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Non-Uniform Bounds in Local Limit Theorems in Case of Fractional Moments. I

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Abstract—Edgeworth-type expansions for convolutions of probability densities and powers of the characteristic functions with non-uniform error terms are established for i.i.d. random variables with finite (fractional) moments of order $s \geq 2$, where s may be noninteger.

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1. INTRODUCTION

Let $(X_n)_{n \geq 1}$ be independent identically distributed random variables with $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 = 1$.

If X_1 has finite moments of all orders, and if the densities ρ_n of the normalized sums $S_n = (X_1 + \dots + X_n)/\sqrt{n}$ exist, they admit a formal Edgeworth-type expansion in powers of $1/\sqrt{n}$

$$\rho_n(x) = \varphi(x) + \sum_{k=1}^{\infty} q_k(x)\, n^{-k/2}. \tag{1.1}$$

Here, $\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ denotes the density of the standard normal law,

$$q_k(x) \;=\; \varphi(x) \sum H_{k+2j}(x)\, \frac{1}{p_1! \dots p_k!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \dots \left(\frac{\gamma_{k+2}}{(k+2)!}\right)^{p_k}, \tag{1.2}$$

where $H_k(x)$ are the Chebyshev–Hermite polynomials and

$$\gamma_k = i^{-k} \frac{d^k}{dt^k}\, \log \mathbf{E}\, e^{itX_1}\Big|_{t=0}$$

denote the cumulants of the underlying distribution. The summation in (1.2) runs over all non-negative integer solutions $(p_1, \dots, p_k)$ to the equation $p_1 + 2p_2 + \dots + kp_k = k$ with $j = p_1 + \dots + p_k$.

A precise asymptotic statement about the formal series (1.1) requires that some moment $\mathbf{E}|X_1|^s$ of order $s \geq 2$ is finite (while the moments of higher orders may be infinite). In this case, the kth order cumulants are well defined for the values $k = 1, \dots, m$, and respectively, the functions q_k are defined for $k \leq m-2$, where $m = [s]$ is the integer part of s. Therefore one needs to evaluate the error of the approximation of ρ_n by the following partial sums of the series (1.1),

$$\varphi_m(x) = \varphi(x) + \sum_{k=1}^{m-2} q_k(x)\, n^{-k/2}, \qquad m = [s].$$

One of the aims of this paper is to prove the following:

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Theorem 1.1. *Assume that* $\mathbf{E}\,|X_1|^s < +\infty$ *for some* $s \geq 2$*. Suppose* S_{n_0} *has a bounded density* ρ_{n_0} *for some* n_0*. Then for all* n *large enough,* S_n *have continuous densities* ρ_n *satisfying, as* $n \to \infty$*,*

$$(1 + |x|^m)\,\big(\rho_n(x) - \varphi_m(x)\big) = o\big(n^{-(s-2)/2}\big) \tag{1.3}$$

uniformly for all x*. Moreover,*

$$(1 + |x|^s)\big(\rho_n(x) - \varphi_m(x)\big) = o\big(n^{-(s-2)/2}\big) + \big(1 + |x|^{s-m}\big)\,\big(O(n^{-(m-1)/2}) + o(n^{-(s-2)})\big). \tag{1.4}$$

In fact, the implied sequence and constants in the error terms hold uniformly over the class of all densities with precribed moment tail function $t \to \mathbf{E}\,|X_1|^s\, 1_{\{|X_1|>t\}}$, parameter n_0 and a bound on the density ρ_{n_0}.

For $|x|$ of order 1, or when $s = m$ is integer, both relations are equivalent. But for large values of $|x|$ and $s > m$, the assertion (1.4) gives an improvement over (1.3), which is essential in some applications.

If $2 \leq s < 3$, (1.4) becomes

$$(1 + |x|^s)\big(\rho_n(x) - \varphi(x)\big) = o\big(n^{-(s-2)/2}\big) + (1 + |x|^{s-2})\, o\big(n^{-(s-2)}\big).$$

In particular, for the smallest value $s = 2$, this contains the Gnedenko local limit theorem $\sup_x |\rho_n(x) - \varphi(x)| \to 0$ as $n \to \infty$.

If $s = m$ is integer and $m \geq 3$, Theorem 1.1 is well known; (1.3)–(1.4) then simplify to

$$(1 + |x|^m)\big(\rho_n(x) - \varphi_m(x)\big) = o\big(n^{-(m-2)/2}\big). \tag{1.5}$$

In this formulation the result is due to Petrov [4] (cf. also Petrov [5], p. 211, or Bhattacharya and Ranga Rao [1], p. 192). Without the term $1 + |x|^m$, (1.5) can be found in the classical book [3]; this weaker variant goes back to the results by Gnedenko [2] and an earlier work by Cramér (who used, according to [3], additional assumptions on the underlying density).

Thus, Theorem 1.1 extends these well-known results to the case, where s is not necessarily integer. The range $2 \leq s < 3$ is of interest as well. Our interest in these somewhat technical extensions, especially (1.4), was motivated by open questions as to the actual rate of convergence in the so-called entropic central limit theorem. Here relation (1.4) led to an unexpected behavior of the error in the approximation of the entropy of sums of independent summands when s increases from 2 to 4. (This error stabilizes at $s = 4$, in contrast to the usual Berry–Esseen-type theorem for distribution functions, where stabilization of errors starts at $s = 3$). In the entropic central limit theorem the classical non-uniform bound (1.5) is not precise enough to derive upper bounds for errors.

Note that the assumption of boundedness of ρ_n in Theorem 1.1 (for some n or, equivalently, for all large n) is necessary for conclusions such as (1.3)–(1.5). It is equivalent to the property that the characteristic function $v(t) = \mathbf{E}\, e^{itX_1}$ is integrable with some power $\nu \geq 1$, i.e.,

$$\int_{-\infty}^{+\infty} |v(t)|^\nu \, dt < +\infty. \tag{1.6}$$

In this case ρ_n are bounded for all $n \geq 2\nu$. The condition (1.6) is sometimes called "smoothness"; it appears naturally in many problems of the asymptotic behaviour of the densities (see, e.g., [7] for detailed discussion).

