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Aspects of Analysis

Functional Inequalities

Lévy Processes

Spectral Analysis

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Isoperimetry and Isoperimetric Functional Inequalities

Lectures by
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Vasilis Kontis

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1. Introduction

These notes were taken from lectures given Prof. S. Bobkov during the *Spring Meeting Seminar* at Imperial College, 17-20 March 2008. Their purpose is to study isoperimetric and functional inequalities and the way in which they are related as well as some of their applications to high-dimensional results, such as the concentration of measure phenomenon. What is more, some recent developments in this area are presented.

The starting point of these notes is the Brunn-Minkowski inequality, which is strongly related to the isoperimetric problem in $\mathbb{R}^n$. This is studied in Section 2. A functional form of the Brunn-Minkowski inequality, known as the Prékopa-Leindler inequality, is then proved in Section 3. In Section 4 an inequality which is very close to the Brunn-Minkowski inequality but involves surface measure rather than volume measure is studied. This has an interesting extension, as its functional form is similar to the Prékopa-Leindler inequality but involving gradient of the functions. An application of such inequalities to the concentration of measure phenomenon is presented in Section 5. We then go back to the Prékopa- Leindler Theorem, which is extended in Section 6 for measures satisfying a certain concavity property. Such measures are studied in more detail in Section 7. An inequality of Brascamp and Lieb is studied in Section 8 and is then extended to a more general class of measures. A refinement of the latter in the class of log-concave measures is presented in Section 9. Section 10 deals with the more specific case of Cauchy measures which are shown to satisfy Poincaré and Log-Sobolev inequalities with a weight. Finally, in Section 11 we study similar inequalities with weights for general measures while in Section 12 we examine what information they carry about functions that satisfy them.

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2. Brunn-Minkowski inequality

The Brunn-Minkowski inequality relates the volume measures of two convex sets. It was first proved by Brunn for convex sets in $\mathbb{R}^3$ (1887) and was extended later by Minkowski to higher dimensions and by Lusternik (1935) who proved it for all measurable sets. It has very strong consequences, both geometric and analytic such as in Isoperimetry or Sobolev Inequalities. Extensive expositions of the Brunn-Minkowski inequality can be found in [DG80], [Gar02].

In the following, we denote the (n-dimensional) Lebesgue measure of a set A by $|A| = vol_n(A) = \lambda(A)$. The Minkowski sum of two convex sets $A, B \in \mathbb{R}^n$ is the set $A + B = \{a + b \,|\, a \in A, b \in B\}$.

Theorem 2.1. (Brunn-Minkowski inequality)
For any convex $A, B \in \mathbb{R}^n$

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n} \tag{2.1}$$

Proof. Let us first look at the case when n=1. We first prove the result for compact sets A and B. Translating if necessary, by invariance of Lebesgue measure we may assume that $\min B = \max A = 0$. We then have that $A + B \supset A \cup B$, which in turn implies

$$|A + B| \geq |A \cup B| = |A| + |B|$$

and the inequality is proved in one dimension. For non-compact A and B, the result follows by an approximation argument, since we have $|A| = \sup\{|K| : K \text{ compact }, K \subset A\}$. The general case in dimension n is a consequence of the homogeneity of Lebesgue measure. It is also a direct consequence of the Prékopa-Leindler inequality: see Corollary 3.2. □

Remark 2.2. *For other direct proofs of the Brunn-Minkowski inequality, the reader is referred to [Gar02] .*

Definition 2.3. *A measure μ is said to be* log-concave *if it satisfies the following property for all measurable sets A and B and $t \in [0, 1]$*

$$\mu(tA + (1 - t)B) \geq \mu(A)^t \mu(B)^{1-t}$$

In other words, taking logarithm we see that $\log \mu$ has to satisfy the usual definition of concavity.

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Let us now see how we can recover the Brunn-Minkowski inequality by the following log-concavity property of Lebesgue measure

$$|tA + sB| \geq |A|^t |B|^s \tag{2.2}$$

where we have written $s = 1 - t$. This inequality is clearly implied by Brunn-Minkowski, since $\left(t|A|^{1/n} + s|B|^{1/n}\right)^n \geq |A|^t |B|^s$ where n is the dimension of the space and $t \in [0, 1], s = 1 - t$ (a more elaborate explanation of this is given later, see Definition (6.1)). Conversely, starting from inequality (2.2) and applying it to the sets

$$\tilde{A} = \frac{1}{|A|^{1/n}} A \text{ and } \tilde{B} = \frac{1}{|B|^{1/n}} B$$

we have that $|\tilde{A}| = |\tilde{B}| = 1$ and choosing $t = \frac{|A|^{1/n}}{|A|^{1/n}+|B|^{1/n}}$ and $s = \frac{|B|^{1/n}}{|A|^{1/n}+|B|^{1/n}}$ (so that t+s=1) we recover the Brunn-Minkowski inequality:

$$\frac{|A + B|}{(|A|^{1/n} + |B|^{1/n})^n} = \left|\frac{A + B}{|A|^{1/n} + |B|^{1/n}}\right| = |t\tilde{A} + s\tilde{B}| \geq |\tilde{A}|^t |\tilde{B}|^s = 1$$
$$\Rightarrow |A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

where in the first equality we used the homogeneity of Lebesgue measure $|sA| = s^n|A|$ in $\mathbb{R}^n$, for $s \in \mathbb{R}$.

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3. Prékopa-Leindler Theorem

In this section, we study a functional form of the Brunn-Minkowski inequality known as the Prékopa-Leindler inequality. Inequalities of this type were extensively studied in the literature, see e.g. S.DasGupta [DG80], S.Dancs and B.Uhrin [DU80], R.Henstock and A.M.Macbeath [HM53], C.Borell [Bor74] and H.J.Brascamp and E.H.Lieb [BL76].

Theorem 3.1. (Prékopa-Leindler, 1971/73)
Let u,v,w: $\mathbb{R}^n \to [0,\infty)$ be non-negative integrable functions such that

$$w(tx+sy) \geq u(x)^t v(y)^s$$

for all $x, y \in \mathbb{R}^n$, where $t \in (0,1)$ is fixed and $s = 1-t$. Then

$$\int w \geq \left(\int u\right)^t \left(\int v\right)^s$$

Remark 3.2. *In particular, for $u = \chi_A$, $v = \chi_B$ and $w = \chi_{tA+sB}$ we obtain the Brunn-Minkowski inequality.*

For the proof of Theorem 3.1 we will need the following lemma which proves the inequality when n=1. The inequality then is proved in any dimension by induction at the end of the section.

Lemma 3.3. *Let u,v,w: $\mathbb{R} \to [0,\infty)$ satisfy*

$$w(tx+sy) \geq \min(u(x), v(y))$$

for all $x, y \in \mathbb{R}^n$, where $t \in (0,1)$ is fixed and $s = 1-t$. If, in addition, $\|u\|_\infty = ess\,sup_{x\in\mathbb{R}} u(x) = \|v\|_\infty$, then

$$\int w \geq t \int u \; + s \int v$$

Proof. Without loss of generality, we may assume that $\|u\|_\infty = \|v\|_\infty = 1$, so that $\|w\|_\infty \geq 1$. Consider, for $0 < \lambda < 1$,

$$A(\lambda) = \{x \in \mathbb{R} | u(x) \geq \lambda\}$$
$$B(\lambda) = \{x \in \mathbb{R} | v(x) \geq \lambda\}$$
$$C(\lambda) = \{x \in \mathbb{R} | w(x) \geq \lambda\}$$

