunn–Minkowski inequality, which we state as a lemma below, cf. [12,13,16,18,19]. [Also note it provides the implication (2.2) $\Rightarrow$ (2.1).]

Lemma 2.3 *Let $t, s > 0$ be fixed, $t + s = 1$, and let $-\infty \leq \kappa \leq \frac{1}{n}$. Assume non-negative measurable functions u, v, w, defined on an open convex set $\Omega \subset \mathbf{R}^n$, satisfy*

$$w(tx + sy) \geq (t^{\kappa_n} u(x) + s v(y)^{\kappa_n})^{1/\kappa_n}, \quad x, y \in \Omega, \tag{2.6}$$

with $\kappa_n = \frac{\kappa}{1 - n\kappa}$, whenever $u(x)v(y) > 0$. Then

$$\int_{\Omega} w(z)\, dz \;\geq \left[t \left(\int_{\Omega} u(x)\, dx \right)^{\kappa} + s \left(\int_{\Omega} v(y)\, dy \right)^{\kappa} \right]^{1/\kappa}. \tag{2.7}$$

Introduce the κ-mean functions

$$M_{\kappa}^{(t)}(a, b) = (ta^{\kappa} + (1 - t)b^{\kappa})^{1/\kappa}, \qquad a, b \geq 0, \quad t \in (0, 1),$$

appearing in (2.6)–(2.7). In particular, $M_0^{(t)}(a, b) = a^t b^{1-t}$ and $M_{-\infty}^{(t)}(a, b) = \min\{a, b\}$.

From Lemma 2.3, we first derive the following assertion of independent interest (which appeared for the particular case $q = 0$ in [7] and before in [6], where it was called "the Brunn–Minkowski inequality for the harmonic mean").

Lemma 2.4 *Let $t, s > 0$, $t + s = 1$, and $q \in [-1, 0]$ be fixed. Assume non-negative measurable functions u, v, w are defined on $(0, +\infty)$ and satisfy for all $x, y > 0$*

$$w\left(M_{-1}^{(t)}(x, y)\right) \geq M_q^{(\lambda)}(u(x), v(y)), \tag{2.8}$$

where $\lambda = ty/(ty + sx)$. Then, if $0 \leq p \leq -1 - 1/q$,

$$\left[\int_0^{+\infty} w(z)\, dz^p\right]^{-1/p} \leq t \left[\int_0^{+\infty} u(x)\, dx^p\right]^{-1/p} + s \left[\int_0^{+\infty} v(y)\, dy^p\right]^{-1/p}. \tag{2.9}$$

In the extreme case $p = 0$, (2.9) should be understood as

$$\int_0^{+\infty} \frac{w(z)}{z} dz \geq \min\left\{\int_0^{+\infty} \frac{u(x)}{x} dx, \int_0^{+\infty} \frac{v(y)}{y} dy\right\}.$$

Proof Since any function of the type $q \to M_q$ is non-decreasing, one may assume $p = -1 - \frac{1}{q}$ or $q = -\frac{1}{p+1}$. Change the variables $x = \frac{1}{\xi}$, $y = \frac{1}{\eta}$ with arbitrary $\xi, \eta > 0$. Then the assumption (2.8) takes the form

$$(t\xi + s\eta)^{1/q}\, w\left(\frac{1}{t\xi + s\eta}\right) \geq \left[t\, \xi\, u(1/\xi)^q + s\, \eta\, v(1/\eta)^q\right]^{1/q}.$$

In other words, the new three functions

$$\bar{u}(\xi) = \xi^{1/q}\, u(1/\xi), \quad \bar{v}(\eta) = \eta^{1/q}\, v(1/\eta), \quad \bar{w}(\zeta) = \zeta^{1/q} w(1/\zeta)$$

satisfy on $\Omega = (0, +\infty)$

$$\bar{w}(t\xi + s\eta) \geq M_q^{(t)}(\bar{u}(\xi), \bar{v}(\eta)).$$

So, the hypothesis (2.6) of Lemma 2.3 is fulfilled in dimension one for these functions with $\kappa_1 = q$, that is, $\kappa = \frac{q}{q+1} = -\frac{1}{p}$. Therefore, we obtain (2.7), which reads as

$$\int_\Omega \bar{w}(\zeta)\, d\zeta \;\geq\; M_{-1/p}^{(t)}\left(\int_\Omega \bar{u}(\xi)\, d\xi, \int_\Omega \bar{v}(\eta)\, d\eta\right). \tag{2.10}$$

Now, if $q > -1$, or equivalently $p > 0$, return to the original variables and note that

$$\int_\Omega \bar{u}(\xi)\, d\xi = \int_0^{+\infty} \xi^{1/q}\, u(1/\xi)\, d\xi = \frac{1}{p} \int_0^{+\infty} u(x)\, dx^p,$$

and similarly for v and w. Thus, (2.10) is exactly the desired inequality (2.9). If $q = -1$ (that is, $p = 0$), (2.10) becomes

$$\int_{\Omega} \bar{w}(\zeta)d\zeta \geq \min\left\{\int_{\Omega} \bar{u}(\xi)d\xi, \int_{\Omega} \bar{v}(\eta)d\eta\right\}.$$

But $\int_{\Omega} \bar{u}(\xi)d\xi = \int_0^{+\infty} \frac{u(x)}{x}\,dx$ and similarly for v and w. Lemma 2.4 is proved. □

Remark 2.5 One may conjecture that, under the additional assumptions on the functions u, v and w that they are non-decreasing on the positive half-axis and satisfy $\|u\|_\infty = \|v\|_\infty$, the inequality (2.10) will remain to hold in the larger interval $0 \leq p \leq -\frac{1}{q}$. This would extend Theorem 2.1 to the values $p \leq n - \frac{1}{\kappa}$.

Proof of Theorem 2.1. Let f be the density of μ, supported on Ω and satisfying (2.2). Since it is even, $\|\lambda x\| = |\lambda|\,\|x\|$, for all $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$. As was already explained, $\|x\| < +\infty$ everywhere. Fix $a, b \in \mathbf{R}^n$, $t \in (0, 1)$, $s = 1 - t$, and apply Lemma 2.4 to the functions on $(0, +\infty)$

$$u(x) = f(xa), \quad v(y) = f(yb), \quad w(z) = f(zc),$$

where $c = ta + sb$. Since

$$M_{-1}^{(t)}(x, y)\, c = \frac{ty}{ty + sx}\,(xa) + \frac{sx}{ty + sx}\,(yb),$$

and f is κ_n-concave, the hypothesis (2.8) is fulfilled with $q = \kappa_n$. The resulting inequality (2.9) of the lemma shows that the function $x \to \|x\|$ is convex. This yields Theorem 2.1. □

Remark 2.6 As the above proof shows, if we drop the symmetry assumption in Theorem 2.1, the functional

$$N_p(x) = \left(\int_0^{+\infty} f(rx)\, dr^p\right)^{-1/p}, \quad x \in \mathbf{R}^n,$$

still shares the properties:

(a) $0 \leq N_p \leq +\infty$, and $N_p(x) = 0$ if and only if $x = 0$;
(b) $N_p(\lambda x) = \lambda N_p(x)$, for all $x \in \mathbf{R}^n$ and $\lambda \geq 0$;
(c) $N_p(tx + sy) \leq tN_p(x) + sN_p(y)$, for all $x, y \in \mathbf{R}^n$ and $t, s > 0$ with $t + s = 1$.

Moreover, N_p is everywhere finite, as long as the supporting open convex set Ω of the measure μ contains the origin. In this case, $\{x \in \mathbf{R}^n : N_p(x) \leq 1\}$ represents a convex body in $\mathbf{R}^n$.

3 Isotropic positions

Theorem 2.1 inspires to introduce the following definition in the class of convex measures. Let us say that a finite, symmetric, full-dimensional convex measure μ with density f on $\mathbf{R}^n$ is q-isotropic, where $-n < q < +\infty$, if the set

$$K_\mu(q) = \left\{ x \in \mathbf{R}^n : \int_0^{+\infty} f(rx)\, dr^{n+q} \geq 1 \right\}$$

represents an isotropic convex body in the classical sense (1.1).

When μ is the Lebesgue measure on a symmetric convex body K, then $K_\mu(q) = K$ for all $q > -n$, so the property of being q-isotropic does not depend on q.

Note $K_\mu(q)$ is always a convex body for $-n < q \leq -1$. Moreover, if μ is κ-concave, $-\infty < \kappa \leq 0$, the latter is true for a larger range $-n < q \leq -1 - \frac{1}{\kappa}$. In particular, there is no upper level restriction on q, when μ is log-concave.

