Journal of Mathematical Sciences, Vol. 179, No. 1, November, 2011

THE BRUNN-MINKOWSKI INEQUALITY IN SPACES WITH BITRIANGULAR LAWS OF COMPOSITION

S. G. Bobkov

University of Minnesota
127 Vincent Hall, Minneapolis, MN, 55455, USA
bobko001@umn.edu

UDC 517.9

Dedicated to N. V. Krylov on the occasion of his 70th birthday

The Brunn–Minkowski inequality and Prékopa–Leindler's theorem are considered with respect to bitriangular laws of composition on Euclidean spaces $\mathbf{R}^n$*. The result is illustrated by an example of the Heisenberg group* $\mathbf{H}^n$*. Bibliography*: 11 *titles.*

1 The Brunn–Minkowski Inequality

The classical Brunn–Minkowski inequality asserts that for all nonempty Borel measurable sets A and B in $\mathbf{R}^n$,

$$|A + B|^{1/n} \geqslant |A|^{1/n} + |B|^{1/n}, \tag{1.1}$$

where $|A|$ stands for the n-dimensional volume (the Lebesgue measure) and $A + B = \{x + y : x \in A,\ y \in B\}$ is the Minkowski sum of the two sets (cf., for example, [1]).

In this note, we show that (1.1) has a natural generalization to certain summation-like operations. We call a binary operation or composition

$$(x, y) = (x_1, \dots, x_n; y_1, \dots, y_n) \to x \oplus y \in \mathbf{R}^n \qquad (x, y \in \mathbf{R}^n)$$

bitriangular if the coordinates of the "sum" have the form

$$(x \oplus y)_k = x_k + y_k + \varphi_{k-1}(x_1, \dots, x_{k-1}; y_1, \dots, y_{k-1}) \tag{1.2}$$

for some functions $\varphi_k : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}$, $k = 1, \dots, n-1$ (with the convention that $\varphi_0 = 0$). We assume that these functions are continuous, which insures that the Minkowski "sum" $A \oplus B = \{x \oplus y : x \in A,\ y \in B\}$ is Lebesgue measurable for all Borel measurable A and B.

Note that such operations do not need be commutative, so the sets $A \oplus B$ and $B \oplus A$ may be different in general.

Theorem 1.1. *Given a bitriangular operation in* $\mathbf{R}^n$*, for all nonempty Borel sets* A *and* B *in* $\mathbf{R}^n$

$$|A \oplus B|^{1/n} \geqslant |A|^{1/n} + |B|^{1/n}. \tag{1.3}$$

Translated from *Problems in Mathematical Analysis* **61**, October 2011, pp. 5–8.

Although this kind of generalization might be of interest in itself, it is motivated in particular by the example of the Heisenberg group $\mathbf{H}^n$. This group is represented as the space $\mathbf{C}^n \times \mathbf{R} \sim \mathbf{R}^{2n+1}$ with the (noncommutative) multiplication

$$[z_1, \dots, z_n, t].[z_1', \dots, z_n', t'] = [z_1 + z_1', \dots, z_n + z_n', t + t' + \varphi],$$

where z_i, z_i' are complex, t, t' are real, and

$$\varphi = 2 \sum_{i=1}^{n} \mathrm{Im}\big(z_i \overline{z_i'}\big).$$

Hence we deal here with the bitriangular law of composition in $\mathbf{R}^{2n+1}$ of the form (1.2) with $\varphi_0 = \dots = \varphi_{2n-1} = 0$ and $\varphi_{2n} = \varphi$.

Corollary 1.2. *For all nonempty Borel sets A and B in the Heisenberg group $\mathbf{H}^n$*

$$|A.B|^{1/(2n+1)} \geqslant |A|^{1/(2n+1)} + |B|^{1/(2n+1)}. \tag{1.4}$$

Note that in analogy with the classical isoperimetric inequality, an isoperimetric problem in $\mathbf{H}^n$ with respect to the Carnot–Carathéodory distance (or, an equivalent gauge distance) suggests to consider a Brunn–Minkowski-type inequality

$$|A.B|^{1/Q} \geqslant |A|^{1/Q} + |B|^{1/Q} \tag{1.5}$$

for the homogeneous dimension $Q = 2n + 2$ of $\mathbf{H}^n$ (which is stronger in comparison with (1.4)). However, as shown by Monti [2], the inequality (1.5) cannot be true already for $n = 1$ (cf. also [3]). Nevertheless, (1.4) does hold, although it does not seem to lead to an isoperimetric inequality in $\mathbf{H}^n$ relating the size to the perimeter of sets.

Below (Section 2), first we recall a simple transportation argument leading to the Brunn–Minkowski inequality (1.3) in the case of convex sets A and B. The general case, together with one functional form of (1.3), is considered separately in Section 3.

2 Triangular Maps and Transportation Argument

Transference plans are a stardard tool to prove various geometric and analytic inequalities. In 1950's, Knothe [4] proposed to use triangular mappings to reach some generalizations of the Brunn–Minkowski inequality. Let us recall and adapt the standard transportation argument to the more general scheme of bitriangular maps.