It then follows by assumption that $tA(\lambda) + sB(\lambda) \subset C(\lambda)$. Using this fact, and recalling that $\int_{-\infty}^{+\infty} w(x)dx = \int_0^\infty |\{w \geq \lambda\}|\, d\lambda$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} w(x)dx &= \int_0^\infty |\{w \geq \lambda\}|\, d\lambda \\
&\geq \int_0^1 |C(\lambda)| d\lambda \\
&\geq \int_0^1 |tA(\lambda) + sB(\lambda)| d\lambda \\
&\geq t\int_0^1 |A(\lambda)| d\lambda + s\int_0^1 |B(\lambda)| d\lambda \\
&= t\int u + s\int v
\end{aligned}$$

□

Corollary 3.4. (Prékopa's Theorem, 1971) *Let μ be an absolutely continuous measure with density p (with respect to the Lebesgue measure) such that*

$$p(tx+sy) \geq p(x)^t p(y)^s$$

for all $x, y \in \mathbb{R}^n$, $t + s = 1$ $(t, s > 0)$. Then, for any measurable sets $A, B \subset \mathbb{R}^n$ and $t \in (0,1)$

$$\mu(tA+sB) \geq \mu(A)^t \mu(B)^s \tag{3.1}$$

In words, a measure μ is log-concave if its density $p = \frac{d\mu}{dx}$ is log-concave (i.e. if $\log(p)$ is concave).

Proof. The result is an immediate consequence of the Prékopa-Leindler Theorem 3.1 applied to the functions

$$u = p\chi_A$$
$$v = p\chi_B$$
$$w = p\chi_{tA+sB}$$

Note that the condition on u,v and w is satisfied by our assumption on p. □

We now provide the inductive step which concludes the proof of Theorem 3.1.

Proof. (of Theorem 3.1) Suppose the result holds in $\mathbb{R}^{n-1}$ and we are given non-negative integrable functions w,u,v on $\mathbb{R}^n$. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$ whose elements are pairs (a, b) with $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$. For a fixed real number c, define $w_c(x) = w(x, c)$ which is now a function on $\mathbb{R}^{n-1}$, and similarly for u_c and v_c. The assumption $w(tx + sy) \geq u(x)^t v(y)^s$ in $\mathbb{R}^n$ translates as $w_{ta+sb}(tx + sy) \geq u_a(x)^t v_b(y)^s$. By the (n-1)-dimensional result, we have

$$W(ta + sb) := \int_{\mathbb{R}^{n-1}} w_{ta+sb} \geq \left(\int_{\mathbb{R}^{n-1}} u_a \right)^t \left(\int_{\mathbb{R}^{n-1}} v_b \right)^s =: U(a)^t V(b)^s$$

Finally, we apply the 1-dimensional result (Lemma 3.3) to the functions W, U and V and using Fubini's Theorem we conclude that

$$\int_{\mathbb{R}} W = \int_{\mathbb{R}^n} w \geq t \left(\int_{\mathbb{R}^n} u \right) + s \left(\int_{\mathbb{R}^n} v \right) = t \left(\int_{\mathbb{R}} U \right) + s \left(\int_{\mathbb{R}} V \right)$$

By the AM-GM inequality we obtain $t \left(\int_{\mathbb{R}^n} u \right) + s \left(\int_{\mathbb{R}^n} v \right) \geq \left(\int_{\mathbb{R}^n} u \right)^t \left(\int_{\mathbb{R}^n} v \right)^s$ $\square$

4. Surface Brunn-Minkowski type inequality

In [Bob07] an inequality closely related to Brunn-Minkowski but expressed in terms of surface measure was studied. A functional form of this, in the spirit of the Prékopa-Leindler inequality, was introduced. In what follows, given a set A, $\boldsymbol{S}(A)$ will denote the size of its boundary, i.e. its *surface measure*.

Theorem 4.1. *If A, B are convex bodies, then for all $t \in (0,1)$, $s = 1-t$,*

$$\boldsymbol{S}(tA+sB)^{1/(n-1)} \geq t\boldsymbol{S}(A)^{1/(n-1)} + s\boldsymbol{S}(B)^{1/(n-1)}$$

In log-concave form, the inequality states that

$$\boldsymbol{S}(tA+sB) \geq \boldsymbol{S}(A)^t \boldsymbol{S}(B)^s$$

Definition 4.2. *We will say that a function u is* quasiconcave *if for all $x, y \in \mathbb{R}$ and all $t \in [0,1]$, $s = 1-t$,*

$$u(tx+sy) \geq \min(u(x), u(y))$$

Remark 4.3. *Note that the above definition of a quasiconcave function is equivalent to the property that the u pre-images of $\{x \in \mathbb{R} \,|\, u(x) \geq \lambda\}$ (for $\lambda \in \mathbb{R}$) are convex sets. For example, any log-concave function is quasi-concave.*

In the same sense that the Prékopa-Leindler inequality is a functional form of the volume Brunn-Minkowski inequality, a functional form of the surface Brunn-Minkowski inequality is as follows:

Theorem 4.4. *Let $u, v, w : \mathbb{R}^n \to [0, \infty)$ be smooth, quasi-concave functions, satisfying*

$$w(tx+sy) \geq u(x)^t v(y)^s$$

for all $x, y \in \mathbb{R}^n$. If, in addition, $w \longrightarrow 0$ as $|z| \longrightarrow 0$, then

$$\int |\nabla w| \geq \left(\int |\nabla u|\right)^t \left(\int |\nabla v|\right)^s$$

Remark 4.5. *The condition $w \longrightarrow 0$ cannot be omitted: to see this, if $u, v \leq 1$, take $w \equiv 1$.*

Lemma 4.6. ([Bal89]) *Suppose that $u, v, w : (0,\infty) \to (0,\infty)$ satisfy*

$$w(x^t y^s) \geq u(x)^t \, v(y)^s$$

for $x, y > 0$, $t, s > 0$ with $s = 1-t$. Then

$$\int_0^\infty |\nabla w| \geq \left(\int_0^\infty |\nabla u|\right)^t \left(\int_0^\infty |\nabla v|\right)^s$$

Proof. Define $\tilde{u} = u(e^{-x})e^{-x}$ and similarly for $\tilde{v}$ and $\tilde{w}$. Note that by a change of variable, we have $\int_0^\infty |\nabla w| = \int_{-\infty}^{+\infty} u(e^{-x})e^{-x}dx = \int_{-\infty}^{+\infty} \tilde{u}$. Finally, observe that $\tilde{w}(tx+sy) \geq \tilde{u}(x)^t \, \tilde{v}(y)^s$. □

We now prove Theorem 4.4:

Proof. Define, for $\lambda > 0$, $A_u(\lambda) = \{x \in \mathbb{R}^n \,|\, u(x) \geq \lambda\}$ and similarly for A_v and A_w. These sets are then convex (by remark (4.3)) and bounded. Moreover, we have

$$A_w(\lambda_1^t \lambda_2^s) \geq tA_u(\lambda_1) + sA_v(\lambda_2) \Rightarrow \underbrace{S(A_w(\lambda_1^t\lambda_2^s))}_{\equiv \tilde{w}(\lambda_1^t\lambda_2^s)} \geq \underbrace{S(A_u(\lambda_1))^t}_{\equiv \tilde{u}(\lambda_1)^t} \underbrace{S(A_v(\lambda_2))^s}_{\equiv \tilde{v}(\lambda_2)^s}$$
$$\Rightarrow \int_0^\infty S(A_w(\lambda))d\lambda \geq \left(\int_0^\infty S(A_u(\lambda))d\lambda\right)^t \left(\int_0^\infty S(A_v(\lambda))d\lambda\right)^s$$

We conclude by the co-area formula (see [Fed69] and Appendix), which states that

$$\int_{\mathbb{R}^n} |\nabla w| dx = \int_{-\infty}^{+\infty} S(\{w \geq \lambda\})d\lambda$$

□

5. Applications to the Gaussian measure and concentration.

Recall that the Gaussian measure on $\mathbb{R}^n$ is defined as the measure with density $\gamma_n(dx) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx$. Brunn-Minkowski-type inequalities have interesting applications to $\mathbb{R}^n$ equipped with this measure.