Nevertheless, this condition may be removed at all, if we require that (1.3)–(1.4) hold true for slightly modified densities rather than for ρ_n.

Theorem 1.2. *Let* $\mathbf{E}\,|X_1|^s < +\infty$ *for some* $s \geq 2$*. Let* c *denote an arbitrary number with* $0 < c < 1$*. Suppose that for* n *large enough* S_n *have absolutely continuous distributions with densities* ρ_n*. Then, for some probability densities* $\widetilde{\rho}_n$*,*

(a) *Relations* (1.3)–(1.4) *hold true for* $\widetilde{\rho}_n$ *in place of* ρ_n;

(b) $\int_{-\infty}^{+\infty} |\widetilde{\rho}_n(x) - \rho_n(x)|\, dx < c^n$ *for all* n *large enough*;

(c) $\widetilde{\rho}_n(x) = \rho_n(x)$ *almost everywhere, if* ρ_n *is bounded* (*a.e.*).

It seems that Theorem 1.2 has not been stated in the literature, even when s is integer. Here, the property c) is added to include Theorem 1.1 in Theorem 1.2 as a particular case.

It turns out that the statement of Theorem 1.2 is more appropriate for a number of applications. For example, it implies that $\rho_n - \varphi_m \to 0$ in the mean, i.e., there is convergence in total variation norm for the corresponding distributions with rate

$$\int_{-\infty}^{+\infty} |\rho_n(x) - \varphi_m(x)|\, dx = o\big(n^{-(s-2)/2}\big). \tag{1.7}$$

For $s = 2$ and $\varphi_2(x) = \varphi(x)$, this statement corresponds to a theorem of Prokhorov [6], while for $s = 3$ and $\varphi_3(x) = \varphi(x)\big(1 + \alpha_3\, \frac{x^3-3x}{6\sqrt{n}}\big)$, to the result of Sirazhdinov and Mamatov [8] (they also covered the case $2 < s < 3$ with O in place of o in (1.7)). If $s \geq 3$ is integer, (1.7) is mentioned in [5] for a more general L^p-convergence, however, under the assumption that the densities ρ_n are bounded.

Theorem 1.2 allows us to study non-uniform convergence in (1.3)–(1.4) as well when excluding exceptional "small" sets (via additional assumptions of entropic type).

The proofs of Theorems 1.1 and 1.2, which are formally based on the application of the inverse Fourier transforms, involve operators, namely, Liouville fractional integrals and derivatives. For this step, we analyse the decay of the Fourier transform for special classes of finite measures with finite fractional moments. (Apparently standard truncation methods are much to density-sensitive and do not provide the required asymptotics.) An essential part of the argument is devoted to the routine analysis of powers of the characteristic functions and more general Fourier transforms in Edgeworth-type expansions. For this step the requirement (1.6) is irrelevant.

In order to describe one of the main intermediate results, which is, as we believe, of an independent interest, let us start with a random variable X, such that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and $\mathbf{E}\,|X|^s < +\infty$, for some $s \geq 2$. Introduce the characteristic function $v(t) = \mathbf{E}\, e^{itX}$, $t \in \mathbf{R}$.

If $m = [s]$, the normalized powers $v_n(t) = v(\frac{t}{\sqrt{n}})^n$, that is, the characteristic functions of S_n, can be approximated by the functions

$$u_m(t) = e^{-t^2/2}\left(1 + \sum_{k=1}^{m-2} P_k(it)\, n^{-k/2}\right).$$

Here we use the classical polynomials

$$P_k(t) \;=\; \sum_{p_1+2p_2+\cdots+kp_k=k} \frac{1}{p_1!\dots p_k!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \cdots \left(\frac{\gamma_{k+2}}{(k+2)!}\right)^{p_k} t^{k+2(p_1+\cdots+p_k)}$$

of degree $3k$, where the summation is performed as in (1.2). Another way to introduce these polynomials is to require that every $q_k(x)$ has the Fourier transform $e^{-t^2/2}P_k(it)$, so that $u_m(t)$ appears as the Fourier transform of $\varphi_m(x)$.

The following statement is standard: in the interval $|t| \leq n^{1/6}$

$$|v_n(t) - u_m(t)| \;\leq\; \frac{\varepsilon_n}{n^{(m-2)/2}}\left(|t|^{m'} + |t|^{m''}\right) e^{-t^2/2},$$

where ε_n do not depend on t and satisfy $\varepsilon_n \to 0$ as $n \to \infty$ (with certain powers m' and m''). Similar bounds also hold for the derivatives of orders $p = 1, \dots, m$, namely

$$\left|\frac{d^p}{dt^p}\, v_n(t) - \frac{d^p}{dt^p}\, u_m(t)\right| \;\leq\; \frac{\varepsilon_n}{n^{(m-2)/2}}\left(|t|^{m'} + |t|^{m''}\right) e^{-t^2/2}.$$

This bound is proved in Petrov [4], cf. also [5], pp. 209–211 (for $m \geq 3$ and $|t| \leq n^{1/7}$). We refine this result with general values of $s \geq 2$ by proving that

$$\left|\frac{d^p}{dt^p}\, v_n(t) - \frac{d^p}{dt^p}\, u_m(t)\right| \leq \frac{\varepsilon_n}{n^{(s-2)/2}}\left(|t|^{m'} + |t|^{m''}\right) e^{-t^2/2}. \tag{1.8}$$

However, the error term in this approximation is still not sufficiently small for our applications, and we have to look for other related representations of v_n. By analogy with u_m, introduce

$$e_m(t) = e^{-t^2/2}\left(1 + \sum_{k=1}^{m-2} P_k(it)\right).$$

Theorem 1.3. *Let* $\mathbf{E}\,|X|^s < +\infty$ $(s \geq 2)$. *For all* $p = 0, 1\dots, m$ *and all* $|t| \leq cn^{1/6}$,

$$\frac{d^p}{dt^p}\,(v_n(t) - u_m(t)) = n\,\frac{d^p}{dt^p}\left[\left(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right)e^{-t^2/2}\right] + r_n \tag{1.9}$$

with

$$|r_n| \leq \left(1 + |t|^{4m^2}\right)e^{-t^2/2}\left(\frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}}\right). \tag{1.10}$$

Here C, c and ε_n are positive constants depending on s and the distribution of X such that $\varepsilon_n \to 0$ as $n \to \infty$.