A map $T = (T_1, \dots, T_n) : G \to \mathbf{R}^n$ defined in some region $G \subset \mathbf{R}^n$ is called *triangular* if for any $k = 1, \dots, n$ the kth coordinate of the map,

$$T_k = T_k(x_1, \dots, x_k), \quad x = (x_1, \dots, x_n) \in G,$$

depends on the first k coordinates, only. Such maps can be constructed to transport any (regularly behaving) probability measure to any other one, and we refer to [5, 6] for discussions and detailed treatment. To avoid unessential technical moments, let us restrict ourselves to open, bounded convex sets A and B in $\mathbf{R}^n$. Then there exists a unique continuous triangular map $T : A \to B$, which pushes forward the normalized Lebesgue measure on A to the normalized Lebesgue measure on B, and such that every coordinate function T_k is increasing with respect to the variable x_k (while the remaining variables are fixed). Moreover, every T_k will be continuously differentiable with respect to x_k. In consequence, the map T satisfies the identity

$$\prod_{k=1}^{n} \frac{\partial T_k(x)}{\partial x_k} = \frac{|B|}{|A|}, \qquad x \in A.$$

A new map $S(x) = x \oplus T(x)$ is also triangular and has coordinate functions

$$S_k(x) = x_k + T_k(x) + \varphi_{k-1}(x_1, \dots, x_{k-1}; T_1(x), \dots, T_{k-1}(x))$$

which are continuously differentiable and increasing with respect to x_k's. Hence, by the change of the variable formula (cf. [5, Lemma 6.4] justifying this step),

$$\begin{aligned} |S(A)| &= \int_A \prod_{k=1}^{n} \frac{\partial S_k(x)}{\partial x_k}\, dx = \int_A \prod_{k=1}^{n} \Big(1 + \frac{\partial T_k(x)}{\partial x_k}\Big)\, dx \geqslant \int_A \bigg[1 + \Big(\prod_{k=1}^{n} \frac{\partial T_k(x)}{\partial x_k}\Big)^{1/n}\bigg]^n dx \\ &= \int_A \bigg[1 + \Big(\frac{|B|}{|A|}\Big)^{1/n}\bigg]^n dx = \Big[|A|^{1/n} + |B|^{1/n}\Big]^n. \end{aligned}$$

Here, we made use of an elementary inequality for nonnegative reals

$$\Big(\prod_{k=1}^{n} (a_k + b_k)\Big)^{1/n} \geqslant \Big(\prod_{k=1}^{n} a_k\Big)^{1/n} + \Big(\prod_{k=1}^{n} b_k\Big)^{1/n}.$$

(Note the latter is a particular case of the Brunn–Minkowski inequality (1.1) for parallepipeds A and B with sides a_k and b_k.) Now, since $S(A) \subset A \oplus B$, we arrive at (1.3) for general (bounded) convex bodies.

3 Generalization of the Prékopa–Leindler Theorem

Theorem 1.1 may be generalized from sets to functions similarly to the classical Prékopa–Leindler theorem.

Theorem 3.1. *Given $t, s > 0$, $t + s = 1$, let $u, v, w : \mathbf{R}^n \to [0, +\infty)$ be measurable functions satisfying*

$$w(tx \oplus sy) \geqslant u(x)^t\, v(y)^s, \qquad x, y \in \mathbf{R}^n. \tag{3.1}$$

Then

$$\int w \geqslant \Big(\int u\Big)^t \Big(\int v\Big)^s. \tag{3.2}$$

In this statement, the underlying functions φ_k, defining the bitringular operation, are allowed to depend on the parameters t, s (and the case $\varphi_k = 0$ returns us to the Prékopa–Leindler theorem).

In particular, the indicator functions $u = 1_A$, $v = 1_B$, $w = 1_{tA \oplus sB}$ satisfy the hypothesis (3.1), and then (3.2) yields

$$|tA \oplus sB| \ \geqslant \ |A|^t \, |B|^s. \tag{3.3}$$

This is a "log-concave" variant of the Brunn–Minkowski inequality. Although formally (3.3) is weaker in comparison with (1.3), the latter may be obtained from (3.3) by applying it to the sets $A' = \frac{1}{\alpha} A$, $B' = \frac{1}{\beta} B$ with $t = \alpha/(\alpha+\beta)$, $s = \beta/(\alpha+\beta)$, where $\alpha = |A|^{1/n}$, $\beta = |B|^{1/n}$. In that case, Theorem 3.1 should be used with the bitriangular operation corresponding to

$$\overline{\varphi}_k(x;y) = \frac{1}{\alpha+\beta}\, \varphi_k((\alpha+\beta)\,x; (\alpha+\beta)\,y).$$

Then

$$tA' \oplus sB' \ = \ \bigg(\frac{1}{\alpha+\beta}\, A \oplus \frac{1}{\alpha+\beta}\, B\bigg) \ = \ \frac{1}{\alpha+\beta}\, (A \oplus B),$$

where on the left-hand side and in the mid the binary operation is applied with the new functions $\overline{\varphi}_k$'s, while on the right-hand side, with original φ_k's.