In the next lines we introduce the infimum- and supremum-convolutions of a function. Given functions $f, g : \mathbb{R}^n \to \mathbb{R}$, let us apply the Prékopa-Leindler theorem to

$$\begin{aligned} u(x) &= \exp\left\{ sg(x) - \frac{1}{2}|x|^2 \right\} \Rightarrow \int u = (2\pi)^{n/2} \mathbb{E} e^{sg} \\ v(y) &= \exp\left\{ tf(y) - \frac{1}{2}|y|^2 \right\} \Rightarrow \int v = (2\pi)^{n/2} \mathbb{E} e^{-tf} \\ w(z) &= \exp\left\{ -\frac{1}{2}|z|^2 \right\} \qquad \Rightarrow \int w = (2\pi)^{n/2} \end{aligned}$$

According to the assumptions of the Prékopa-Leindler Theorem, we require that

$$\begin{gathered} \frac{1}{2}|tx + sy|^2 \geq t\left(sg(x) - \frac{1}{2}|x|^2 \right) + s\left(-tf(y) - \frac{1}{2}|y|^2 \right) \\ \Updownarrow \\ \frac{1}{2}|x-y|^2 \geq g(x) - f(y) \end{gathered}$$

We see therefore that the optimal choices for g and f are

$$\begin{aligned} g(x) = (Qf)(x) &= \inf_y \left(f(y) + \frac{|x-y|^2}{2} \right) \\ f(x) = (Pg)(x) &= \sup_x \left(g(x) - \frac{|x-y|^2}{2} \right) \end{aligned}$$

Definition 5.1. *Qf is called the* infimum-convolution *of f while Pg is the* supremum-convolution *of g.*

With Qf and Pg defined as above, the following inequalities hold true:

Corollary 5.2. ([Mau91]) *For all measurable functions f,g and $t \in [0,1]$, s=1-t*

$$\left(\mathbb{E} e^{sQf} \right)^t \left(\mathbb{E} e^{-tf} \right)^s \leq 1$$

and

$$\left(\mathbb{E} e^{sg} \right)^t \left(\mathbb{E} e^{-tPg} \right)^s \leq 1$$

The use of infimum and supremum convolutions has various interesting applications, a few of which we present in what follows.

The concentration of measure phenomenon is a property of measure which was studied extensively in the literature. For an introduction the reader is referred to [Led01]. Given a measurable, non-empty subset $A \subset \mathbb{R}^n$, apply the above Corollary to

$$f(x) = \begin{cases} 0, & x \in A \\ +\infty, & x \notin A \end{cases}$$

With this function, we have

$$\begin{aligned} Qf(x) &= \inf_y \left(f(y) + \frac{1}{2}|x-y|^2 \right) \\ &= \frac{1}{2} \operatorname{dist}(A,x)^2 \end{aligned}$$

Taking $t = s = \frac{1}{2}$ gives

$$\mathbb{E} e^{\frac{1}{4}\operatorname{dist}(A,x)^2} \leq \frac{1}{\gamma_n(A)}$$

Now Chebyshev's inequality for γ_n states that

$$\gamma_n\left(\{ x : |f(x)| \geq t \} \right) \leq \frac{1}{t^2} \int f^2 d\gamma_n$$

which implies (for $t = e^{h/2}$, $f = e^{\operatorname{dist}(x,A)/2}$)

$$1 - \gamma_n(A^h) \leq \frac{1}{\gamma_n(A)} e^{-\frac{1}{4}h^2}$$

where $A^h = \{ x : \operatorname{dist}(x,A) < h \}$ denotes the open h-neighbourhood of A.

Being restricted to convex functions, we have the following result of Tsirel'son (1983):

Theorem 5.3. *Given a Gaussian random process $(X_t)_{t \in T}$ with variances $\sigma_t^2 = Var(X_t)$*

$$\mathbb{E} e^{\sup_t \{ X_t - \frac{\sigma_t^2}{2} \}} \leq e^{\mathbb{E} \sup_t X_t}$$

Further interesting applications include Poincaré-type inequalities and logarithmic Sobolev inequalities. These are studied in Sections 11 and 12 (see also Appendix). The former is sometimes referred to as Spectral Gap inequality and reads:

$$Var_{\gamma_n}(f) \leq \mathbb{E}_{\gamma_n}|\nabla f|^2$$

Remark 5.4. *We conclude this section by noting that more generally, we may consider convolutions with respect to convex V and put $Q_t f = \inf_y\{f(y)+tV(\frac{x-y}{t})\}$. This gives a semigroup, with generator $\frac{\partial Q_t}{\partial t}|_{t=0} = V(x)$. In particular, the semigroup $Q_t f = \inf_y\{f(y) + |x-y|^2/2t\}$ is generated by $L = -(1/2)\|\nabla f\|^2$ which appears on the right hand side of inequalities like the Spectral Gap inequality [BL08].*

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6. Dimensional Prékopa-Leindler theorem

Let us now state a dimensional form of the Prékopa-Leindler theorem which is expressed in terms of the *generalised mean* of two numbers.

Definition 6.1. *For $a, b > 0$ we define the* generalised mean *of a and b by*

$$M_\kappa^{(t)}(a,b) = (ta^\kappa + sb^\kappa)^{1/\kappa}$$

where $\kappa \in [-\infty, +\infty]$.

When $\kappa = +\infty$ the above expression is understood as the maximum of a and b, while for $\kappa = -\infty$ as their minimum. Moreover, when $\kappa = 0$ we have $M_\kappa^{(t)}(a,b) = a^t b^s$, the geometric mean of a and b and for $\kappa = 1$ we get their arithmetic mean $ta + sb$. Finally, note that putting $\kappa = -1$ we obtain the harmonic mean of a and b, which is $\frac{1}{t/a+s/b}$.
More generally, one can show that $M_\kappa^{(t)}(a,b) < M_\lambda^{(t)}(a,b)$ whenever $\kappa < \lambda$. Differentiating in κ we have

$$\begin{aligned}\frac{\partial}{\partial\kappa} M_\kappa^{(t)}(a,b) &= \frac{\partial}{\partial\kappa}(ta^\kappa + sb^\kappa)^{1/\kappa} \\ &= (ta^\kappa + sb^\kappa)^{1/\kappa}\left\{\frac{1}{\kappa}\cdot\frac{ta^k\log a + sb^\kappa \log b}{ta^\kappa + sb^\kappa} - \frac{1}{\kappa^2}\log(ta^\kappa + sb^\kappa)\right\}\end{aligned}$$

Thus to prove that the function is increasing, we need to show that

$$\begin{aligned} &\frac{1}{\kappa}\cdot\frac{ta^k\log a + sb^\kappa\log b}{ta^\kappa + sb^\kappa} \geq \frac{1}{\kappa^2}\log(ta^\kappa + sb^\kappa) \\ \iff \quad & ta^k \log a^\kappa + sb^\kappa \log b^\kappa \geq (ta^\kappa + sb^\kappa)\log(ta^\kappa + sb^\kappa)\end{aligned}$$

The latter statement is Jensen's inequality applied to the convex function $x \log x$ and this concludes the proof.
Furthermore, we have the following generalisation of the AM-GM inequality:

Lemma 6.2. *Let $\frac{1}{\kappa} = \frac{1}{\kappa'} + \frac{1}{\kappa''}$, $\kappa' + \kappa'' > 0$. Then, for all $a_1, b_1, a_2, b_2 > 0$, $t \in (0,1)$*

$$M_{\kappa'}^{(t)}(a_1,b_1) M_{\kappa''}^{(t)}(a_2,b_2) \geq M_\kappa^{(t)}(a_1a_2, b_1b_2)$$

Proof. This follows from Jensen's inequality ([HLP52]) which states that if A, B, C and D are positive and $\kappa, \kappa', \kappa''$ as above

$$(A+B)^{1/\kappa'}(C+D)^{1/\kappa''} \geq A^{1/\kappa'}C^{1/\kappa''} + B^{1/\kappa'}C^{1/\kappa''}$$

This gives the result when $A = ta_1^{\kappa'}, B = sb_1^{\kappa'}, C = ta_2^{\kappa''}$ and $D = sa_2^{\kappa''}$. □

The following theorem is a dimensional form of the Prékopa-Leindler Theorem.