Thus, the closeness of e_m to v near zero determines the rate of approximation of v_n's by the functions u_m's (which have a different formal nature). This representation will be of use in the proof of Theorems 1.1–1.2, since, as we will see, the Liouville integrals may be applied to give a pointwise bound on the (inverse) Fourier transforms within the class of the functions of the form $\hat{V}(\frac{t}{\sqrt{n}})\,e^{-t^2/2}$, such as in (1.9).

Note that for $s \geq 3$ the expression in the last brackets of (1.10) is dominated by $Cn^{-(m-1)/2}$, while in the range $2 \leq s < 3$ the second summand $\varepsilon_n n^{-(s-2)}$ dominates the first one. In any case, with respect to the growing parameter n, the bound (1.9) is sharper than the one given in (1.8). This observation explains the improvement of (1.4) compared with relation (1.3).

We also remark that Theorem 1.3 holds for a more general class of functions, including Fourier–Stieltjes transforms $v(t)$ of finite (signed) measures with finite sth moment, such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. For example, the approximating functions u_m and e_m are not positive definite, but belong to this class.

The exposition of this paper, which is divided in two parts, is based on chapters of auxiliary results organized in accordance with the following table.

Contents for Part I

Contents for Part II

2. DIFFERENTIABILITY WITH IMPROVED REMAINDER TERMS

For our purposes, we use the following terminology.

Definition 2.1. Let a complex-valued function $y = y(t)$ be defined in some interval $a < t < b$, and let $s \geq 0$. We say that y is s-times differentiable if it has continuous derivatives up to order $m = [s]$ in (a, b) and for any $t_0 \in (a, b)$, as $t \to t_0$,

$$y^{(m)}(t) = y^{(m)}(t_0) + o(|t - t_0|^{s-m}). \tag{2.1}$$

The case $s = 0$ corresponds to continuity, while the case of a positive integer $s = m$ to the property of just having continuous derivatives up to order m.

The following obvious characterization will be an important tool in the derivation of the Edgeworth-type expansions for charactersitic functions. It is obtained from (2.1) by the repeated integration over the variable t near t_0.

Proposition 2.2. *Let y have continuous derivatives of order up to $m = [s]$ in (a, b). The function y is s-times differentiable on (a, b) if and only if for any point $t_0 \in (a, b)$ and all $p = 0, \dots, m$, as $t \to t_0$,*

$$\frac{d^p}{dt^p}\, y(t) = \frac{d^p}{dt^p} \sum_{k=0}^{m} \frac{y^{(k)}(t_0)}{k!}\, (t - t_0)^k + o(|t - t_0|^{s-p}). \tag{2.2}$$

One can also provide quantitative estimates on the remainder term in (2.2), if we start with

$$|y^{(m)}(t) - y^{(m)}(t_0)| \leq |t - t_0|^{s-m} \varepsilon(|t - t_0|),$$

where $\varepsilon = \varepsilon(w)$ is a nondecreasing function in $w \geq 0$ such that $\varepsilon(w) \to 0$ as $w \to 0$. Then

$$\left| \frac{d^p}{dt^p}\, y(t) - \frac{d^p}{dt^p} \sum_{k=0}^{m} \frac{y^{(k)}(t_0)}{k!}\, (t - t_0)^k \right| \leq |t - t_0|^{s-p}\, \varepsilon(|t - t_0|)$$

for any $p = 0, \dots, m$.

By the chain rule given below as Lemma 2.4, we have the following:

Proposition 2.3. *If y is s-times differentiable on (a, b), $s \geq 0$, and $z = z(y)$ is analytic in some domain containing all values $y(t)$, then the superposition $z(y(t))$ is also s-times differentiable on (a, b).*

Lemma 2.4. *Under the conditions of Proposition 2.3, $z(y(t))$ has derivatives up to order $m = [s]$ on (a, b), given by*

$$\frac{d^p}{dt^p}\, z(y(t)) = p! \sum \frac{d^{k_1 + \dots + k_p}\, z(y)}{dy^{k_1 + \dots + k_p}} \bigg|_{y = y(t)} \prod_{r=1}^{p} \frac{1}{k_r!} \left(\frac{1}{r!}\, \frac{d^r y(t)}{dt^r} \right)^{k_r}, \tag{2.3}$$

for all $p = 1, \dots, m$, where the summation is performed over all nonnegative integer solutions $(k_1, \dots, k_p)$ to the equation $k_1 + 2k_2 + \dots + pk_p = p$.

Proof of Proposition 2.3. By definition, for all $t_0 \in (a, b)$ and $r = 1, \dots, m$,

$$\frac{d^r y(t)}{dt^r} = c_r + o(|t - t_0|^{s-m}) \quad \text{as } t \to t_0,$$

where $c_r = y^{(r)}(t_0)$. Raising these equalities to the k_r-powers and then multiplying them, we get a similar representation

$$\prod_{r=1}^{m} \frac{1}{k_r!} \left(\frac{1}{r!}\, \frac{d^r y(t)}{dt^r} \right)^{k_r} = c_k + o(|t - t_0|^{s-m})$$

with some constants c_k depending on the m-tuples $k = (k_1, \dots, k_m)$. In addition, putting $j = k_1 + \dots + k_m$, we have

$$\begin{aligned} z^{(j)}(y(t)) &= z^{(j)}(y(t_0)) + O(|y(t) - y(t_0)|) \\ &= z^{(j)}(y(t_0)) + o(|t - t_0|^{s-m}). \end{aligned}$$

Inserting these relations in (2.3) with $p = m$ we obtain that $[z(y(t))]^{(m)} = c + o(|t - t_0|^{s-m})$ with some constant c.