If $K_\mu(q)$ is a convex body, but not isotropic, one can find a linear invertible map $T : \mathbf{R}^n \to \mathbf{R}^n$ with $|\det(T)| = 1$, which puts $K_\mu(q)$ in the isotropic position, i.e., such that the convex body $T(K_\mu(q))$ is isotropic. And then the image of the measure, $T(\mu)$, will be q-isotropic, since

$$K_{T(\mu)}(q) = T(K_\mu(q)).$$

Thus, the linear maps, which preserve the Lebegue measure on $\mathbf{R}^n$ and put convex measures and associated convex bodies in the q-isotropic positions, are the same.

If $q = 2$ and μ has a finite second moment, we return to the usual isotropy definition (1.4) for measures, which requires that $L^2(\mu)$-norms of the linear functionals, $\|\langle x, \theta \rangle\|_2 = (\int \langle x, \theta \rangle^2 d\mu(x))^{1/2}$, are constant on the unit sphere. This is due to the general identity

$$\int |\langle x, \theta \rangle|^2 d\mu(x) = \int_{K_\mu(2)} |\langle x, \theta \rangle|^2 dx.$$

Since $K_\mu(2)$ might not be a convex body in general, one could work with $L^q(\mu)$ -"norms" of the linear functionals $x \to \langle x, \theta \rangle$, other than $L^2(\mu)$-norms, even if $q < 1$ (for example, for $q = 0$ or $q = -1/2$). Indeed, if μ is κ-concave, $-\infty < \kappa < 0$, the bound on the density (2.4) ensures that $\int |x|^q d\mu(x) < +\infty$, whenever $0 < q < -1/\kappa$. This was already noted by Borell [12]. In that case

$$\|\langle x, \theta \rangle\|_q = \left(\int |\langle x, \theta \rangle|^q d\mu(x) \right)^{1/q}, \quad q < -1/\kappa,$$

is finite for all directions θ. Hence, one may wonder whether or not, after some linear transformation of μ these L^q-quantities will be equal or almost equal to each other along all directions. This turns out to be true and may be shown on the basis of the body case by virtue of the norms associated to μ.

Theorem 3.1 *Let μ be a finite, symmetric, full-dimensional κ-concave measure on $\mathbf{R}^n$, $-\infty < \kappa \le 0$. If it is q-isotropic with $-1 < q \le -1 - 1/\kappa\,(q < +\infty)$, then for all unit vectors θ, θ',*

$$\frac{1}{C}\,\|\langle x, \theta' \rangle\|_q \le \|\langle x, \theta \rangle\|_q \le C\,\|\langle x, \theta' \rangle\|_q, \tag{3.1}$$

where the constant C depends q, only.

Proof In accordance with Theorem 2.1, consider the family of the norms on $\mathbf{R}^n$,

$$\|x\|_q = \left(\int_0^{+\infty} f(rx)\, dr^{n+q} \right)^{-1/(n+q)}, \quad -n < q \le -1 - \frac{1}{\kappa} \quad (q < +\infty),$$

and denote by $K(q) = K_\mu(q)$ the corresponding unit balls. Note the Lebesgue measure, restricted to $K(q)$, with its density $f_q = 1_{K(q)}$ generates the same norm $\|x\|_q$ as μ. Using the polar coordinates $x = ru$ ($u \in S^{n-1}$, $r > 0$), write

$$\begin{aligned}
\int |\langle x, \theta \rangle|^q \, d\mu(x) &= \int_{\mathbf{R}^n} |\langle x, \theta \rangle|^q \, f(x)\, dx \\
&= \omega_n \int_{S^{n-1}} |\langle u, \theta \rangle|^q \left(\int_0^{+\infty} r^q f(ru)\, dr^n \right) d\sigma_{n-1}(u) \\
&= \frac{n\omega_n}{n+q} \int_{S^{n-1}} |\langle u, \theta \rangle|^q \left(\int_0^{+\infty} f(ru)\, dr^{n+q} \right) d\sigma_{n-1}(u) \\
&= \frac{n\omega_n}{n+q} \int_{S^{n-1}} |\langle u, \theta \rangle|^q \, \|u\|_q^{-n-q} \, d\sigma_{n-1}(u),
\end{aligned}$$

where ω_n denotes the volume of the unit Euclidean ball in $\mathbf{R}^n$. Similarly,

$$\int_{K(q)} |\langle x, \theta \rangle|^q \, dx = \frac{n\omega_n}{n+q} \int_{S^{n-1}} |\langle u, \theta \rangle|^q \, \|u\|_q^{-n-q} \, d\sigma_{n-1}(u).$$

Therefore, for all θ we have $\int |\langle x, \theta \rangle|^q \, d\mu(x) = \int_{K(q)} |\langle x, \theta \rangle|^q \, dx$. From this, if $q \neq 0$, for all $\theta, \theta' \in S^{n-1}$,

$$\frac{\|\langle x, \theta \rangle\|_{L^q(\mu)}}{\|\langle x, \theta' \rangle\|_{L^q(\mu)}} = \frac{\|\langle x, \theta \rangle\|_{L^q(\mu_q)}}{\|\langle x, \theta' \rangle\|_{L^q(\mu_q)}}, \tag{3.2}$$

where μ_q denotes the normalized Lebesgue measure on the set $K(q)$. But, in the convex body case (cf. [24,25]), L^2-norms of the linear functionals are equivalent to L^q-norms for all $q > -1$ in the sense that

$$\frac{1}{C_q} \, \|\langle x, \theta \rangle\|_2 \leq \|\langle x, \theta \rangle\|_q \leq C_q \, \|\langle x, \theta \rangle\|_2.$$

[Here the norms are with respect to the uniform distribution over $K(q)$.] Hence, when $K(q)$ is isotropic, we get (3.1) from (3.2). □

As another application, let us now turn to the comparison of the sections of a measure and the Hensley-type theorem. A main observation in this section is:

Theorem 3.2 *Let d be integer,* $1 \leq d \leq n-1$. *For any finite, symmetric, full-dimensional convex measure* μ *on* $\mathbf{R}^n$ *with density* f*, there exists a unique symmetric convex body* K_d *in* $\mathbf{R}^n$*, such that*

$$\int_H f(x)dx = \mathrm{vol}_{n-d}(K_d \cap H) \tag{3.3}$$

for any linear subspace H *in* $\mathbf{R}^n$ *of codimension* d*. Namely,* $K_d = K_\mu(-d)$.

Introduce the quantity

$$\mu^+(H) = \int_H f(x)\,dx$$

similarly to the slices case $d = 1$. Note the integral is finite in view of the general estimate $f(x) \leq C(1+|x|)^{-n}$ on the density of a convex measure [cf. (2.4)].

Proof The associated convex body,

$$K_d = K_\mu(-d) = \left\{ x \in \mathbf{R}^n : \int_0^{+\infty} f(rx)\,dr^{n-d} \geq 1 \right\},$$

is defined for the norm $\|x\| = \left(\int_0^{+\infty} f(rx)\,dr^{n-d}\right)^{-1/(n-d)}$. Given a linear subspace H in $\mathbf{R}^n$ of codimension d, one may use the polar coordinates $x = r\theta, r > 0, |\theta| = 1, \theta \in H$, to write that

$$\begin{aligned} \mu^+(H) = \int_H f(x)dx &= \omega_{n-d} \int_{S^{n-1}\cap H} \left(\int_0^{+\infty} f(r\theta)dr^{n-d} \right) d\sigma_{n-d-1}(\theta) \\ &= \omega_{n-d} \int_{S^{n-1}\cap H} \|\theta\|^{-(n-d)} d\sigma_{n-d-1}(\theta), \end{aligned}$$

where σ_{n-d-1} is the normalized Lebesgue measure on the unit sphere of H. Similarly, by the definition of K_d,

$$\mathrm{vol}_{n-d}(K_d \cap H) = \omega_{n-d} \int_{S^{n-1} \cap H} \|\theta\|^{-(n-d)} d\sigma_{n-d-1}(\theta).$$

This proves the desired relation (3.3).

Uniqueness is provided by the Funk (also called Funk–Minkowski) theorem: The function of the form $H \to \mathrm{vol}_{n-d}(K \cap H)$, defined on the set of all linear subspaces of $\mathbf{R}^n$ of codimension d, uniquely determines a symmetric convex body K. See [22, Theorem 7.2.3], where this property is proved by using the injectivity of the spherical Radon transform, or [29, Corollary 3.10], with the proof based on the Fourier transform. □

Remark Relation (3.3) with $K_d = \{x : \int_0^{+\infty} f(rx)\, dr^{n-d} \geq 1\}$ continues to hold without the symmetry assumption. More precisely, if the supporting open convex set of the measure μ contains the origin, then K_d is a convex body (cf. Remark 2.6) and (3.3) is fulfilled.

Under the same assumptions on the measure as in Theorem 3.2, we obtain the following corollary (containing Theorem 1.1 in case $d = 1$).