Theorem 6.3. *Let $-\infty \leq \kappa \leq \frac{1}{n}$ and $t, s > 0$, $s = 1 - t$. Suppose that $u, v, w : \mathbb{R}^n \to [0, \infty)$ satisfy*

$$w(tx + sy) \geq (tu(x)^{\kappa_n} + sv(y)^{\kappa_n})^{1/\kappa_n}$$

for all $x, y \in \mathbb{R}^n$, where $\kappa_n = \frac{\kappa}{1-n\kappa}$.
Then

$$\int w \geq \left[t\left(\int u\right)^\kappa + s\left(\int v\right)^\kappa\right]^{\frac{1}{\kappa}}$$

Remark 6.4. *Note that*

$$\begin{aligned}\kappa = \frac{1}{n} &\Rightarrow \kappa_n = +\infty \\ \kappa = 0 &\Rightarrow \kappa_n = 0 \\ \kappa = -\infty &\Rightarrow \kappa_n = -\frac{1}{n}\end{aligned}$$

We can now prove Theorem 6.3: Note that the basic case n=1 follows from Lemma 3.3. Write $a = \sup u < \infty$, $b = \sup v < \infty$, $\rho = M_{\kappa_n}^{(t)}(a,b)$, $\eta = \frac{ta^{\kappa_n}}{\rho^{\kappa_n}} = \frac{ta^{\kappa_n}}{ta^{\kappa_n} + sb^{\kappa_n}} \in (0,1)$. Apply Lemma 3.3 to the functions

$$\begin{aligned}U(x) &= \frac{1}{a}u\left(\frac{a^{\kappa_n}}{\rho^{\kappa_n}}x\right) \\ V(x) &= \frac{1}{b}v\left(\frac{b^{\kappa_n}}{\rho^{\kappa_n}}y\right)\end{aligned}$$

and note that $\|U\|_\infty = \|V\|_\infty = 1$. We then have

$$\begin{aligned}W(\eta x + (1-\eta)y) &= w\left(t\frac{a^{\kappa_n}}{\rho^{\kappa_n}}x + s\frac{b^{\kappa_n}}{\rho^{\kappa_n}}y\right) \\ &\geq M_{\kappa_n}^{(t)}\left(u\left(\frac{a^{\kappa_n}}{\rho^{\kappa_n}}x\right), v\left(\frac{b^{\kappa_n}}{\rho^{\kappa_n}}y\right)\right) \\ &= M_{\kappa_n}^{(t)}(aU(x), bV(y)) \\ &\geq M_{\kappa_n}^{(t)}(a,b)\min(U(x), V(y))\end{aligned}$$

Therefore,

$$\begin{aligned}
\int W &\geq \eta \int M_{\kappa_n}^{(t)}(a,b)\ U(x)dx + (1-\eta)\int M_{\kappa_n}^{(t)}(a,b)\ V(y)dy \\
&= M_{\kappa_n}^{(t)}(a,b)\left(\frac{t}{a}\int u + \frac{s}{b}\int v\right) \\
&= M_{\kappa_n}^{(t)}(a,b) M_1^{(t)}\left(\frac{1}{a}\int u, \frac{1}{b}\int v\right) \\
&\geq M_{\kappa}^{(t)}\left(\int u, \int v\right)
\end{aligned}$$

where the last step follows by applying Lemma 6.2 with $\kappa' = \kappa_n$, $\kappa'' = 1$.

□

7. Hierarchy of convex measures

Suppose that μ is concentrated on some open convex set $\Omega \subset \mathbb{R}^n$ and is finite on compact sets. The following notion of κ-concavity will allow us to generalise our results:

Definition 7.1. *Let $\kappa \in [-\infty, +\infty]$. The measure μ is called* κ-concave *if it satisfies*

$$\begin{aligned} \mu(tA + sB) &\geq (t\mu(A)^\kappa + s\mu(B)^\kappa)^{1/\kappa} \\ &\equiv M_k^{(t)}(\mu(A), \mu(B)) \end{aligned}$$

for all measurable $A, B \subset \mathbb{R}^n$ of positive measure and all $t \in (0,1)$.

Remark 7.2. *If $\mu = \delta_x$ then $\kappa = +\infty$. Otherwise, we necessarily have $\kappa \leq 1$. Moreover, we must always have $\kappa \leq 1/\dim(\Omega)$.*

The following special cases arise: When $\kappa = 1/n$ (and μ is absolutely continuous), then μ is the Lebesgue measure restricted on Ω (the fact that the Lebesgue measure is κ-concave with $\kappa = 1/n$ is the Brunn-Minkowski inequality). When $\kappa = 0$, μ is log-concave. The case $\kappa = -\infty$ describes all convex measures μ .

Theorem 7.3. (Borell's Characterisation) *If μ is convex, then μ is concentrated on some convex set $\Omega \subset \mathbb{R}^n$ where it is absolutely continuous with respect to the Lebesgue measure on Ω. Moreover, μ is κ-concave if and only if $\kappa \leq 1/\dim(\Omega)$ and μ has a density $p(x)$ on Ω satisfying*

$$\begin{aligned} p(tx + sy) &\geq (tp(x)^{\kappa_n} + sp(y)^{\kappa_n})^{1/\kappa_n} \\ &= M_{\kappa_n}^{(t)}(p(x), p(y)) \end{aligned}$$

Thus

$$\mu \text{ is } \kappa\text{-concave} \iff p \text{ is } \kappa_n\text{-concave}$$

Proof. This is a direct consequence of Theorem 6.3 applied to the functions $u = \chi_A$, $v = p\chi_B$ and $w = \chi_{tA+sB}$. $\square$

Remark 7.4. *Note that*

$$p \text{ is } \kappa_n\text{-concave} \iff \begin{cases} p^{\kappa_n} \text{ is concave when } \kappa_n > 0 \\ \log p \text{ is concave when } \kappa_n = 0 \\ p^{\kappa_n} \text{ is convex when } \kappa_n < 0 \end{cases}$$

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The weakest case arises when $\kappa = -\infty$, so that $\kappa_n = -1/n$. In this case μ has a density $\frac{d\mu}{dx} = \frac{1}{V(x)^n}$ for some positive convex function V defined on some convex $\Omega \subset \mathbb{R}^n$. This is the general form of convex n-dimensional measures.

Example 7.5. *Given $\beta > n/2$, $\mu = \nu_\beta$ with density $p(x) = \frac{1}{Z}\frac{1}{(1+|x|^2)^\beta}$, $x \in \mathbb{R}$ (Z being some normalising factor chosen so that $\mu(\mathbb{R}^n) = 1$) is called the* generalised Cauchy measure *on $\mathbb{R}^n$ with parameter β (the condition $\beta > n/2$ ensures that ν_β is a probability measure) . This measure is κ-concave if and only if*

$$\begin{aligned} & p(x)^{\kappa_n} \text{ is convex} \\ \iff & \left(1 + |x|^2\right)^{-\beta\kappa_n} \text{ is convex} \\ \iff & \left(\sqrt{1 + |x|^2}\right)^{-2\beta\kappa_n} \text{ is convex} \\ \iff & -2\beta\kappa_n \geq 1 \to \kappa_n = -1/2\beta \\ & 1/\kappa_n = -2\beta \to \kappa = \frac{1}{2\beta - n} \end{aligned}$$

The standard case is when $\beta = \frac{n+1}{2}$, $d = 1$.