In addition, the right-hand side of (2.3) represents a continuous function in t, so necessarily $c = [z(y)]^{(m)}(t_0)$. This means that (2.1) is fulfilled, and Proposition 2.3 is proved. □

3. DIFFERENTIABILITY OF FOURIER–STIELTJES TRANSFORMS

A large variety of examples of s-times differentiable functions appear as Fourier–Stieltjes transforms of finite measures on the real line with finite absolute sth moment.

Proposition 3.1. *Let X be a random variable with characteristic function $v(t) = \mathbf{E}\, e^{itX}$. If $\mathbf{E}\,|X|^s < +\infty$, $s \geq 0$, then v is s-times differentiable on the real line. Moreover, its $m = [s]$ derivatives are representable, as $t \to 0$, by*

$$v^{(p)}(t) = \sum_{k=0}^{m-p} \mathbf{E}\,(iX)^{p+k}\, \frac{t^k}{k!} + o(|t|^{s-p}), \qquad p = 0, \dots, m. \tag{3.1}$$

One can state a similar proposition for more general Fourier–Stieltjes transforms

$$v(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x),$$

where F is a function of bounded variation on the real line such that $\int |x|^s\, d\,|F|(x) < +\infty$ (where $|F|$ denotes the variation of F viewed as a positive finite measure). On the other hand, such a more general statement may be obtained from Proposition 3.1 as well. Indeed, one can always represent F as a linear combination $c_1 F_1 - c_2 F_2$ of two orthogonal probability distributions (with $c_1, c_2 \geq 0$). Then $|F| = c_1 F_1 + c_2 F_2$, so $\int |x|^s\, dF_i(x) < +\infty$. Applying Proposition 3.1 to F_i, we obtain a similar statement for F.

Proof of Proposition 3.1. By the moment assumption, the characteristic function v has m continuous derivatives described by

$$v^{(p)}(t) = \mathbf{E}\,(iX)^p = \int_{-\infty}^{+\infty} e^{itx}\, (ix)^p\, dF(x), \qquad p = 0, 1, \dots, m,$$

where F is the distribution function of X. In particular,

$$v^{(m)}(t) = \int_{-\infty}^{+\infty} e^{itx}\, (ix)^m\, dF(x). \tag{3.2}$$

Hence the relation (3.1) would follow immediately from the formula (2.2) of Proposition 2.2, once it has been established that v is s-times differentiable. Namely, we need to see that, for any $t_0 \in \mathbf{R}$, as $t \to t_0$,

$$v^{(m)}(t) = v^{(m)}(t_0) + o(|t - t_0|^{s-m}). \tag{3.3}$$

The formula (3.2) is telling us that $v^{(m)}$ represents the Fourier–Stieltjes transform of the finite measure $F_m(dx) = (ix)^m\, dF(x)$ with $\int |x|^{s-m}\, d\,|F_m|(x) < +\infty$. Representing $F_m = c_1 G + c_2 H$ with distribution functions G and H, we can also represent $v^{(m)}$ as a linear combination of the two Fourier–Stieltjes transforms with $\int |x|^{s-m}\, d\,G(x) < +\infty$ and similarly for H. Hence, in order to show that v is s-times differentiable, it is enough to consider in (3.3) the case $m = 0$ and $0 \le s < 1$ only.

Thus Proposition 3.1 has been reduced to the case $\mathbf{E}\,|X|^s < +\infty$, $0 \le s < 1$, when one needs to show that

$$v(t) = v(t_0) + o(|t - t_0|^s).$$

Moreover, without loss of generality, it suffices to consider the point $t_0 = 0$ only, in which case we need to show the relation $v(t) = 1 + o(|t|^s)$. The case $s = 0$ is immediate, so let $s > 0$ and write

$$1 - v(t) = \int_{-\infty}^{+\infty} (1 - e^{itx})\, dF(x).$$

For definiteness, let $t > 0$. Since in general $|1 - e^{ix}| \le \min\{2, |x|\}$, $x \in \mathbf{R}$, we have

$$\begin{aligned} |1 - v(t)| &\le 2 \int_{|x| \ge 2/t} dF(x) + t \int_{|x| < 2/t} |x|\, dF(x) \\ &= 2 \int_{x \ge 2/t} dG(x) + t \int_{x < 2/t} x\, dG(x), \end{aligned}$$

where G is the distribution of $|X|$. By assumption,

$$\psi(x) = \mathbf{E}\,|X|^s\, 1_{\{|X| \ge x\}} = \int_x^{+\infty} y^s\, dG(y) \to 0 \quad \text{as } x \to +\infty.$$

We have $\psi(x) \ge x^s \int_x^{+\infty} dG(y)$, so $\int_x^{+\infty} dG(y) = o(x^{-s})$, that is, $\int_{x \ge 2/t} dG(x) = o(t^s)$ as $t \to 0$. Finally, by integration by parts,

$$\int_{x < 2/t} x\, dG(x) \le \int_0^{2/t} (1 - G(x))\, dx \le \int_0^{2/t} \frac{\psi(x)}{x^s}\, dx = t^{s-1} \int_0^2 \frac{\psi(y/t)}{y^s}\, dy.$$

Hence

$$t \int_{x < 2/t} x\, dG(x) \le t^s \int_0^2 \frac{\psi(y/t)}{y^s}\, dy.$$

But the last integral tends to zero, as $t \to 0$, by the Lebesgue dominated convergence theorem.