Hence, since $|A'| = |B'| = 1$, the inequality (3.3) with these sets and parameters turns into (1.3).

Proof of Theorem 3.1. In dimension $n = 1$, Theorem 3.1 represents the usual one-dimensional Prékopa–Leindler theorem [7]–[9]. A simple inductive proof of the multi-dimensional case in the Prékopa–Leindle theorem is described in several books (cf., for example, [10, 11]), and here we follow a standard argument with necessary modifications.

Thus, let $n \geqslant 2$. Assume that Theorem 3.1 is true for dimensions less than n. For fixed vectors $a = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $b = (y_1, \dots, y_{n-1}) \in \mathbf{R}^{n-1}$ consider the three functions on the real line

$$\overline{u}(x_n) = u(a, x_n), \quad \overline{v}(y_n) = v(b, y_n),$$

and

$$\overline{w}(z_n) = w(ta \oplus sb,\, z_n + \varphi_{n-1}(ta, sb)),$$

where $ta \oplus sb$ is defined in $\mathbf{R}^{n-1}$ in the usual way for the collection $\varphi_0, \dots, \varphi_{n-2}$.

Note that, by the very definition of the bitriangular operation, we always have the identity

$$t(a, x_n) \oplus s(b, y_n) = (ta \oplus sb,\, tx_n + sy_n + \varphi_{n-1}(ta, sb)).$$

Hence for all real x_n, y_n

$$\overline{w}(tx_n + sy_n) = w(t(a, x_n) \oplus s(b, y_n)) \geqslant u(a, x_n)^t \, v(b, y_n)^s = \overline{u}(x_n)^t \, \overline{v}(y_n)^s,$$

where we have applied the hypothesis (3.1).

Thus, the triple $(\overline{u}, \overline{v}, \overline{w})$ also satisfies the hypothesis (3.1) – in the one-dimensional case. Using the equality

$$\int_{-\infty}^{+\infty} \overline{w}(z_n)\, dz_n = \int_{-\infty}^{+\infty} w(ta \oplus sb,\, z_n)\, dz_n,$$

we therefore obtain

$$\int_{-\infty}^{+\infty} w(ta \oplus sb,\, z_n)\, dz_n \ \geqslant \ \bigg(\int_{-\infty}^{+\infty} u(a, x_n)\, dx_n\bigg)^t \bigg(\int_{-\infty}^{+\infty} v(b, y_n)\, dy_n\bigg)^s.$$

But this means that the new three functions on $\mathbf{R}^{n-1}$

$$f(a) = \int_{-\infty}^{+\infty} u(a, x_n)\, dx_n, \quad g(b) = \int_{-\infty}^{+\infty} v(b, y_n)\, dy_n, \quad h(c) = \int_{-\infty}^{+\infty} w(c, z_n)\, dz_n$$

satisfy (3.1) in dimension $n-1$. It remains to apply the induction hypothesis to the triple (f, g, h), and the desired conclusion (3.2) follows by the Fubini theorem.

Theeorem 3.1 and thus Theorem 1.1 are proved in full generality.

Acknowledgments

The work is supported in part by DMS-1106530.

The author would like to thank Boguslaw Zegarlinski for discussions of the Heisenberg group.

References

1. Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer, Berlin etc. (1988).
2. R. Monti, "Brunn–Minkowski and isoperimetric inequality in the Heisenberg group," *Ann. Acad. Sci. Fenn. Math.* **28**, No. 1, 99–109 (2003).
3. G. P. Leonardi and S. Masnou, "On the isoperimetric problem in the Heisenberg group $\mathbf{H}^n$," *Ann. Mat. Pura Appl. (4)* **184**, No. 4, 533–553 (2005).
4. H. Knothe, "Contributions to the theory of convex bodies," *Michigan Math. J.* **4**, 39–52 (1957).
5. S. G. Bobkov, "Large deviations via transference plans," *Adv. Math. Res.* **2**, 151–175 (2003).
6. S. G. Bobkov, "Large deviations and isoperimetry over convex probability measures with heavy tails," *Electron. J. Probab.* **12**, 1072–1100 (2007).
7. A. Prékopa, "Logarithmic concave measures with applications to stochastic programming," *Acta Sci. Math. (Szeged)* **32**, 301–316 (1971).
8. A. Prékopa, "On logarithmic concave measures and functions," *Acta Sci. Math. (Szeged)* **34**, 335–343 (1973).
9. L. Leindler, "On a certain converse of Hölder's inequality. II," *Acta Sci. Math. (Szeged)* **33**, No. 3-4, 217–223 (1972).
10. G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry,* Cambridge Univ. Press, Cambridge (1989).
11. M. Ledoux, *The Concentration of Measure Phenomenon*, Am. Math. Soc., Providence, RI (2001).

Submitted on September 12, 2011