Remark 7.6. *The measure ν_β may be characterised as the distribution of the random vector $X = (X_1, ..., X_n)$ constructed as follows. Consider two random vectors Y and ξ, Y having a standard Gaussian distribution and ξ having χ_d distribution with d degrees of freedom*[1]*. Then, the vector $X = \frac{Y}{\xi}$ has ν_β distribution, $\nu_\beta = \mathcal{L}\left(\frac{Y}{\xi}\right)$. Thus, in $\mathbb{R}^\infty$ there is $\mathbb{P}$ so that all its finite n-dimensional distributions are $\approx \nu_\beta$.*

[1] In other words, ξ has density $\chi_d(r) = c_d r^{d-1} e^{-r^2/2}$, where c_d is a constant and $r > 0$.

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8. Brascamp-Lieb inequality and a generalisation.

In this section we consider a probability measure μ (concentrated on some open convex subset $\Omega \subset \mathbb{R}^n$ as before) with density

$$p(x) = 1/V(x)^\beta$$

for some $\beta\gamma \geq n$, where V is a positive convex function. Note that by the above definition, μ is $\kappa-$concave with $\kappa = -\frac{1}{\beta-n}$. The Brascamp-Lieb inequality is an extension of the Poincaré inequality stated for log-concave measures, i.e. of the form $\mu(dx) = e^{-V}dx$ with some convex function V satisfying certain assumptions. In this section we study an extension of this result to the more general class of measures with densities $\mu(dx) = V(x)^{-\beta}$. We begin by the following theorem, which makes use of the fundamental dimensional Prékopa-Leindler result.

Suppose that V is in $C^2(\Omega)$ and define the distance-like function

$$d_V(x, y) = V(y) - V(x) - \langle V'(x), y - x\rangle$$

For example, we could take $V(x) = 1 + |x|^2$ which gives $d_V(x, y) = |x - y|^2$.

Theorem 8.1. *Assume $f, g : \Omega \to \mathbb{R}$ are measurable functions satisfying*

$$f(x)V(x) \leq g(y)V(y) + d_V(x, y) \tag{8.1}$$

for $x, y \in \Omega$. If, in addition, f is μ-integrable and $g \geq -1$, we have

$$1 + \frac{\beta}{\beta - n}\int f d\mu \leq \left(\int (g + 1)^{-\beta} d\mu\right)^{-\frac{1}{\beta-n}}$$

When $\beta \to \infty$ this yields the following

Corollary 8.2. *Let μ be log-concave on Ω with density $e^{-W(x)}$, for some convex $W \in C^1(\Omega)$.*
If

$$f(x) \leq g(y) + d_W(x, y) \quad \text{for all } x, y \in \Omega$$

then

$$\int e^f d\mu \leq e^{\int g d\mu}$$

Remark 8.3. *Note that the corollary was already obtained for γ_n:*

$$f = Qg \Rightarrow \mathbb{E}(e^{Qg}) \leq e^{\mathbb{E}g}$$

Proof. Assume $\sup f < \infty$, $\inf g > -1$. For $t \in (0, 1)$ and $s = 1 - t$ define

$$T_s(x, y) = \frac{tV(x) + sV(y) - V(tx + sy)}{ts}$$

As $s \to 0$, we have $T_s(x, y) \to d_V(x, y)$. Take open convex $\Omega_0 \subset \Omega$ with $\overline{\Omega_0} \subset \Omega$ and let $\kappa = -\frac{1}{\beta-n}$, $\kappa_n = \frac{\kappa}{1-n\kappa} = -\frac{1}{\beta}$. For $\varepsilon > 0$ small enough, let $f_\varepsilon(x) = f(x) - \frac{\varepsilon}{V(x)}$ and apply the dimensional Prékopa-Leindler theorem to the functions

$$u(x) = (1 - sf_\varepsilon(x))^{\frac{1}{\kappa_n}} \frac{p(x)}{\mu(\Omega_0)}$$
$$v(y) = (1 + tg(y))^{\frac{1}{\kappa_n}} \frac{p(x)}{\mu(\Omega_0)}$$
$$w(z) = \frac{p(z)}{\mu(\Omega_0)}$$

For these u, v, w, the hypothesis of the Prékopa-Leindler Theorem can be written as

$$f(x)v(x) \leq g(y)v(y) + T_s(x, y) + \varepsilon$$

which is true for small s and ε. Thus, writing $\mu_0 = \frac{1}{\mu(\Omega_0)}\mu|_{\Omega_0}$ we obtain

$$1 \leq t\left(\int (1 - sf_\varepsilon)^{\frac{1}{\kappa_n}} d\mu_0\right)^\kappa + s\left(\int (1 + tg)^{\frac{1}{\kappa_n}} d\mu_0\right)^\kappa$$

which, as $s \to 0$, gives

$$1 + \frac{\kappa}{\kappa_n}\int f_\varepsilon \, d\mu_0 \leq \left(\int (1 + g)^{\frac{1}{\kappa_n}} d\mu_0\right)^\kappa$$

Finally, letting $\varepsilon \to 0$ and $\Omega_0 \uparrow \Omega$, we get

$$1 + \frac{\kappa}{\kappa_n}\int f d\mu_0 \leq \left(\int (1 + g)^{\frac{1}{\kappa_n}} d\mu_0\right)^\kappa$$

which is the claim written in terms of κ and κ_n. □

Let us restrict ourselves now to log-concave probability measures on $\mathbb{R}^n$ with support on convex $\Omega \subset \mathbb{R}^n$ and density $p(x) = e^{-W(x)}$, where $W(x) \in C^2(\Omega)$ is convex and has positive and invertible second derivative $W''(x) > 0$. Then, p can be written in the form $1/V(x)^\beta$ with $V(x) = e^{-W/\beta}$, for some $\beta\gamma \geq n$, where V is a positive convex function. Recall that such μ is $\kappa-$concave with $\kappa = -\frac{1}{\beta-n}$. For these measures we have the following Brascamp-Lieb inequality:

Theorem 8.4. (Brascamp-Lieb, 1976) *For any bounded smooth g on Ω,*

$$Var_\mu(g) = \int g^2 d\mu - \left(\int \ g d\mu\right)^2 \leq \int \left\langle (W'')^{-1}\nabla g, \nabla g\right\rangle d\mu \tag{8.2}$$

Example 8.5. *Consider n-dimensional Gaussian measure $\mu = \gamma_n$. A computation shows that $W'' = I$. In this case, Theorem 8.4 gives $Var_\mu(g) \leq \int |\nabla g|^2 \, d\gamma_n$. This inequality is known as* Spectral Gap *(or* Poincaré*) inequality.*

Example 8.6. *Suppose that there exits a constant c such that $W'' \geq cI$. In this case, we obtain $Var_\mu(g) \leq \frac{1}{c}\int |\nabla g|^2 d\mu$, that is, the Poincaré inequality for the measure μ holds (with constant c).*

Suppose now that $\mu(dx) = V(x)^{-\beta}$. Theorem 8.4 can be generalised as follows:

Theorem 8.7. *([BL]) If $\beta \geq n$, for any bounded smooth g with mean 0 on Ω, we have*

$$Var_\mu(g) \leq \frac{1}{\beta+1}\int \frac{\langle (V'')^{-1}\nabla G, \nabla G\rangle}{V} d\mu \tag{8.3}$$

where G=gV.