Proposition 3.1 is proved. □

4. CUMULANTS. THE FUNCTIONS $\psi_z(t) = \frac{1}{2}\, t^2 + \frac{1}{z^2}\, \log v(tz)$

If a complex-valued function v on the real line has m continuous derivatives and $v(0) \neq 0$, then $v(t) \neq 0$ in some interval $(-c, c)$. Moreover, the principal value of the logarithm $\log v(t)$ is well defined in that interval and represents a function, which has also m continuous derivatives. The corresponding derivatives at the origin (with a proper normalization),

$$\gamma_k = \frac{d^k}{i^k dt^k}\, \log v(t)\big|_{t=0}, \qquad k = 0, 1, \dots, m,$$

will be called the generalized cumulants or just cumulants, associated to v.

This terminology is standard, when v represents the characteristic function of a random variable (with m finite absolute moments). However, we shall have more general classes of functions.

Applying Propositions 2.2–2.3, we arrive at:

Proposition 4.1. *Let v be s-times differentiable on the real line, $s \geq 0$, not vanishing in some interval $(-c, c)$. Then, $\log v$ is s-times differentiable in $(-c, c)$. In particular, as $t \to 0$,*

$$\frac{d^p}{dt^p} \log v(t) = \frac{d^p}{dt^p} \sum_{k=0}^{m} \frac{\gamma_k}{k!} (it)^k + o(|t|^{s-p}), \qquad p = 0, 1, \ldots, m, \quad m = [s]. \tag{4.1}$$

Note that if v has $m+1$ continuous derivatives, then, by the usual Taylor's theorem, the remainder term in (4.1) can be sharpened, and we have

$$\frac{d^p}{dt^p} \log v(t) = \frac{d^p}{dt^p} \sum_{k=0}^{m} \frac{\gamma_k}{k!} (it)^k + O(|t|^{(m+1)-p}), \qquad p = 0, 1, \ldots, m. \tag{4.2}$$

If $v(t) = \mathbf{E}\, e^{itX}$ is the characteristic function of a random variable X, the assumptions of Proposition 4.1 are fulfilled as long as $\mathbf{E}\,|X|^s < +\infty$ (Proposition 3.1). Then γ_k are usual cumulants with $\gamma_0 = 0$. In the particular case $p = 0$, formula (4.1) takes the form

$$\log v(t) = \sum_{k=1}^{m} \frac{\gamma_k}{k!} (it)^k + o(|t|^s). \tag{4.3}$$

However, the case $p \geq 1$ in (4.1) cannot be deduced directly from (4.3).

Let us recall how to relate the cumulants to the moments $\alpha_k = \mathbf{E}X^k$. Applying Lemma 2.4 with $z(y) = \log y$, $y = v(t)$, at the point $t = 0$, one obtains a well-known identity

$$\gamma_p \; = \; p! \sum (-1)^{k_1 + \cdots + k_p - 1} \, (k_1 + \cdots + k_p - 1)! \prod_{r=1}^{p} \frac{1}{k_r!} \left(\frac{\alpha_r}{r!}\right)^{k_r},$$

where the summation extends over all non-negative integer solutions $(k_1, \ldots, k_p)$ to the equation $k_1 + 2k_2 + \cdots + pk_p = p$. Note that γ_p depends on the first p moments of X only. For example,

$$\gamma_1 = \alpha_1, \qquad \gamma_3 = \alpha_3 - 3\alpha_1\alpha_2 + \alpha_1^3,$$

$$\gamma_2 = \alpha_2 - \alpha_1^2, \qquad \gamma_4 = \alpha_4 - 3\alpha_2^2 - 4\alpha_1\alpha_3 + 12\,\alpha_1^2\alpha_2 - 6\alpha_1^4.$$

Under standard moment assumptions, such as $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, we have $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = \alpha_3$, $\gamma_4 = \alpha_4 - 3$. For any normal random variable, $\gamma_k = 0$ for all $k \geq 3$.

Now, returning to the general setting as in Proposition 4.1, assume that $s \geq 2$, and $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Then $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = 1$, so $\log v(t) = -\frac{t^2}{2} + o(|t|^2)$. Therefore, it is natural to center and normalize this function by introducing the family of the functions

$$\psi_z(t) = \frac{1}{2}\, t^2 + \frac{1}{z^2} \, \log v(tz), \qquad |tz| < c,$$

where $z \neq 0$ is a given parameter. Clearly, ψ_z is s-times differentiable in this t-interval, and

$$\psi_z(0) = \psi_z'(0) = \psi_z''(0) = 0, \qquad \psi_z^{(p)}(0) = \gamma_p \, i^p z^{p-2} \quad (p = 3, \ldots, m).$$

Moreover, reformulating Proposition 4.1 in terms of the functions $t \to \frac{1}{z^2} \log v(tz)$ with fixed $z \neq 0$, we get:

Corollary 4.2. *Let $v(t)$ be s-times differentiable on the real line, $s \geq 2$, not vanishing for $|t| \leq c$, and such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Given $z \neq 0$ in the interval $|tz| \leq c$ for all $p = 0, 1, \ldots, m$, $m = [s]$,*

$$\frac{d^p}{dt^p} \psi_z(t) = \frac{d^p}{dt^p} \sum_{k=3}^{m} \frac{\gamma_k}{k!} (it)^k \, z^{k-2} + |t|^{s-p} \, |z|^{s-2} \varepsilon(tz), \tag{4.4}$$

where $\varepsilon = \varepsilon(t)$ is defined and continuous in $|t| \leq c$ and satisfies $\varepsilon(t) \to 0$ as $t \to 0$.

Also, as remarked after Proposition 4.1, cf. (4.2), when v has $m+1$ continuous derivatives, a representation with sharper remainder term holds,

$$\frac{d^p}{dt^p}\,\psi_z(t) = \frac{d^p}{dt^p}\,\sum_{k=3}^{m}\frac{\gamma_k}{k!}\,(it)^k\,z^{k-2} + A\,|t|^{(m+1)-p}\,|z|^{m-1},$$

where $A = A(t,z)$ is a bounded function in the domain $|tz| \leq c$.