Corollary 3.3 *If μ is q-isotropic with $q = -d$, for all linear subspaces H_1, H_2 in $\mathbf{R}^n$ of codimension d,*

$$\frac{\mu^+(H_1)}{\mu^+(H_2)} \leq C_d, \tag{3.4}$$

where the constant C_d depends on d, only. In particular, $C_1 = \sqrt{6}$.

Indeed, when K is an isotropic symmetric convex body in $\mathbf{R}^n$ with volume $\mathrm{vol}_n(K) = 1$, Theorem $1'$ in [28] asserts that, for any linear subspace H in $\mathbf{R}^n$ of codimension d,

$$C'_d \leq L_K^d \mathrm{vol}_{n-d}(K \cap H) \leq C''_d, \tag{3.5}$$

where L_K is an isotropic constant of the body, and C'_d, C''_d are certain positive (explicit) constants, depending on d, only. In particular, $C'_1 = \frac{1}{2\sqrt{3}}$, $C''_1 = \frac{1}{\sqrt{2}}$, which are optimal [in this case, treated by D. Hensley separately, (3.5) follows from the functional inequality (3.9) below]. Therefore, for all linear subspaces H_1 and H_2 of codimension d,

$$\frac{\mathrm{vol}_{n-d}(K \cap H_1)}{\mathrm{vol}_{n-d}(K \cap H_2)} \leq C_d \tag{3.6}$$

with $C_d = C''_d/C'_d$. In particular, $C_1 = \sqrt{6}$. When, however, d grows, Hensley's constant C_d grows faster than $d^{(d^2+d)/2}$. It was improved by K. Ball in [7], where (3.6) is obtained with $C_d = (d(d+1)(d+2))^{d/2}\omega_d/(2^d d!^{1/2})$, so that $C_1 = \sqrt{6}$ and $C_d < (\frac{1}{2}\pi e^2 d)^{d/2}$ in general.

Note the assumption on the volume may be removed in (3.6). It remains to apply the equality (3.3) to H_1, H_2 and deduce (3.4) from (3.6) with $K = K_d$.

A similar comparison statement continues to hold for other positions. However, we do not know whether the involved constant may be made universal. Let us look at what one can get for 0-isotropic convex measures. We need:

Lemma 3.4 *Let ξ be a non-degenerate real-valued random variable with a symmetric κ-concave probability distribution, $-\infty < \kappa \le 1$. If g is its density, then, whenever $q > -1$ and $q < -1/\kappa$ (in case $\kappa \le 0$),*

$$c_1(q) \le g(0)\|\xi\|_q \le c_\kappa(q), \tag{3.7}$$

where $c_\kappa(q) > 0$ depends on (κ, q), only.

Proof The distribution ν of ξ is concentrated on some symmetric interval $(-a, a)$. Introduce the distribution function $G(x) = \nu(-\infty, x] = \mathbf{P}\{\xi \le x\}$ and its inverse $G^{-1} : (0,1) \to (-a, a)$. Define $I(t) = g(G^{-1}(t))$ so that

$$G^{-1}(t) = \int_{1/2}^{t} \frac{ds}{I(s)}, \quad 0 < t < 1.$$

As mentioned in Remark 2.2, the function $I^{1/(1-\kappa)}$ is concave on (0, 1), which is equivalent to the κ-concavity of ν. For normalization convenience, assume $g(0) = 1$, so that $I(1/2) = g(0) = 1$. Then, by the concavity, for all $s \in (0, 1)$,

$$(2\min\{s, 1-s\})^{1-\kappa} \le I(s) \le 1.$$

Therefore,

$$\left|\int_{1/2}^{t} ds\right| \le |G^{-1}(t)| \le \left|\int_{1/2}^{t} \frac{ds}{(2\min\{s, 1-s\})^{1-\kappa}}\right|.$$

Since the distribution of G^{-1} under the Lebesgue measure on (0, 1) coincides with ν, we conclude (with a standard modification in case $q = 0$) that

$$\left(\int_0^1 \left|\int_{1/2}^{t} ds\right|^q dt\right)^{1/q} \le \|\xi\|_q \le \left(\int_0^1 \left|\int_{1/2}^{t} \frac{ds}{(2\min\{s, 1-s\})^{1-\kappa}}\right|^q dt\right)^{1/q}. \tag{3.8}$$

Here the right-hand side represents the constant $c_\kappa(q)$, and for $\kappa = 1$ it coincides with the left-hand side of (3.8), representing $c_1(q)$. The requirement $q > -1$ is needed for $c_1(q) > 0$. If $\kappa \geq 0$, we have $c_\kappa(q) < +\infty$ for all q. However, when κ is negative, this constant is finite if and only if $q < -1/\kappa$. The lemma is proved. □

Remark As follows from (3.8), the optimal value of $c_\kappa(q)$ in (3.7) is attained regardless of q, when the distribution ν of the random variable ξ corresponds to the associated function $I(s) = (2\min\{s, 1-s\})^{1-\kappa}$. In particular, $c_1(q)$ is attained for the uniform distribution on $[-\frac{1}{2}, \frac{1}{2}]$ and $c_0(q)$ for the two-sided exponential distribution with density $g(x) = e^{-2|x|}$. If $\kappa < 0$, the extreme measure ν may be viewed as a symmetrized Pareto distribution with parameter $\alpha = -1/\kappa$.

Let us also mention that in the log-concave case (κ=0) and when q=2, we have $c_1(q)^2 = \|\xi\|_2^2 = \frac{1}{12}$ (where ξ is uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$) and $c_0(q)^2 = \|\xi\|_2^2 = \frac{1}{2}$ (with ξ exponentially distributed). Therefore, the inequality (3.7) of Lemma 3.4 becomes

$$\frac{1}{12} \leq g(0)^2 \int_{-\infty}^{+\infty} x^2 g(x)dx \leq \frac{1}{2}, \tag{3.9}$$

which thus holds true for any symmetric log-concave probability density g on the line. This is exactly what Hensley proved towards his inequality (3.5) for the case of $(n-1)$-dimensional sections of convex bodies.

Now, combining Theorems 3.1 and 3.4, we obtain:

Corollary 3.5 *Let μ be a finite, 0-isotropic, symmetric, κ-concave measure on $\mathbf{R}^n$, $n \geq 2$, with $\kappa \geq -1$. For all hyperplanes H_1, H_2, passing through the origin,*

$$\frac{\mu^+(H_1)}{\mu^+(H_2)} \leq C,$$

where C is a universal constant.

Indeed, without loss of generality let $\mu(\mathbf{R}^n) = 1$. If $H = \{x \in \mathbf{R}^n : \langle x, \theta\rangle = 0\}$, where θ is a unit vector, then $\mu^+(H) = g(0)$, where g is the density of the random variable $\xi(x) = \langle x, \theta\rangle$ under the probability measure μ. Since μ is symmetric and κ-concave, all linear functionals have symmetric κ-concave distributions on $\mathbf{R}$. Hence, by Lemma 3.4 with $\kappa = -1$ and $q = 0$,

$$\frac{c_1(0)}{\|\xi\|_0} \leq \mu^+(H) \leq \frac{c_{-1}(0)}{\|\xi\|_0}.$$

Applying this to H_1 and H_2, it remains to make use of Theorem 3.1 with $q = 0$.

4 The slicing problem

The well-known slicing problem, raised and deeply studied by Bourgain [15], is to determine whether or not, for any convex body K in $\mathbf{R}^n$ of unit volume, there is a hyperplane H such that $\mathrm{vol}_{n-1}(K \cap H) \geq \ell$, for some universal constant $\ell > 0$. Equivalently, restricting ourselves to symmetric isotropic convex bodies, one wonders if one can bound the isotropic constant L_K from above by a (dimension free) universal constant.

The problem has various equivalent formulations. What also seems rather interesting, it may be generalized and formulated as a problem about isotropic log-concave measures. This direction was examined by Ball in the same paper [7] with the help of the norms

$$\|x\| = \left(\int_0^{+\infty} f(rx)dr^{n+q} \right)^{-1/(n+q)} \tag{4.1}$$

in the particular case $q = 2$. In that case, if a symmetric, log-concave probability measure μ on $\mathbf{R}^n$ with density f is 2-isotropic, this norm generates an isotropic convex body K. More precisely, it was shown that the "isotropic constant" of the measure,

$$L_\mu^2 = f(0)^{\frac{2}{n}} \int \langle x, \theta \rangle^2 \, d\mu(x) \qquad (\theta \in S^{n-1}), \tag{4.2}$$

can be bounded from above by $C^2 L_K^2$, up to some universal factor C.

The purpose of the present section is to extend this type of relationship between L_K and L_μ to other convex measures. However, the definition (4.2) does not make sense in general, and it is more natural to speak about the size of the slices of the measure, $\mu^+(H)$, similarly to the original formulation of the slicing problem for convex bodies.