Theorem 8.7 is indeed a generalisation of Theorem 8.4: To see this, take $V = c_\beta{}^{1/\beta}\left(1+\frac{1}{\beta}W\right)^\beta$ in (8.3) : in the limit as $\beta \to \infty$ we obtain (8.2).

Proof. Given $\varepsilon > 0$, $x \in \Omega$, let

$$\begin{aligned} F_\varepsilon(x) &= \inf_y \left\{G(y) + \frac{1}{\varepsilon}d_V(x,y)\right\} \\ f_\varepsilon(x) &= \frac{F_\varepsilon(x)}{V(x)} \end{aligned}$$

so that the infimum-convolution inequality (8.1) is satisfied with $\varepsilon f_\varepsilon$ and εg in place of f and g respectively:

$$\varepsilon f_\varepsilon(x)V(x) \geq \varepsilon g(y)V(y) + d_V(x,y)$$

Theorem 8.1 then says that, if we further assume $\varepsilon g \geq -1$, we have:

$$1 + \varepsilon\frac{\kappa}{\kappa_n}\int f_\varepsilon d\mu \leq \left(\int (1+\varepsilon g)^{\frac{1}{\kappa_n}}\, d\mu\right)^\kappa \tag{8.4}$$

with $\kappa = -\frac{1}{\beta-n}$, so that $\kappa_n = -1/\beta$. We claim that (8.4) yields (8.3) in the limit as $\varepsilon \to 0$. To this end, use Taylor expansion in ε to obtain

$$d_V(x, x+\varepsilon h) = \frac{\varepsilon^2}{2}\langle V''(x)h, h\rangle + |h|^2\, o(\varepsilon^2)$$

so that by definition of F_ε,

$$\begin{aligned} F_\varepsilon(x) &= \inf_{x+\varepsilon h\in\Omega}\left\{G(x+\varepsilon h) + \frac{1}{\varepsilon}d_V(x, x+\varepsilon h)\right\} \\ &= \inf_{x+\varepsilon h\in\Omega}\left\{G(x) + \varepsilon\langle\nabla G(x), h\rangle + \frac{\varepsilon}{2}\langle V''(x)h,h\rangle + |h|^2\, o(\varepsilon^2)\right\} \\ &\geq \inf_{h\in\mathbb{R}^n}\left\{G(x) + \varepsilon\langle\nabla G(x), h\rangle + \frac{\varepsilon}{2}\langle V''(x)h,h\rangle + |h|^2\, o(\varepsilon^2)\right\} \\ &= G_\varepsilon(x) + o(\varepsilon) \end{aligned}$$

where $G_\varepsilon(x) = G(x) - \frac{\varepsilon}{2}\langle (V'')^{-1}\nabla G(x), \nabla G(x)\rangle$.
This implies that

$$f_\varepsilon(x) = g(x) - \frac{\varepsilon}{2}\frac{\langle (V'')^{-1}\nabla G(x), \nabla G(x)\rangle}{V(x)} + o(\varepsilon)$$

Next, we replace the above expression for f_ε in (8.4) and perform Taylor expansion in the right-hand side to get:

$$\begin{aligned} &1 + \varepsilon\frac{\kappa}{\kappa_n}\int g d\mu - \varepsilon^2\frac{\kappa}{2\kappa_n}\left(\int \frac{\langle (V'')^{-1}\nabla G, \nabla G\rangle}{V}d\mu + o(\varepsilon)\right) \leq \\ &\quad 1 + \varepsilon\frac{\kappa}{\kappa_n}\int g d\mu + \varepsilon^2\frac{\kappa}{2\kappa_n}\left(\frac{1}{\kappa_n} - 1\right)\int g^2 d\mu + \frac{\kappa(\kappa-1)}{2}\left(\frac{\varepsilon}{\kappa_n}\int g d\mu\right)^2 + o(\varepsilon^3) \\ &\quad = 1 + \varepsilon^2\frac{\kappa}{2\kappa_n}\left(\frac{1}{\kappa_n} - 1\right)\int g^2 d\mu + o(\varepsilon^3), \quad \text{using} \int g d\mu = 0 \end{aligned}$$

Finally, dividing by $\varepsilon^2\kappa/\kappa_n$ and rearranging, we arrive at

$$\begin{aligned} &0 \leq \frac{1}{2}\left\{\int \frac{\langle (V'')^{-1}\nabla G, \nabla G\rangle}{V}d\mu + \left(\frac{1}{\kappa_n} - 1\right)\int g^2 d\mu\right\} + o(\varepsilon) \\ &\Rightarrow \int \frac{\langle (V'')^{-1}\nabla G, \nabla G\rangle}{V}d\mu \geq \left(1 - \frac{1}{\kappa_n}\right)\int g^2 d\mu \quad \text{, as } \varepsilon \to 0 \\ &\Rightarrow \int \frac{\langle (V'')^{-1}\nabla G, \nabla G\rangle}{V}d\mu \geq (1+\beta)\int g^2 d\mu \end{aligned}$$

□

9. Refinement of Brascamp-Lieb inequality in the log-concave case

Recall the generalised form of the Brascamp-Lieb inequality (Theorem 8.7) for measures on $\mathbb{R}^n$ with density $p(x) = \frac{1}{V(x)^\beta}$ for some $\beta \geq n$:

$$Var_\mu(g) \leq \frac{1}{\beta+1} \int_{\mathbb{R}^n} \frac{\langle (V'')^{-1} \nabla G, \nabla G\rangle}{V} d\mu \tag{9.1}$$

where $G = gV$.

Here, we aim to apply this result to the log-concave case. Recall that Theorem 8.4 of Brascamp-Lieb was stated for this type of measures. Using the extended inequality above, a refinement can be obtained. To this end we write $V(x) = e^{-W(x)/\beta}$, i.e. $p(x) = e^{W(x)}$. We then have

$$\begin{aligned} V' &= \frac{1}{\beta} e^{W/\beta} W' \\ V'' &= \frac{1}{\beta^2} e^{W/\beta} W' \otimes W' + \frac{1}{\beta} e^{W/\beta} W'' \end{aligned}$$

where the operation $\otimes$ is defined by $(u \otimes v)_{ij} = u_i v_j$, so

$$\begin{aligned} V'' &= \frac{1}{\beta} e^{W/\beta} \left(\frac{1}{\beta} W' \otimes W' + W'' \right) \\ &\equiv \frac{1}{\beta} e^{W/\beta} R_{w,\beta} \end{aligned}$$

and taking inverse

$$(V'')^{-1} = \beta e^{-W/\beta} R_{w,\beta}^{-1} = \beta V R_{W,\beta}^{-1}$$

Now since $G = gV$, we have $G' = Vg' + gV'$ hence

$$\begin{aligned} \frac{\langle (V'')^{-1} G', G'\rangle}{V} &= \beta \langle R_{W,\beta}^{-1} (Vg' + gV'), Vg' + gV'\rangle \\ &\leq \beta r \langle R_{W,\beta}^{-1} g', g'\rangle V^2 + \beta s \langle R_{W,\beta}^{-1} V', V'\rangle g^2 \end{aligned}$$

for any $r > 1$, $s = \frac{r}{r-1}$ conjugate to r (such that $\frac{1}{r} + \frac{1}{s} = 1$), where the last inequality follows from the general fact that for $u, v \in \mathbb{R}^n$ and r as above

$$\langle A(u+v), u+v\rangle \leq r\langle Au, u\rangle + s\langle Av, v\rangle$$