In the case of characteristic functions, Corollary 4.2 admits a slight refinement.

Corollary 4.3. *Let X be a random variable with characteristic function v and with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, $\mathbf{E}\,|X|^s < +\infty$, $s \geq 2$. Then, given $z \neq 0$, the relation* (4.4) *holds in the interval* $|tz| < \sqrt{2}$.

Indeed, by Taylor's theorem, $|1 - v(t)| \leq \frac{1}{2}\,t^2$ for all $t \in \mathbf{R}$. Hence one may choose in Corollary 4.2 any value $0 < c < \sqrt{2}$.

5. THE CASE OF MOMENTS OF ORDER $2 \leq s < 3$

In case $2 \leq s < 3$, Corollary 4.2 is simplified, since then there are no terms in the sum (4.4). In particular, when $s = 2$, we have

$$\psi_z(t) = t^2\,\varepsilon_0(tz), \quad \psi_z'(t) = |t|\,\varepsilon_1(tz), \quad \psi_z''(t) = \varepsilon_2(tz) \tag{5.1}$$

with some functions $\varepsilon_i(z) \to 0$ as $z \to 0$. This leads to the following observation, which is classical in case of characteristic functions.

Proposition 5.1. *Assume that $v(t)$ has two continuous derivatives with $v(0) = 1$, $v'(0) = 0$ and $v''(0) = -1$. There is a function $T_z \to +\infty$ as $z \to 0$ $(0 < |z| \leq 1)$ such that uniformly in the intervals $|t| \leq T_z$*

$$\left|\frac{d^p}{dt^p}\,v(tz)^{1/z^2} - \frac{d^p}{dt^p}\,e^{-t^2/2}\right| \leq\; e^{-t^2/2}\,\varepsilon(z), \qquad p = 0,1,2, \tag{5.2}$$

where $\varepsilon(z) \to 0$ as $z \to 0$.

Proof. For completeness we include a well-known argument. Let $|t| \leq c$ be an interval, where the function $v(t)$ is not vanishing. Choose the function $T = T_z$ satisfying $T_z|z| \leq c$ whenever $0 < |z| \leq 1$ and $T_z|z| \to 0$ as $z \to 0$. These conditions will be assumed from now on. Moreover, for any continuous function $V(t)$, one can choose $T_z \to +\infty$ as $z \to 0$ such that

$$\sup_{|t| \leq T_z} |V(t)\psi_z(t)| \to 0 \quad \text{as } z \to 0,$$

and similarly for the first two derivatives of ψ_z.

For the proof, it is enough to see that, whenever $\varepsilon(z) \to 0$ and $W(t) \geq 0$ is continuous and increasing in $t \geq 0$, one can choose $T_z \to +\infty$ such that $W(T_z)\,\sup_{|t| \leq T_z}|\varepsilon(tz)| \to 0$ as $z \to 0$. Here, we may assume in the following that $\varepsilon(z) \geq 0$ is even and also increasing in $z > 0$. Then the latter statement may be simplified to $W(T_z)\,\varepsilon(T_z z) \to 0$, which is obviously true with a sufficiently slowly growing T_z.

In particular, in view of (5.1), with some $T_z \to +\infty$ as $z \to 0$ $(0 < |z| \leq 1)$, we have

$$\varepsilon(z) = \sup_{|t| \leq T_z}\big(|\psi_z(t)| + |\psi_z'(t)| + |\psi_z''(t)|\big) \to 0 \quad \text{as } z \to 0. \tag{5.3}$$

Now, write $v(tz)^{1/z^2} = g(t)e^{\psi_z(t)}$, where $g(t) = e^{-t^2/2}$. Applying (5.3), we get

$$|v(tz)^{1/z^2} - g(t)| \leq g(t)\,|e^{\psi_z(t)} - 1| \leq Cg(t)|\psi_z(t)|$$

for all $|t| \leq T_z$ with some constant C. Since also $\psi_z(t) \to 0$ uniformly in that interval, we arrive at the desired conclusion in case $p = 0$.

Writing $(v(tz)^{1/z^2})' = g'(t)e^{\psi_z(t)} + g(t)\psi_z'(t)e^{\psi_z(t)}$ and using the previous step, we get

$$|(v(tz)^{1/z^2})' - g'(t)| \leq C|t|g(t)\,|\psi_z(t)| + Cg(t)\,|\psi_z'(t)|.$$

Since $\psi_z'(t) \to 0$ and $t\psi_z(t) \to 0$ uniformly in that interval with an appropriate choice of T_z, we arrive at the conclusion in case $p = 1$. The case $p = 2$ is similar. $\square$

Now, let us turn to the range $2 < s < 3$. In this case, we obtain up to polynomial factors in front of $e^{-t^2/2}$ in (5.2) more information about the possible growth of T_z.

Proposition 5.2. *Let $v(t)$ be s-times differentiable, $2 < s < 3$, not vanishing for $|t| \le c$ $(c > 0)$, and such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Given $0 < |z| \le 1$,*

$$\left| \frac{d^p}{dt^p}\, v(tz)^{1/z^2} - \frac{d^p}{dt^p}\, e^{-t^2/2} \right| \le \big(|t|^{s-2} + |t|^{s+2}\big)\, e^{-t^2/2}\, \varepsilon(z), \qquad p = 0, 1, 2,$$

uniformly for $|t| \le c\,|z|^{-(s-2)/s}$ with some function $\varepsilon(z) \to 0$ as $z \to 0$.

Proof. By Corollary 4.2, in the intervals $|t| \le T_z = c\,|z|^{-(s-2)/s}$ with $0 < |z| \le 1$

$$|\psi_z(t)| \le |t|^s\, |z|^{s-2}\, \varepsilon(z),$$
$$|\psi_z'(t)| \le |t|^{s-1}\, |z|^{s-2}\, \varepsilon(z),$$
$$|\psi_z''(t)| \le |t|^{s-2}\, |z|^{s-2}\, \varepsilon(z)$$

with some bounded function $\varepsilon(z) \to 0$ as $z \to 0$. Indeed, the conditions $0 < |z| \le 1$ and $|t| \le T_z$ insure that $|tz| \le c\,|z|^{2/s} \le c$ and also $|tz| \to 0$ as $z \to 0$, uniformly in $|t| \le T_z$.