Thus, assume we have a finite, symmetric, full-dimensional convex measure μ on $\mathbf{R}^n$, $n \geq 2$, with density f, satisfying (for normalization reason) $f(0) = 1$. What can one say about possible values of $\mu^+(H)$? As we know from Theorem 1.1, all slices have approximately equal size, as long as the measure μ is (-1)-isotropic. So this may naturally be assumed, but we do not do this.

Introduce the best constant $\ell = \ell_\mu$ in the (isoperimetric-type) relation

$$\mu^+(H) \geq \ell \left(\int_{\mathbf{R}^n} f(x)dx \right)^{(n-1)/n}, \tag{4.3}$$

serving for all hyperplanes H, passing through the origin. In particular, when μ is the Lebesgue measure, restricted to a symmetric convex body K in $\mathbf{R}^n$, the above inequality defines the best constant $\ell = \ell_K$ in the inequality

$$\mathrm{vol}_{n-1}(K \cap H) \geq \ell \mathrm{vol}_n(K)^{(n-1)/n}. \tag{4.4}$$

Within universal factors, the quantity ℓ_K is known to be reciprocal to L_K, if K is isotropic. Indeed, by Hensley's inequality (3.9), applied to the densities of the linear functionals over the uniform distribution in K, we always have $\frac{1}{12} \le \ell_K^2 L_K^2 \le \frac{1}{2}$. With a similar argument, this relation extends to symmetric 2-isotropic log-concave measures as $\frac{1}{12} \le \ell_\mu^2 L_\mu^2 \le \frac{1}{2}$, but not to the larger class of convex measures.

Theorem 4.1 *Given a finite, symmetric, full-dimensional, κ-concave measure μ on $\mathbf{R}^n$, $-\infty < \kappa < 0$, with density f such that $f(0) = 1$, we have*

$$\ell_\mu \ge c\ell_K, \tag{4.5}$$

where K is the associated unit ball for the norm (4.1) with $q = -1$, and where

$$c = \frac{\alpha}{n}\left(\frac{\Gamma(\alpha)\, n!}{\Gamma(n+\alpha)}\right)^{1/n}, \qquad \alpha = -\frac{1}{\kappa}. \tag{4.6}$$

The log-concave case may also be included in this statement by letting $\kappa \to 0$, in which case $\lim_{\alpha\to+\infty} c = n!^{1/n}/n$. Then we get that

$$\ell_\mu \ge \frac{n!^{1/n}}{n}\ell_K \ge \frac{1}{e}\,\ell_K.$$

This is an equivalent formulation of K. Ball's theorem ($L_\mu \le CL_K$), up to a universal factor coming in the passage from slices sizes to L^2-norms of linear functionals.

When κ is fixed and n grows to infinity, the constant c is equivalent to $\alpha/(en)$, so the dimension may essentially influence on the size of $\mu^+(H)$. Note, however, the value of c, given in (4.6), is optimal for (4.5), as one can see on the example of the measure μ with density $f(x) = (1 + |x|)^{-(n+\alpha)}$, $x \in \mathbf{R}^n$.

Proof Let $K = \{x \in \mathbf{R}^n : \|x\| \le 1\}$ be the convex body, generated by the norm

$$\|x\| = \left(\int_0^{+\infty} f(rx)dr^{n-1}\right)^{-1/(n-1)}.$$

Given a linear subspace H in $\mathbf{R}^n$ of dimension $n-1$, we may use the polar coordinates $x = r\theta$, $r > 0$, $|\theta| = 1$, as in the proof of Theorem 3.2, and apply the definition (4.4) to write that

$$\begin{aligned}\mu^+(H) = \mathrm{vol}_{n-1}(K \cap H) &\ge \ell_K \mathrm{vol}_n(K)^{(n-1)/n} \\ &= \ell_K\left(\omega_n \int_{S^{n-1}} \|\theta\|^{-n} d\sigma_{n-1}(\theta)\right)^{(n-1)/n}.\end{aligned}$$

Since also

$$\int_{\mathbf{R}^n} f(x)\,dx = \omega_n \int_{S^{n-1}} \left(\int_0^{+\infty} f(r\theta)\,dr^n \right) d\sigma_{n-1}(\theta),$$

in order to get an inequality of the form (4.3) and then (4.5), we are reduced to the bound

$$\int_{S^{n-1}} \|\theta\|^{-n}\,d\sigma_{n-1}(\theta) \,\geq\, c^{n/(n-1)} \int_{S^{n-1}} \left(\int_0^{+\infty} f(r\theta)\,dr^n \right) d\sigma_{n-1}(\theta).$$

Moreover, it is enough to require a similar inequality for integrands at every $\theta \in S^{n-1}$, that is,

$$\int_0^{+\infty} f(r\theta)\,dr^{n-1} \,\geq\, c \left(\int_0^{+\infty} f(r\theta)\,dr^n \right)^{(n-1)/n}. \tag{4.7}$$

Our task is now completely in dimension one. Recall that $f(0) = 1$ is assumed, and that f is κ_n-concave, according to Borell's characterization (2.2) of κ-concave measures, where $\kappa_n = \kappa/(1 - n\kappa)$. In addition, by the symmetry hypothesis, $f(r\theta)$ is non-increasing in $r > 0$ along every direction. So,

$$f(r\theta) = (1 + \psi_\theta(r))^{1/\kappa_n}, \quad r > 0,$$

for some convex, non-decreasing function $\psi_\theta : [0, +\infty) \to [0, +\infty)$ with $\psi_\theta(0) = 0$. □

To derive (4.7) and thus complete the proof of the theorem, we appeal to:

Lemma 4.2 *Let $q > p > 0$ and let $Q : [0, +\infty) \to [0, +\infty)$ be non-increasing, not identically zero. In the class of all convex functions $\psi(r) \geq 0$ in $r \geq 0$, such that $\psi(0) = 0$, the best constant c in*

$$\int_0^{+\infty} Q(\psi(r))\,dr^p \,\geq\, c \left(\int_0^{+\infty} Q(\psi(r))\,dr^q \right)^{p/q} \tag{4.8}$$

is attained at the function $\psi(r) = r$.

In case of the exponential function $Q(r) = e^{-r}$, this statement is proved in [7], Lemma 4 (with an additional assumption $p \geq 1$, which can actually be relaxed to $p > 0$). But the argument may easily be extended to cover the case of the general Q. It is based on Lemma 3 in [7], which asserts the following (under the same assumptions as in Lemma 4.2): If

$$\int_0^{+\infty} Q(\psi(r))dr^p = \int_0^{+\infty} Q(r)dr^p, \tag{4.9}$$

and the integrals are finite, then for all $t \geq 0$,

$$\int_t^{+\infty} Q(\psi(r))dr^p \leq \int_t^{+\infty} Q(r)dr^p. \tag{4.10}$$

Assuming this assertion, take a convex, non-negative function ψ on $[0, +\infty)$ with $\psi(0) = 0$, and define $\lambda > 0$ by $\int_0^{+\infty} Q(\psi(r))\, dr^p = \int_0^{+\infty} Q(r/\lambda)dr^p$, so that

$$\lambda^{-p} = \frac{\int_0^{+\infty} Q(r)dr^p}{\int_0^{+\infty} Q(\psi(r))dr^p} \tag{4.11}$$

(without loss of generality, assume the integrals are finite). Then (4.9) is fulfilled for $Q_\lambda(r) = Q(r/\lambda)$ in place of Q and $\psi_\lambda = \lambda\psi$ in place of ψ. Using a general identity

$$\int_0^{+\infty} h(r)dr^q = \frac{q(q-p)}{p} \int_0^{+\infty} s^{q-p-1} \left(\int_s^{+\infty} h(r)dr^p \right) ds,$$

we can apply the conclusion (4.10) to $(Q_\lambda, \psi_\lambda)$ and obtain that

$$\begin{aligned} \int_0^{+\infty} Q(\psi(r))dr^q &= \int_0^{+\infty} Q_\lambda(\psi_\lambda(r))\, dr^q \\ &= \frac{q(q-p)}{p} \int_0^{+\infty} s^{q-p-1} \left(\int_s^{+\infty} Q_\lambda(\psi_\lambda(r))\, dr^p \right) ds \\ &\leq \frac{q(q-p)}{p} \int_0^{+\infty} s^{q-p-1} \left(\int_s^{+\infty} Q_\lambda(r)\, dr^p \right) ds \;=\; \lambda^q \int_0^{+\infty} Q(r)\, dr^q. \end{aligned}$$

In view of (4.11), we arrive at

$$\frac{\left(\int_0^{+\infty} Q(\psi(r))\, dr^q\right)^{1/q}}{\left(\int_0^{+\infty} Q(\psi(r))\, dr^p\right)^{1/p}} \leq \frac{\left(\int_0^{+\infty} Q(r)\, dr^q\right)^{1/q}}{\left(\int_0^{+\infty} Q(r)\, dr^p\right)^{1/p}},$$

which is exactly the statement of Lemma 4.2.