Note that

$$\begin{aligned} \frac{\langle (V'')^{-1} V', V'\rangle}{V} &= \langle (\beta W'' + W' \otimes W' \ )^{-1} W', W'\rangle \\ &\leq 1 \text{ , by convexity} \end{aligned}$$

In the case where n=1, we have $\frac{(W')^2}{\beta W'' + (W')^2} \leq 1$.
Assuming $\int g d\mu = 0$, we get

$$\begin{aligned} (\beta+1) \int g^2 d\mu &\leq r \int g^2 d\mu + s \int \langle (V'')^{-1} g', g'\rangle d\mu \\ &\Updownarrow \\ (\beta + 1 - r) \int g^2 d\mu &\leq \beta s \int \langle {R_{W,\beta}}^{-1} \nabla g, \nabla g\rangle d\mu \end{aligned}$$

Corollary 9.1. *Assume $\beta \geq n$. For any bounded smooth g, the following inequality holds*

$$Var_\mu(g) \leq C_\beta \int_{\mathbb{R}^n} \langle {R_{W,\beta}}^{-1} \nabla g, \nabla g\rangle d\mu$$

with $C_\beta = \left(\sqrt{\beta+1} - 1\right)^2 \beta^{-1}$ $(1 < \beta < b)$ and $R_{W,\beta} = \beta^{-1} W' \otimes W' + W''$.
In particular, when $n = \beta = 1$, we have

$$Var_\mu(g) \leq b \int_{-\infty}^{+\infty} \frac{|\nabla g(x)|}{W'(x)^2 + W''(x)} d\mu(x)$$

We finish this section by presenting two examples and applying the above result.

Example 9.2. *Suppose that $\Omega = (0, +\infty)$, $p(x) = e^{-x}$ and $W(x) = x$. Then the Corollary implies that the following Poincaré inequality holds:*

$$Var_\mu(g) \leq b \int_0^{+\infty} g'(x)^2 d\mu(x)$$

Example 9.3. *In $\Omega = \mathbb{R}$ equipped with Gaussian measure $\gamma(dx) = p(x)dx$, with $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, we have a Poincaré type inequality with weight $1 + x^2$:*

$$Var_\mu(g) \leq b \int_{-\infty}^{+\infty} \frac{g'(x)^2}{1+x^2} d\gamma(x)$$

10. Application to Cauchy measures

In this section we study Cauchy measures ν_β with density $p(x) = Z^{-1}\left(1+|x|^2\right)^{-\beta}$, where Z is the normalising factor. Functional inequalities for such measures were considered in [BL].

Theorem 10.1. *The Cauchy measures ν_β ($\beta > n$) satisfy the Poincaré inequality with weight*

$$Var_{\nu_\beta}(g) \leq \frac{1}{\beta-3}\int |\nabla g(x)|^2\left(1+|x|^2\right)d\nu_\beta(x)$$

for all bounded smooth functions g on $\mathbb{R}^n$ ($n \geq 3$).

Proof. Taking $V(x) = 1+|x|^2$ (so that $p(x) = Z^{-1}V(x)^{-\beta}$) we see that $V'' = 2I$ and if we assume that $\int g d\mu = 0$, the Brascamp-Lieb inequality gives

$$(\beta+1)\int_{\mathbb{R}_n} g^2 d\mu \leq \int_{\mathbb{R}^n}\frac{\left|\nabla\left(g(x)\left(1+|x|^2\right)\right)\right|^2}{2\left(1+|x|^2\right)}d\nu_\beta$$

and since

$$\begin{aligned}\left|\nabla\left(\ g(x)\left(1+|x|^2\right)\right)\right|^2 &= \left|\left(1+|x|^2\right)\nabla g(x) + 2g(x)x\right|^2 \\ &\leq 2|\left(1+|x|^2\right)^2|\nabla g(x)|^2 + 8g(x)^2|x|^2\end{aligned}$$

using the fact $|x|^2 \leq 1+|x|^2$ we conclude that

$$(\beta+1)\int_{\mathbb{R}^n} g^2 d\nu_\beta \ \leq \int_{\mathbb{R}^n}|\nabla g(x)|^2\left(1+|x|^2\right)d\nu_\beta(x) + 4\int_{\mathbb{R}^n} g(x)^2 d\nu_\beta(x)$$

□

Remark 10.2. *The integrand on the RHS can be bounded so that the inequality holds with a better constant $1/2\beta$ ([BL]). Moreover, the weight $(1+|x|^2)$ is optimal.*

One can show ([BL]) that the Cauchy distributions also satisfy a Logarithmic Sobolev inequality with weight

$$Ent_{\nu_\beta}\left(g^2\right) \leq \frac{1}{\beta-1}\int_{\mathbb{R}^n}|\nabla g(x)|^2\left(1+|x|^2\right)^2 d\nu_\beta(x)$$

After rescaling, the Cauchy distributions approximate the Gaussian distribution in the limit as $\beta \to \infty$. The above inequality thus yields the Logarithmic Sobolev inequality for the Gaussian measure

$$Ent_{\gamma_n}(g^2) \leq 2\int |\nabla g|^2 d\gamma_n$$

11. Weighted Cheeger and weighted Poincaré-type inequalities

In this section we consider probability measures on $\mathbb{R}^n$ with density of the form

$$\rho(x) = V(x)^{-\beta}$$

$x \in \Omega, \beta \geq n$, where $V : \Omega \to (0, \infty)$ is convex. Note that such a measure is κ-concave with $\kappa = -1/(\beta - n)$.

Theorem 11.1. ([BL]) *Let $\beta > n$. For any smooth bounded g on Ω*

$$Var_\mu(g) \leq C \frac{\beta - n + 1}{\beta - n} \int |\nabla g(x)|^2 \left(r^2 + |x|^2\right) d\mu(x)$$

where $\mu(\{|x| \leq r\}) = \frac{2}{3}$.

Proof. We first consider g with $\mathrm{m}(g) = 0$, where m denotes a median of g under the measure μ. We may assume $g \geq 0$ (otherwise, split it into its positive and negative parts $g = g_+ - g_-$). For such g we derive an inequality of the form

$$\int |g| d\mu \leq D \int |\nabla g|(r + |x|) d\mu(x)$$

Equivalently, we can state this as the following inequality on sets:

$$\forall A \subset \mathbb{R}^n \text{ with } \mu(A) \in (0, 1/2] : \ \mu(A) \leq D\nu^+(A)$$

where ν is the measure with density $\nu(dx) = (r + |x|)\, d\mu(x)$ and ν^+ is the surface measure defined by

$$\nu^+(A) = \liminf_{\varepsilon \downarrow 0} \frac{\nu(A + \varepsilon B) - \nu(A)}{\varepsilon} = \int_{\partial A} (r + |x|)\, \rho(x) d\lambda_{n-1}(x)$$

where B is the unit ball centred at the origin in $\mathbb{R}^n$ and $d\lambda_{n-1}$ denotes (n-1)-dimensional Lebesgue measure. Similarly, we write

$$\mu^+(A) = \int_{\partial A} \rho(x) d\lambda_{n-1}(x)$$

For $x \in \partial A$, denote by $n_A(x)$ the normal to the surface of A at the point x. Note that

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left[\mu\left((1-\varepsilon)A + \varepsilon B\right) - \mu(A)\right] &= r\mu^+(A) - \int_{\partial A} \langle n_A(x), x\rangle\, \rho(x) d\lambda_{n-1}(x) \\
&\leq r\mu^+(A) - \int_{\partial A} |x| \rho(x) d\lambda_{n-1}(x) \\
&= \int_{\partial A} (r + |x|)\, \rho(x) d\lambda_{n-1}(x) \\
&= \nu^+(A)
\end{aligned}$$