Now, by the first inequality,

$$|\psi_z(t)| \le |t|^s\, |z|^{s-2}\, \varepsilon(z) \le C \qquad (|t| \le T_z)$$

with some constant C. Hence using the same notation and arguments as in the proof of Proposition 5.1, for all $|t| \le T_z$,

$$|v(tz)^{1/z^2} - g(t)| \le C' g(t) |\psi_z(t)| \le C' g(t) \cdot |t|^s\, |z|^{s-2}\, \varepsilon(z).$$

Since also $|\psi_z'(t)| \le |t|^{s-1}\, |z|^{s-2}\, \varepsilon(z)$, with some constants C, C' we get

$$\begin{aligned} |(v(tz)^{1/z^2})' - g'(t)| &\le C|t| g(t)\, |\psi_z(t)| + C g(t)\, |\psi_z'(t)| \\ &\le C' \big(|t|^{s+1} + |t|^{s-1}\big)\, g(t)\, \varepsilon(z). \end{aligned}$$

Finally, writing

$$(v(tz)^{1/z^2})'' = g''(t) e^{\psi_z(t)} + 2g'(t)\psi_z'(t) e^{\psi_z(t)} + 2g(t)(\psi_z'' + \psi_z'(t)^2)\, e^{\psi_z(t)}$$

and using $|\psi_z''(t)| \le |t|^{s-2}\, |z|^{s-2}\, \varepsilon(z)$, we get that up to some constants

$$\begin{aligned} |(v(tz)^{1/z^2})'' - g''(t)| &\le C t^2 g(t)\, |\psi_z(t)| + C|t| g(t)\, |\psi_z'(t)| + C g(t)\, \big(|\psi_z''(t)| + |\psi_z'(t)|^2\big) \\ &\le C' \Big(|t|^{s+2} + |t|^s + |t|^{s-2} + |t|^{2(s-1)}\Big)\; g(t)\; |z|^{s-2} \varepsilon(z). \end{aligned}$$

All powers of $|t|$ vary from $s-2$ to $s+2$, so Proposition 5.2 is proved. □

6. DEFINITION OF THE EXPANSION POLYNOMIALS P_k

The polynomials P_k introduced in Section 1 appear not only in connection with characteristic functions, but in a more general setting as well.

Namely, let $v(t)$ be a complex-valued function on the real line, which is s-times differentiable $(s \ge 2)$ and such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Then v has cumulants γ_k, $k = 1, \dots, m$, where $m = [s]$. Moreover, $\gamma_1 = 0$ and $\gamma_2 = 1$.

Assume v is not vanishing in the interval $|t| \le c$, and let us return to the functions

$$\psi_z(t) = \frac{1}{2}\, t^2 + \frac{1}{z^2}\, \log v(tz),$$

where $z \neq 0$ is viewed as a (small) parameter and $|tz| \leq c$. Recall that, by Corollary 4.2,

$$e^{t^2/2}\, v(tz)^{1/z^2} = e^{\psi_z(t)} = \exp\bigg\{\sum_{k=1}^{m-2} \frac{\gamma_{k+2}}{(k+2)!}\,(it)^{k+2}\,z^k + |t|^s |z|^{s-2}\,\varepsilon(tz)\bigg\}, \tag{6.1}$$

where $\varepsilon(t)$ is defined and continuous in $|t| \leq c$ and satisfies $\varepsilon(t) \to 0$ as $t \to 0$. Moreover, if v has $m+1$ derivatives, the remainder term here may be replaced with $A\,|t|^{(m+1)-p}\,|z|^{m-1}$, where $A = A(t,z)$ is bounded in the domain $|tz| \leq c$.

The sum in (6.1) is vanishing in case $2 \leq s < 3$. To study this representation in the case $s \geq 3$, introduce the polynomials

$$W_z(t) = \sum_{k=1}^{m-2} \frac{\gamma_{k+2}}{(k+2)!}\,(it)^{k+2}\,z^k.$$

By a formal Taylor's representation with respect to the (complex) variable z, we have

$$e^{W_z(t)} = 1 + \sum_{k=1}^{\infty} a_k(it)\,z^k \tag{6.2}$$

with some coefficients $a_k(it)$. To justify this step and precisely determine the coefficients, write

$$e^{W_z(t)} = \sum_{p=0}^{\infty} \frac{W_z(t)^p}{p!},$$

which can further be expanded as

$$\sum_{p=0}^{\infty} \sum_{p_1+\dots+p_{m-2}=p} \frac{(\frac{\gamma_3}{3!})^{p_1} \dots (\frac{\gamma_m}{m!})^{p_{m-2}}}{p_1! \dots p_{m-2}!}\,(it)^{3p_1+4p_2+\dots+mp_{m-2}}\,z^{p_1+2p_2+\dots+(m-2)p_{m-2}}. \tag{6.3}$$

The whole double sum is absolutely summable for all complex numbers t and z. Indeed, let $C = \sum_{k=3}^m \frac{|\gamma_k|}{k!}$. For any fixed integer $p \geq 0$, the finite sum of the absolute values of the terms in (6.3) is bounded by

$$\frac{1}{p!}\left(\sum_{k=1}^{m-2} \frac{|\gamma_{k+2}|}{(k+2)!}\,|t|^{k+2}\,|z|^k\right)^p \leq \frac{1}{p!}\,C^p \left(\max_{1 \leq k \leq m-2} |t|^{k+2}\,|z|^k\right)^p.$$

Assume without loss of generality (in order to get some quantitative bounds) that $|t^3 z| \leq 1$ and $|z| \leq 1$. Then,

$$|t|^{k+2}\,|z|^k \leq |z|^{-(k+2)/3}\,|z|^k = |z|^{(2k-2)/3} \leq 1.$$

Hence the above sum is bounded by $C^p/p!$ Furthermore, note that $|W_z(t)| \leq C$.