In particular, take in Lemma 4.2 the function $Q(r) = (1+r)^{-\beta}$ with a parameter $\beta > 0$. Then $\int_0^{+\infty} Q(r)\,dr^p = \Gamma(\beta - p)\,\Gamma(p+1)/\,\Gamma(\beta)$, whenever $0 < p < \beta$. If $p = n-1$, $q = n$, the best constant $c = c_n(\beta)$ in (4.8) is therefore given by

$$\frac{\int_0^{+\infty} Q(r)\,dr^{n-1}}{\left(\int_0^{+\infty} Q(r)\,dr^n\right)^{(n-1)/n}} = \frac{\beta - n}{n}\left(\frac{n!^{1/n}\,\Gamma(\beta - n)}{\Gamma(\beta)}\right)^{1/n}.$$

Finally, since in (4.7) our parameter is $\beta = -1/\kappa_n = -(1-n\kappa)/\kappa = n + \alpha$, where $\alpha = -1/\kappa$, $c_n(\beta)$ is exactly the constant (4.6).

This finishes the proof of Theorem 4.1.

Note, when c is optimal [and is defined by (4.6)], the equality in (4.7) and therefore in (4.5) is attained in case $f(r\theta) = (1+r)^{1/\kappa_n}$, i.e., when $f(x) = (1+|x|)^{-(n+\alpha)}$.

5 The floating body

Given a probability measure μ on $\mathbf{R}^n$ and a number $\delta \in (0, \frac{1}{2})$, the floating body F_δ of μ at level $1-\delta$ may be defined as

$$F_\delta = \bigcap_{\mu(H) \geq 1-\delta} H, \tag{5.1}$$

where the intersection is running over all closed half-spaces H in $\mathbf{R}^n$ with measure $\mu(H) \geq 1-\delta$. In connection with the general problem on the conditions, ensuring that F_δ is non-empty, the floating body was considered by many authors (cf. e.g. [26,35,38] and remarks below).

If μ is absolutely continuous, the restriction on half-spaces may be replaced with $\mu(H) = 1-\delta$. In general, F_δ represents a compact convex set. If μ is symmetric, it is symmetric, as well (and therefore non-empty).

In particular, when K is a symmetric convex body with its uniform distribution, we arrive at the notion of a floating body of K, cf. [34]. In this case a closely related is the notion of the flotation surface, when in each direction one cuts from K a segment of a fixed proportional "volume", cf. [30,31]. More precisely, to involve the general symmetric measure case, for every unit vector θ choose the minimal $r = r_\delta(\theta) \geq 0$, such that $\mu\{\langle x, \theta\rangle \leq r\} \geq 1-\delta$. The flotation surface of μ is then defined as the collection

$$S_\delta = \{r_\delta(\theta)\,\theta : \theta \in S^{n-1}\}.$$

The two concepts were given a considerable attention in Convex Geometry in the late 1980s, when C. Schütt raised the following natural question in the convex body case: Is it true that every half-space H in $\mathbf{R}^n$ with measure $\mu(H) = 1-\delta$ serves as a tangent plane at some boundary point of F_δ? Or, equivalently, one wonders if the flotation surface S_δ represents the boundary of the floating body F_δ.

An affirmative answer to this question was given by Meyer and Reisner in [32], and independently by K. Ball with a different proof. K. Ball's argument covers the case of an arbitrary log-concave probability measure and is described in [33]. In the next section we discuss further extensions of this result to the class of (some) convex measures.

However, first let us look at the meaning of the definition (5.1). As emphasized in [34], when a symmetric convex body K is isotropic, its floating body F_δ is almost a Euclidean ball, centered at the origin and with some radius r, depending only on K, mainly through its isotropic constant L_K. More precisely, if (for normalization reason) K has volume one, for some function $c = c(\delta)$, we have

$$\frac{L_K}{c} B \subset F_\delta \subset c L_K B, \tag{5.2}$$

where B is the unit Euclidean ball. Therefore, whether or not F_δ looks like a ball may serve as indication that the original body K is (almost) isotropic. Note that without the isotropy assumption (5.2) implies that F_δ is close to some ellipsoid.

This view may appropriately be extended to general convex measures in the following observation.

Theorem 5.1 *Let μ be a symmetric, full-dimensional, κ-concave probability measure on $\mathbf{R}^n$ with $\kappa > -\infty$. There exists an ellipsoid $\mathcal{E}$, such that for all $\delta \in (0, \frac{1}{2})$,*

$$\frac{1}{c}\mathcal{E} \subset F_\delta \subset c\mathcal{E}, \tag{5.3}$$

where c depends on δ and κ, only. Moreover, if μ is (-1)-isotropic, then for $\mathcal{E}$ one can take a Euclidean ball.

Proof Fix δ and assume μ is (-1)-isotropic. Introduce the half-spaces $H_\theta = \{x \in \mathbf{R}^n : \langle x, \theta\rangle \leq r\}$, where $r = r_\delta(\theta)$, as before. Thus, $F_\delta = \bigcap_{\theta \in S^{n-1}} H_\theta$. So, $x \in F_\delta$ if and only if $\langle x, \theta\rangle \leq r_\delta(\theta)$ for all $\theta \in S^{n-1}$, that is,

$$\|x\| \equiv \sup_{\theta \in S^{n-1}} \langle x, \frac{\theta}{r_\delta(\theta)}\rangle \leq 1.$$

Therefore, F_δ may be described as the dual to the unit ball

$$G_\delta = \{x \in \mathbf{R}^n : \|x\|_* \leq 1\} = \text{clos conv}\left\{\frac{\theta}{r_\delta(\theta)} : \theta \in S^{n-1}\right\}$$

for the norm $\|\cdot\|$, where clos conv denotes the closed convex hull of a set.

Now, consider the linear functionals $\varphi_\theta(x) = \langle x, \theta\rangle$ and their densities g_θ under the measure μ, together with the distribution functions $G_\theta(\lambda) = \mu\{\varphi_\theta \leq \lambda\}$. By the very definition, $r_\delta(\theta)$ represents the quantile of φ_θ under μ of order $1 - \delta$, that is, $r_\delta(\theta) = G_\theta^{-1}(1 - \delta)$. As we have seen in the proof of Lemma 3.4, for any $t \in (\frac{1}{2}, 1)$,

$$\int_{1/2}^{t} ds \le \frac{G_\theta^{-1}(t)}{g_\theta(0)} \le \int_{1/2}^{t} \frac{ds}{(2\min\{s, 1-s\})^{1-\kappa}}.$$

Applying this with $t = 1 - \delta$, we get

$$\frac{1-2\delta}{2}\, g_\theta(0) \le r_\delta(\theta) \le \frac{(2\delta)^\kappa - 1}{-2\kappa}\, g_\theta(0). \tag{5.4}$$

Note the both sides of (5.4) coincide in the critical case $\kappa = 1$ (which is only possible for the uniform distribution on interval of the real line). Thus, for all $\theta, \theta' \in S^{n-1}$,

$$\frac{r_\delta(\theta)}{r_\delta(\theta')} \le \frac{(2\delta)^\kappa - 1}{-\kappa(1-2\delta)}\, \frac{g_\theta(0)}{g_{\theta'}(0)}.$$

But, by Corollary 3.3 with $d = 1$, we have $\frac{g_\theta(0)}{g_{\theta'}(0)} \le \sqrt{6}$. So, defining $c = c(\delta, \kappa) > 0$ by

$$c^2 = \frac{(2\delta)^\kappa - 1}{-\kappa(1-2\delta)}\, \sqrt{6}, \tag{5.5}$$

one can choose $r > 0$ such that $\frac{1}{c} \le \frac{r}{r_\delta(\theta)} \le c$, for all $\theta \in S^{n-1}$. As a consequence, $B(0, 1/(rc)) \subset G_\delta \subset B(0, c/r)$, where $B(0, \rho)$ stands for the Euclidean ball with center at the origin and radius ρ. Equivalently, for the dual sets we obtain that

$$\frac{1}{c} B(0, r) \subset F_\delta \subset c B(0, r).$$

The statement of Theorem 5.1 follows with $\mathcal{E} = B(0, r)$. □

Remark As follows from (5.5) with $\kappa = 0$, in the log-concave case one may take in Theorem 5.1 $c^2 = (\sqrt{6}\, \log \frac{1}{2\delta})/(1-2\delta)$, which is of order $\log \frac{1}{\delta}$, as $\delta \to 0$. If $\kappa < 0$, c is of order $\delta^{\kappa/2}$. Also, if $\kappa = 0$, (5.4) takes the form

$$\frac{1-2\delta}{2}\, g_\theta(0) \le r_\delta(\theta) \le \frac{1}{2} \log \frac{1}{2\delta}\, g_\theta(0).$$

If μ is isotropic in the sense that $\int \langle x, \theta\rangle^2 d\mu(x) = \sigma^2$ for all unit vectors θ, then $g_\theta(0)$ is of order $1/\sigma$ within universal factors. Hence, for $0 < \delta \le \delta_0 < 1/2$, there exist some constants $C_0 = C_0(\delta_0)$, $C_1 = C_1(\delta_0)$, such that $C_0\sigma \le r_\delta(\theta) \le C_1\sigma \log \frac{1}{\delta}$. Therefore,

$$C_0 \sigma B \subset F_\delta \subset C_1 \sigma \log \frac{1}{\delta} B,$$

where B is the unit Euclidean ball, and we recover (5.2).