We know from Borell's characterisation (Theorem 7.3) that μ satisfies a Brunn-Minkowski type inequality

$$\mu(\alpha A + (1-\alpha)B) \geq \left(\alpha \mu(A)^\kappa + (1-\alpha)\mu(B)^\kappa\right)^{1/\kappa}$$

Therefore,

$$\begin{aligned}
\nu^+(A) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{\left((1-\varepsilon)\mu(A)^\kappa + \varepsilon \mu(B_r)^\kappa\right)^{1/\kappa} \ - \mu(A)\right\} \\
&= \mu(A)^{1-\kappa} \frac{\mu(A)^\kappa - \mu(B_r)^\kappa}{-\kappa}
\end{aligned}$$

Thus, to prove the theorem, it suffices to show that

$$\mu(A)^{1-\kappa} \frac{\mu(A)^\kappa - \mu(B_r)^\kappa}{-\kappa} \geq \frac{1}{D} \mu(A)$$

for $\mu(A) \leq 1/2$. For simplicity, let $\mu(A) = t$, $p_r = \mu(B_r)$. We then need to show

$$\frac{t^\kappa - p_r^\kappa}{-\kappa} \geq \frac{1}{D} t^\kappa$$

The critical case is when $t = \frac{1}{2}$, when we have

$$\frac{1}{D} = \frac{1 - (2p_r)^\kappa}{\kappa} \geq \frac{\log(2p_r)}{1 - \kappa}$$

Thus, if $p_r = \mu(B_r) > 1/2$, then

$$\int |g| d\mu \leq D \int |\nabla g(x)|\, (r + |x|)\, d\mu(x) \tag{11.1}$$

holds with

$$D = \frac{1-\kappa}{\log 2p_r} = \frac{1}{\log 2p_r}\left(1+\frac{1}{\beta-n}\right) = \frac{1}{\log 2p_r}\,\frac{\beta-n+1}{\beta-n}$$

Finally, to reach the Poincaré inequality apply (11.1) to g^2 : Using Cauchy-Schwarz we get

$$\int g^2 d\mu \leq 2D\int |\nabla g| g(x)\,(r+|x|)\,d\mu(x) \leq 2D\sqrt{\int |\nabla g|^2\,(r+|x|)^2\,d\mu(x)}\sqrt{\int g^2 d\mu}$$

Dividing both sides by $\sqrt{\int g^2 d\mu}$ and then squaring, we arrive at

$$\int g^2 d\mu \leq 4D^2\int |\nabla g|^2\,(r+|x|)^2\,d\mu(x) \leq 8D^2\int |\nabla g|^2\left(r^2+|x|^2\right)d\mu(x)$$

For general functions g, we may always replace g by $g-m$ and use the fact that

$$Var_\mu(g) = \inf_a \int (g-a)^2 d\mu \leq \int (g-m)^2 d\mu$$

Since $|\nabla(g-m)| = |\nabla g|$, we conclude by

$$\int (g-m)^2 d\mu(x) \leq 8D^2\int |\nabla g|^2\left(r^2+|x|^2\right)d\mu(x)$$

□

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12. Growth of moments under Poincaré and Sobolev inequalities

We have now established both Poincaré and Sobolev-type inequalities for the Cauchy distributions. More generally, suppose that a measure μ satisfies

$$Var_\mu(g) \leq \int |\nabla g|^2 w^2 d\mu \tag{12.1}$$

$$Ent_\mu(g) \leq 2\int |\nabla g|^2 w^2 d\mu \tag{12.2}$$

with some weight $w : \mathbb{R}^n \to [0, \infty)$. The following theorem tells us what information we can extract from those about the moments of Lipschitz functions.

Theorem 12.1. *Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ has Lipschitz norm $\|f\|_{Lip} \leq 1$ and $\int f d\mu = 0$. If $\|w\|_p < \infty$, $p \geq 2$, then under the Poincaré inequality* (12.1)*, we have*

$$\|f\|_p \leq \frac{p}{\sqrt{2}}\|w\|_p$$

Moreover, under the logarithmic Sobolev inequality (12.2)*, one has*

$$\|f\|_p \leq \sqrt{p-1}\|w\|_p$$

Example 12.2. *If $w \equiv c \equiv const$, then $\|f\|_p \leq \tilde{C}p \Rightarrow \int e^{\lambda|f|} < \infty$*

Appendix A: Coarea Formula Suppose that $m > n$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function. The co-area formula states that for every real valued $g \in L^1(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} g(x) J_n f(x) d_n x = \int_{\mathbb{R}^n} \int_{f^{-1}\{y\}} g(x)\ d\mathcal{H}^{m-n}(x) d_n y$$

where $J_n f$ is the Jacobian of f, $d_n x$ denotes n-dimensional Lebesgue measure and $\mathcal{H}^n$ n-dimensional Hausdorff measure (for a definition, see [Fed69]; in $\mathbb{R}^n$ this can be thought of as simply Lebesgue measure).

The above formula is particularly useful when f takes values in $\mathbb{R}$, in which case the Jacobian is just length of gradient and the formula reads

$$\int_{\mathbb{R}^m} g(x) |\nabla f(x)| d_n x = \int_{-\infty}^{\infty} \int_{f^{-1}\{y\}} g(x)\ d\mathcal{H}^{m-1}(x) dy$$

This is in fact equivalent to the (seemingly weaker) formula with $g \equiv 1$

$$\int_{\mathbb{R}^m} |\nabla f(x)| d_n x = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(f^{-1}(y)) dy$$

where in the RHS we are integrating over the measure of the level sets $\{y = f(x)\}$ of f.

Starting from the last formula with g being the characteristic function of a Lebesgue measurable set A and then approximating any integrable g by simple functions one can prove the formula for general g.

Appendix B: Sobolev and Poincaré inequalities

We say that a measure μ satisfies a Logarithmic Sobolev inequality if for all functions such that the RHS is finite

$$Ent_\mu(f^2) = \int \left(f^2 \log f^2\right) d\mu - \left(\int f^2 d\mu\right) \log\left(\int f^2 d\mu\right) \leq C \int |\nabla f|^2 d\mu$$

with some constant C independent of f. The left hand side is always non-negative, as can be seen by Jensen's inequality applied to the convex function $x \log x$.

The measure μ is said to satisfy Poincaré or Spectral Gap inequality if

$$Var_\mu(f) = \int \left(f - \int f d\mu\right)^2 d\mu \leq D \int |\nabla f|^2 d\mu$$

for all f such that the RHS is finite, with some constant D independent of the f.

If μ satisfies Logarithmic Sobolev inequality, then it must satisfy a Poincaré inequality. This follows by the observation that for the function $\tilde{f} = 1 + \varepsilon f$ we have

$$Ent_\mu(\tilde{f}^2) = 2\varepsilon^2 Var_\mu(f) + O(\varepsilon^3)$$

something which can be seen by a Taylor expansion of the function $(1+\varepsilon)\log(1+\varepsilon)$ about $\varepsilon = 0$. Moreover, by definition of $\tilde{f}$ it follows that $\nabla \tilde{f} = \varepsilon \nabla f$ and letting $\varepsilon \to 0$ in the Logarithmic Sobolev inequality one obtains the Poincaré inequality.

An important fact about the Logarithmic Sobolev inequality is that it is equivalent to the Hypercontractivity property of the μ-invariant semigroup corresponding to the Energy Form $\int |\nabla f|^2 d\mu$ on the RHS of the inequality ([Gro75]). It is moreover known that if μ satisfies such an inequality, then it has Gaussian concentration, i.e. $\exists \alpha : \int e^{\alpha |x|^2} d\mu < \infty$ (by the so-called *Herbst argument*). On the other hand, the Poincaré inequality implies exponential concentration.

For a detailed introduction the reader is referred to [GZ03], [SC02].

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