Now, let us comment on the conditions, ensuring that F_δ is non-empty, when a probability measure μ on $\mathbf{R}^n$ is not necessarily symmetric. In general, $F_\delta \neq \emptyset$, as long as $\delta \leq 1/(n+1)$. In case of the plane, this result was obtained by Neumann [35] in 1945. Higher dimensions were treated by Rado [38], who involved a larger class of subadditive measures; see also [8] for related results.

When μ is a uniform distribution in a convex body K, this statement may considerably be sharpened: F_δ is non-empty for $\delta = (n/(n+1))^n$. Again, for dimension $n = 2$ this was proved in [35], while for higher dimensions—by Grünbaum [26] (and independently by Hammer [27]), with similar proofs, based on the Schwarz symmetrization. More precisely, it was shown that F_δ contains the baricenter (or centroid) of K.

Note, regardless of the dimension, we have $F_\delta \neq \emptyset$ for $\delta = 1/e$ in the convex body case. This observation may properly be generalized to involve arbitrary κ-concave probability measures. The following nice theorem is due to Caplin and Nalebuff, cf. [17], Proposition 3 (where the condition $\kappa > -1$ was somehow hidden).

Theorem 5.2 *For any κ-concave probability measure μ on $\mathbf{R}^n$ with $-1 < \kappa \leq 1$, the floating body F_δ contains the baricenter of the measure, as long as*

$$\delta \leq (1+\kappa)^{-1/\kappa}. \tag{5.6}$$

The condition $\kappa > -1$ ensures, in particular, that $\int |x| \, d\mu(x) < +\infty$ [cf. (2.4)], so the baricenter $\int x \, d\mu(x)$ is well-defined. In the case of the uniform distribution in a convex body, one has $\kappa = 1/n$ and $(1+\kappa)^{-1/\kappa} = (n/(n+1))^n$, and we obtain Grünbaum's theorem. In the log-concave case, Theorem 5.2 also appeared in [10], Lemma 3.3. Let us explain how to extend its argument to involve general κ-concave measures.

Since the κ-concavity property is invariant under projections, the distributions of the linear functionals under μ represent κ-concave probability measures on the real line. Therefore, (5.6) is reduced to the following one-dimensional statement: For any random variable ξ with a non-degenerate κ-concave distribution,

$$(1+\kappa)^{-1/\kappa} \leq \Pr\{\xi \leq \mathbf{E}\xi\} \leq 1 - (1+\kappa)^{-1/\kappa}. \tag{5.7}$$

If $\kappa = 1$, ξ has to be uniformly distributed on a finite interval (a, b), and then both sides of (5.7) coincide. So, assume $-1 < \kappa < 1$. Introduce the distribution function $G(x) = \Pr\{\xi \leq x\} = \int_a^x g(z)\, dz$, $a < x < b$, with its inverse $G^{-1} : (0, 1) \to (a, b)$, and define the associated function $I = g(G^{-1})$. As we discussed in Remark 2.2, the function $I^{1/(1-\kappa)}$ is concave on (0,1), so for all $\alpha \in (0, 1)$,

$$\frac{I(u)^{1/(1-\kappa)}}{1-u} \leq \frac{I(\alpha)^{1/(1-\kappa)}}{1-\alpha}, \quad \text{for} \quad u \in (0, \alpha],$$
$$\frac{I(u)^{1/(1-\kappa)}}{1-u} \geq \frac{I(\alpha)^{1/(1-\kappa)}}{1-\alpha}, \quad \text{for} \quad u \in [\alpha, 1).$$

Since $G^{-1}(s) - G^{-1}(\alpha) = \int_\alpha^s \frac{du}{I(u)}$, $0 < s < 1$, we get that

$$\begin{aligned}
\mathbf{E}\xi &= \int_0^1 G^{-1}(s)ds = \int_0^1 \left[G^{-1}(\alpha) + \int_\alpha^s \frac{du}{I(u)} \right] ds \\
&= G^{-1}(\alpha) + \int_\alpha^1 \frac{1-u}{I(u)}\, du - \int_0^\alpha \frac{u}{I(u)}\, du \\
&\le G^{-1}(\alpha) + \frac{(1-\alpha)^{1-\kappa}}{I(\alpha)} \int_\alpha^1 (1-u)^\kappa\, du - \frac{(1-\alpha)^{1-\kappa}}{I(\alpha)} \int_0^\alpha \frac{u}{(1-u)^{1-\kappa}}\, du \\
&= G^{-1}(\alpha) + \frac{(1-\alpha)^{1-\kappa}}{I(\alpha)} \left[\frac{1}{1+\kappa} - \frac{1-(1-\alpha)^\kappa}{\kappa} \right].
\end{aligned}$$

The expression in the square brackets is vanishing for $\alpha = 1 - (1+\kappa)^{-1/\kappa}$, and then $\mathbf{E}\xi \le G^{-1}(\alpha)$ or $G(\mathbf{E}\xi) \le \alpha$. This is exactly the right inequality in (5.7). The left inequality in (5.7) is obtained from the right by applying it to $-\xi$.

Remark Proposition 3 of [17] asserts that $\Pr\{\xi \le \mathbf{E}\xi\} \ge (1+\kappa)^{-1/\kappa}$ for any κ-concave distribution function $G(x) = \Pr\{\xi \le x\}$. This condition is somewhat weaker than the κ-concavity of the distribution of ξ.

Both inequalities in (5.7) are sharp, since on the right-hand side there is equality for ξ having a special distribution. Namely, introduce a κ-concave probability measure μ_κ on $(0, +\infty)$ by requiring that its associated function is $I_\kappa(t) = (1-t)^{1-\kappa}$. Its distribution function is given by

$$G(x) = \mu_\kappa(0, x) = 1 - (1 - \kappa x)^{1/\kappa}, \quad 0 < x < c_\kappa.$$

More precisely, when $0 < \kappa \le 1$, μ_κ is supported on the finite interval $(0, \frac{1}{\kappa})$. If $\kappa = 1$, we obtain a uniform distribution on the unit interval $(0, 1)$. When $-\infty < \kappa \le 0$, μ_κ is not supported on a finite interval, so that $c_\kappa = +\infty$. If $\kappa = 0$, we obtain the one-sided exponential distribution with density e^{-x}. Now, if $\kappa > -1$ and ξ is distributed according to μ_κ, then $\mathbf{E}\xi = 1/(1+\kappa)$ and $G(\mathbf{E}\xi) = 1 - (1+\kappa)^{-1/\kappa}$.

6 The flotation surface

Let μ be a symmetric, full-dimensional κ-concave probability measure on $\mathbf{R}^n$. As we have mentioned, by a theorem of Meyer-Reisner and Ball [32,33], in the log-concave case, for any $\delta \in (0, \frac{1}{2})$,

$$S_\delta = \partial F_\delta. \tag{6.1}$$

That is, the boundary of the floating body of μ at a given level is exactly the corresponding flotation surface of the measure. Here we follow K. Ball's argument to involve in this statement more convex measures. Our basic tool will be Lemma 2.4 (needed with $p = 1$).

Theorem 6.1 *For any symmetric, full-dimensional, κ-concave probability measure μ on $\mathbf{R}^n$, the identity (6.1) is valid whenever $\kappa \geq -1$.*

We do not know whether any restriction on κ may be removed. For readers convenience, first let us remind basic steps as described in [33]. Introduce the symmetric strips

$$B(a) = \{x \in \mathbf{R}^n : |\langle x, a\rangle| \leq 1\}, \quad a \in \mathbf{R}^n,$$

and their measures $W(a) = \mu(B(a))$. Note $W(0) = 1$ and $W(a) \to 0$, as $|a| \to +\infty$ (which is true for any absolutely continuous μ).

Theorem 6.1 will follow from:

Lemma 6.2 *If $\kappa \geq -1$, the function $1/W(a)$ is convex on $\mathbf{R}^n$.*

As soon as we accept this property, one may define a norm

$$\|a\| = \min\{\rho \geq 0 : W(a/\rho) \geq 1 - 2\delta\}$$

with the unit ball $G = \{a \in \mathbf{R}^n : W(a) \geq 1 - 2\delta\}$. Writing $a = \frac{1}{r}\theta$ with $r > 0$, $\theta \in S^{n-1}$, we have, by symmetry of the measure and the very definition of r_δ,

$$\begin{aligned} a \in G &\iff W(a) \geq 1 - 2\delta \iff \mu\{|\langle x, \theta\rangle| \leq r\} \geq 1 - 2\delta \\ &\iff \mu\{\langle x, \theta\rangle \leq r\} \geq 1 - \delta \iff r \geq r_\delta(\theta). \end{aligned}$$

Therefore, $\partial G = \{\frac{\theta}{r_\delta(\theta)} : \theta \in S^{n-1}\}$ and $G = G_\delta$. From this, the dual norm is

$$\|x\|_* = \sup_{\|a\|=1} |\langle x, a\rangle| = \sup\left\{\frac{1}{r_\delta(\theta)}\,|\langle x, \theta\rangle| : \theta \in S^{n-1}\right\},$$

which is exactly the norm with unit ball F_δ (We considered it in the beginning of the proof of Theorem 5.1). Therefore, at every point $x \in \partial F_\delta$ there is a supporting hyperplane $\langle x, a\rangle = 1$ with some $a \in \partial G$. Since $a = \frac{\theta}{r_\delta(\theta)}$, the equation of the plane is $\langle x, \theta\rangle = r_\delta(\theta)$, for some $\theta \in S^{n-1}$, which is the equation for the boundary of one of the half-spaces H, participating in the intersection $F_\delta = \cap_{\mu(H)=1-\delta} H = \cap_\theta H_\theta$. Moreover, by the convexity of G, any ∂H_θ is a tangent plane at some boundary point of F_δ.

This would complete the proof of the theorem.

Remark As have seen, what is needed for the proof of Theorem 6.1 is only the convexity of the set G. This means that the function W should be quasi-concave.

Proof of Lemma 6.2. Since projections of κ-concave measures are κ-concave, the statement of the lemma is entirely two-dimensional. So one may assume for simplicity that $n = 2$ (of course, the symmetry of the measure plays a crucial role).

We follow the construction, described in [33]. Thus, fix non-collinear non-zero vectors $a, b \in \mathbf{R}^2$ and let $c = \frac{a+b}{2}$. In particular, $\rho = (|a|^2|b|^2 - \langle a, b\rangle^2)^{1/2} > 0$. Put

$$a' = \frac{\langle a, b\rangle a - |a|^2 b}{\rho}, \quad b' = \frac{|b|^2 a - \langle a, b\rangle b}{\rho}, \quad c' = \frac{a' + b'}{2}.$$

Then $|a'| = |a|$, $|b'| = |b|$, $|c'| = |c|$, and $\langle a, a'\rangle = \langle b, b'\rangle = \langle c, c'\rangle = 0$. In addition,

$$\langle a', a - b\rangle = \langle b', a - b\rangle = \langle c', a - b\rangle = \rho.$$

Introduce the line $H = \{x \in \mathbf{R}^2 : \langle x, a - b\rangle = 0$ and the segment on it

$$D = \{x \in H : |\langle x, a\rangle| \le 1\} = \{x \in H : |\langle x, b\rangle| \le 1\} = \{x \in H : |\langle x, c\rangle| \le 1\}.$$

Then we have the representation of the strips on the plane as the union of the disjoint segments $B(a) = \cup_{t \in \mathbf{R}}(ta' + D)$, and similarly for the vectors b and c.

Now, let f be the density of μ on the plane, chosen to be κ_2-concave [recall that $\kappa_2 = \kappa/(1 - 2\kappa)$ as in the Borell description (2.2)]. Given a line J, parallel to H, denote by μ_J the measure on it with density f with respect to Lebesgue measure on J. Note, by the same characterization (2.2) in dimension one, μ_J is κ'-concave, if and only if f is κ_1'-concave with $\kappa_1' = \kappa'/(1 - \kappa')$. Equalizing $\kappa_2 = \kappa_1'$, we conclude that every μ_J is κ'-concave with $\kappa' = \kappa/(1 - \kappa)$. □

By Fubini's theorem and the symmetry of μ,

$$W(a) = \mu(B(a)) = \frac{2\rho}{|a - b|} \int_0^{+\infty} \mu_{ta'+H}(ta' + D)dt, \tag{6.2}$$

and similarly for b and c with the same coefficient in front of the integral. Define the functions on $(0, +\infty)$

$$u(t) = \mu_{ta'+H}(ta' + D), \quad v(t) = \mu_{tb'+H}(tb' + D), \quad w(t) = \mu_{tc'+H}(tc' + D).$$

For all $t, s > 0$, we have $\frac{t}{t+s}(sa' + D) + \frac{s}{t+s}(tb' + D) = \frac{2ts}{t+s} c' + D$, so, by the κ'-concavity of μ_J's,

$$w\left(\frac{2ts}{t+s}\right) \ge M_{\kappa'}^{(\frac{t}{t+s})}(u(s), v(t)).$$

Hence, the hypothesis (2.8) of Lemma 2.4 is fulfilled with its notations $t = \frac{1}{2}$ and $q = \kappa'$. So, if $\kappa \le 0$, the concluding inequality (2.9) is valid for all p, whenever

$0 \le p \le -\frac{1}{q} - 1 = -\frac{1}{\kappa}$. In particular, when $\kappa \ge -1$, one can take $p = 1$, which gives

$$\left[\int_0^{+\infty} w(t)dt\right]^{-1} \le \frac{1}{2}\left[\int_0^{+\infty} u(t)dt\right]^{-1} + \frac{1}{2}\left[\int_0^{+\infty} v(t)dt\right]^{-1}.$$

In view of (6.2), this is exactly the desired inequality $W(c)^{-1} \le \frac{1}{2}\, W(a)^{-1} + \frac{1}{2}\, W(b)^{-1}$.
Lemma 6.2 follows.

7 Santaló-type inequalities

In this section we conclude with remarks on the Santaló-type inequalities for the family of convex measures. Given a symmetric convex body K in $\mathbf{R}^n$, the Santaló or Blashke-Santaló inequality asserts (cf. [36,37]) that

$$\mathrm{vol}_n(K)\mathrm{vol}_n(K^o) \le \mathrm{vol}_n(B)^2, \tag{7.1}$$

where

$$K^o = \{x \in \mathbf{R}^n : \langle x, y\rangle \le 1,\ \text{for all } y \in K\}$$

is the polar body, and where B stands for the unit Euclidean ball in $\mathbf{R}^n$ with center at the origin. Thus, the left-hand side of (7.1) is maximized for Euclidean balls.

In [6], K. Ball found an interesting application of (7.1) to the class of symmetric log-concave functions. Namely, introduce the Legendre transform,

$$\mathcal{L}v(y) = \sup_{x \in \mathbf{R}^n} [\langle x, y\rangle - v(x)], \quad y \in \mathbf{R}^n,$$

which may be defined for all functions v on $\mathbf{R}^n$ with values in $[-\infty, +\infty]$. The correspondence $v \to \mathcal{L}v$ is known to have a number of remarkable properties and, from the other side, it appears naturally in various characterization problems, related to abstract duality transforms on different classes of functions. For recent developments in this direction we refer the interested reader to the works of Artstein and Milman [2–4].

In particular, consider the class $\mathcal{F}_n$ of all symmetric lower-semicontinuos convex functions $v : \mathbf{R}^n \to [0, +\infty]$ with $v(0) = 0$. In this class the Legendre transform acts as bijection, and $\mathcal{L}(\mathcal{L}v) = v$, for all $v \in \mathcal{F}_n$. The result of K. Ball, obtained on the basis of (7.1), is that

$$\int e^{-v(x)}dx \int e^{-w(y)}dy \le (2\pi)^n,$$

for any v in $\mathcal{F}_n$ with $w = \mathcal{L}v$. Only recently, in [1], this inequality was understood as the Santaló-type inequality for the canonical polarity in the class $\mathcal{F}_n$. In modern language and notations, it is now stated as

$$\int f(x)dx \int f^o(y)dy \leq (2\pi)^n, \tag{7.2}$$

where $f = e^{-v}$ is an arbitrary integrable symmetric log-concave function with $f(0) = 1$ (that is, with $v \in \mathcal{F}_n$), and $f^o = e^{-w}$ is its dual. Note the Blashke-Santaló inequality (7.1) is included in the functional form (7.2), when the latter is restricted to the special functions $f(x) = e^{-\|x\|^2/2}$, $f^o(y) = e^{-\|x\|_*^2/2}$, where $\|\cdot\|$ and $\|\cdot\|_*$ are the norms, generated by the convex bodies K and K^o, respectively.

The inequality (7.2) may be properly modified to cover non-symmetric log-concave functions (cf. e.g. [1]). On the other hand, it may further be extended to a larger class of functions, serving as densities of finite convex measures on $\mathbf{R}^n$. More precisely, we have:

Theorem 7.1 *Given $v \in \mathcal{F}_n$ with the Legendre transform $w = \mathcal{L}v$, for any $\beta > n$,*

$$\int \frac{dx}{(1+v(x))^\beta} \int \frac{dy}{(1+w(y))^\beta} \leq \left(\int \frac{dz}{(1+|z|^2)^{\beta/2}}\right)^2. \tag{7.3}$$

If $\beta > n$, the functions of the form $f = (1+v)^{-\beta}$ represent densities of finite κ-concave measures on $\mathbf{R}^n$, satisfying $f(0) = 1$, with $\kappa = -1/(\beta - n)$. In analogy with the log-concave case, $f^o = (1+w)^{-\beta}$ may be viewed as the density, dual to f within the same class of κ-concave measures (with an additional property that $f(0) = 1$).

As an equivalent variant of (7.3), we also have

$$\int \frac{dx}{(1+v(x)/\beta)^\beta} \int \frac{dy}{(1+w(y)/\beta)^\beta} \leq \left(\int \frac{dz}{(1+|z|^2/\beta)^{\beta/2}}\right)^2,$$

which in the limit, as $\beta \to +\infty$, turns into (7.2). Therefore, Theorem 7.1 implies the Blashke–Santaĺo inequality. Note, however, in contrast with (7.2) the inequality (7.3) does not say anything about the extremal situation (the extremal role of Cauchy measures is only asymptotical).

As for the proof (7.3) may easily be derived from a recent result of Fradelizi and Meyer, which says the following (cf. [21, Proposition 3]). Let $f_1, f_2 : \mathbf{R}^n \to [0, +\infty)$ and $\rho : [0, +\infty) \to [0, +\infty)$ be measurable functions, satisfying

$$f_1(x) f_2(y) \leq \rho(\langle x, y\rangle)^2 \tag{7.4}$$

for all $x, y \in \mathbf{R}^n$, such that $\langle x, y\rangle > 0$. Then, if f_1 is even,

$$\int f_1(x)\,dx \int f_2(y)\,dy \leq \left(\int \rho(|z|^2)\,dz\right)^2. \tag{7.5}$$

Take $f_1(x) = (1 + v(x))^{-\beta}$, $f_2(y) = (1 + w(y))^{-\beta}$, $\rho(t) = (1 + t)^{-\beta/2}$. Then the hypothesis (7.4) reads as

$$(1 + v(x))(1 + w(y)) \geq 1 + \langle x, y \rangle, \quad \text{given that } \langle x, y \rangle > 0.$$

It is obviously fulfilled, since, by the definition of the Legendre transform, we have $v(x) + w(y) \geq \langle x, y \rangle$. The resulting inequality (7.5) is exactly (7.3), where the right-hand side is finite for $\beta > n$.

Acknowledgments I would like to express my gratitude to Vitali Milman for numerous discussions and remarks along many places of the paper. Thanks to Alexander Koldobsky for the references on the Funk theorem and to the referee for valuable remarks and references.

References

1. Artstein, S., Klartag, B., Milman, V.: The Santaló point of a function, and a functional form of the Santaló inequality. Mathematika, **51**(1–2) (2004), 33–48 (2005)
2. Artstein, S., Milman, V.: The concept of duality in asymptotic geometric analysis, and the characterization of the Legendre transform. Preprint (2007)
3. Artstein, S., Milman, V.: A characterization of the concept of duality. Electron. Res. Announc. Math. Sci. **14**, 42–59 (2007) (electronic)
4. Artstein, S., Milman, V.: The concept of duality for measure projections of convex bodies. J. Funct. Anal. **254**(10), 2648–2666 (2008)
5. Avriel, M.: r-convex functions. Math. Programming **2**, 309–323 (1972)
6. Ball, K.M.: Isometric Problems in ℓ_p and Sections of Convex Sets. PhD Dissertation, Cambridge (1986)
7. Ball, K.M.: Logarithmically concave functions and sections of convex bodies. Studia Math. **88**, 69–84 (1988)
8. Birch, B.J.: On $3N$ points in a plane. Proc. Cambridge Philos. Soc. **55**, 289–293 (1959)
9. Bobkov, S.G.: Isoperimetric and analytic inequalities for log-concave probability measures. Ann. Probab. **27**(4), 1903–1921 (1999)
10. Bobkov, S.G.: On concentration of distributions of random weighted sums. Ann. Probab. **31**(1), 195–215 (2003)
11. Bobkov, S.G.: Large deviations and isoperimetry over convex probability measures with heavy tails. Electron. J. Probab. **12**, 1072–1100 (2007)
12. Borell, C.: Convex measures on locally convex spaces. Ark. Math. **12**, 239–252 (1974)
13. Borell, C.: Convex set functions in d-space. Period. Math. Hungar. **6**(2), 111–136 (1975)
14. Bourgain, J.: On high-dimensional maximal functions associated to convex bodies. Am. J. Math. **108**(6), 1467–1476 (1986)
15. Bourgain, J.: On the distribution of polynomials on high-dimensional convex sets. Geometric Aspects of Functional Analysis (1989–1990), Lecture Notes in Mathematics, vol. 1469, pp. 127–137. Springer, Berlin (1991)
16. Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn–Minkowski and Pre'kopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Funct. Anal. **22**(4), 366–389 (1976)
17. Caplin, A., Nalebuff, B.: Aggregation and social choice: a mean voter theorem. Econometrica **59** (1), 1–23 (1991)
18. Das Gupta, S.: Brunn–Minkowski inequality and its aftermath. J. Multivar. Anal. **10**(3), 296–318 (1980)
19. Dancs, S., Uhrin, B.: On a class of integral inequalities and their measure-theoretic consequences. J. Math. Anal. Appl. **74**(2), 388–400 (1980)
20. Fradelizi, M.: Hyperplane sections of convex bodies in isotropic position. Beiträge Algebra Geom. **40**(1), 163–183 (1999)
21. Fradelizi, M., Meyer, M.: Some functional forms of Blaschke-Santaló inequality. Math. Z. **256**(2), 379–395 (2007)

22. Gardner, R.J.: Geometric Tomography. Encyclopedia of Mathematics and its Applications, vol. 58, xvi+424 pp. Cambridge University Press, Cambridge (1995)
23. Giannopoulos, A.A., Milman, V.D.: Extremal problems and isotropic positions of convex bodies. Israel J. Math. **117**, 29–60 (2000)
24. Gorin, E.A., Favorov, S. Yu.: Generalizations of the Khinchin inequality. (Russian) Teor. Veroyatnost. i Primenen. **35**(4), 762–767 (1990); translation in Theory Probab. Appl., **35**(4), 766–771 (1990)
25. Guédon, O.: Kahane–Khinchine type inequalities for negative exponent. Mathematika **46**(1), 165–173 (1999)
26. Grünbaum, B.: Partitions of mass-distributions and of convex bodies by hyperplanes. Pac. J. Math. **10**, 1257–1261 (1960)
27. Hammer, P.C.: Volumes Cut from Convex Bodies by Planes. Preprint (this reference is given in [26])
28. Hensley, D.: Slicing convex bodies—bounds for slice area in terms of the body's covariance. Proc. Am. Math. Soc. **79**(4), 619–625 (1980)
29. Koldobsky, A.L.: Fourier analysis in convex geometry. Mathematical Surveys and Monographs, vol. 116, vi+170 pp. American Mathematical Society, Providence (2005)
30. Leichtweiss, K.: On the affine surface of convex bodies (in German). Manuscripta Math. **56**(4), 429–464 (1986)
31. Leichtweiss, K.: On a formula of Blaschke on the affine surface area (in German). Studia Sci. Math. Hungar. **21**(3–4), 453–474 (1986)
32. Meyer, M., Reisner, S.: A geometric property of the boundary of symmetric convex bodies and convexity of flotation surfaces. Geom. Dedicata **37**(3), 327–337 (1991)
33. Meyer, M., Reisner, S.: Characterizations of affinely-rotation-invariant log-concave measures by section-centroid location. Geom. Aspects of Funct. Anal. (1989–1990), Lecture Notes in Mathematics, vol. 1469, pp. 145–152. Springer, Berlin (1991)
34. Milman, V.D., Pajor, A.: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geom. Aspects of Funct. Anal. (1987–1988), Lecture Notes in Mathematics, vol. 1376, pp. 64–104. Springer, Berlin (1989)
35. Neumann, B.H.: On an invariant of plane regions and mass distributions. J. Lond. Math. Soc. **20**, 226–237 (1945)
36. Pisier, G.: The Volume of Convex Bodies and Banach Space Geometry. Cambridge University Press, Cambridge (1989)
37. Santaló, L.A.: An affine invariant for convex bodies of n-dimensional space (Spanish). Port. Math. **8**, 155–161 (1949)
38. Rado, R.: A theorem on general measure. J. Lond. Math. Soc. **21**, 291–300 (1946)