Thus the total sum of the absolute values is bounded by e^C, and one may freely choose the order of summation. Collecting the coefficients in (6.3) in front of z^k, we arrive at (6.2) with

$$a_k(it) = \sum_{p_1+2p_2+\dots+(m-2)p_{m-2}=k} \frac{1}{p_1! \dots p_{m-2}!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \dots \left(\frac{\gamma_m}{m!}\right)^{p_{m-2}} (it)^{3p_1+4p_2+\dots+mp_{m-2}},$$

where the summation is extended over all non-negative integer solutions $(p_1, \dots, p_{m-2})$ to the equation $p_1 + 2p_2 + \dots + (m-2)p_{m-2} = k$. Note that

$$3p_1 + 4p_2 + \dots + mp_{m-2} = k + 2(p_1 + p_2 + \dots + p_{m-2}).$$

Hence replacing it with t,

$$a_k(t) = \sum_{p_1+2p_2+\dots+(m-2)p_{m-2}=k} \frac{1}{p_1! \dots p_{m-2}!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \dots \left(\frac{\gamma_m}{m!}\right)^{p_{m-2}} t^{k+2(p_1+\dots+p_{m-2})}. \tag{6.4}$$

In addition, if $k \le m-2$, the condition $p_1 + 2p_2 + \cdots + (m-2)p_{m-2} = k$ implies that $p_{k+1} = \cdots = p_{m-2} = 0$. Therefore, in this case a_k depends on the first k cumulants $\gamma_3, \dots, \gamma_{k+2}$ only. More precisely,

$$a_k(t) \;=\; \sum_{p_1+2p_2+\cdots+kp_k=k} \frac{1}{p_1!\dots p_k!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \cdots \left(\frac{\gamma_{k+2}}{(k+2)!}\right)^{p_k} t^{k+2(p_1+\cdots+p_k)}, \quad 1 \le k \le m-2.$$

For example, if $m = 3$, we have $a_1(t) = \frac{\gamma_3}{6}\, t^3$. In case $m = 4$, $a_1(t) = \frac{\gamma_3}{6}\, t^3$ and

$$a_2(t) = \frac{\gamma_3^2}{72}\, t^6 + \frac{\gamma_4}{24}\, t^4.$$

In general, subject to $p_1 + 2p_2 + \cdots + kp_k = k$, the expression $k + 2(p_1 + \cdots + p_k)$ does not exceed $3k$ and reaches this value (when $p_1 = k$, $p_2 = \cdots = p_k = 0$), so a_k represents a polynomial in t of degree exactly $3k$.

Definition 6.1. Given an integer $m \ge 3$ and complex numbers $\gamma_3, \dots, \gamma_m$, one defines P_k ($1 \le k \le m-2$) as the polynomial a_k introduced above, namely,

$$P_k(t) \;=\; \sum_{p_1+2p_2+\cdots+kp_k=k} \frac{1}{p_1!\dots p_k!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \cdots \left(\frac{\gamma_{k+2}}{(k+2)!}\right)^{p_k} t^{k+2(p_1+\cdots+p_k)}.$$

With this definition, the representation (6.2) may also be written as

$$\exp\left\{\sum_{k=3}^{m} \frac{\gamma_k}{k!}\,(it)^k z^{k-2}\right\} \;=\; 1 + \sum_{k=1}^{m-2} P_k(it)\, z^k + \sum_{k=m-1}^{\infty} a_k(it) z^k, \tag{6.5}$$

where a_k's are described in (6.4).

7. ASSOCIATED PROJECTION OPERATORS

Let us note first that every polynomial a_k in (6.4) contains terms involving powers of t not smaller than $k+2$ (since necessarily $p_1 + \cdots + p_{m-2} \ge 1$). This observation may be used to obtain an initial trivial bound for the last sum in (6.5) in case of small values of t.

Lemma 7.1. *Given complex numbers $\gamma_3, \dots, \gamma_m$, for all complex z and t, $|t| \le 1$,*

$$\sum_{k=m-1}^{\infty} |a_k(it) z^k| \le \left(e^{C(z)} - 1\right) |t|^{m+1}, \tag{7.1}$$

where $C(z) = \sum_{k=3}^{m} \frac{|\gamma_k|}{k!}\, |z|^{k-2}$.

Indeed, using (6.4) and $|t| \le 1$, we see that each term in the sum of (7.1) is bounded by $|t|^{m+1}$ up to the factor

$$\sum_{p_1+2p_2+\cdots+(m-2)p_{m-2}=k} \frac{1}{p_1!\dots p_{m-2}!} \left(\frac{|\gamma_3|\,|z|}{3!}\right)^{p_1} \cdots \left(\frac{|\gamma_m|\,|z|^{m-2}}{m!}\right)^{p_{m-2}}.$$

Summation of these expressions over all $k \ge 1$ results in $e^{C(z)} - 1$.

The bound of Lemma 7.1 is needed in order to express the "cumulants" γ_k directly in terms of the associated polynomials P_j.

Lemma 7.2. *Given complex numbers $\gamma_3, \dots, \gamma_m$, for all complex z and $k = 3, \dots, m$,*

$$\frac{\gamma_k}{k!}\, z^{k-2} \;=\; \frac{d^k}{i^k\, dt^k}\, \log\left(1 + \sum_{j=1}^{m-2} P_j(it) z^j\right)\Big|_{t=0}. \tag{7.2}$$

To see this, note that, by (6.5) and (7.1), we obtain that, as $t \to 0$,

$$\sum_{k=3}^{m} \frac{\gamma_k}{k!}\,(it)^k z^{k-2} = \log\Big(1 + \sum_{j=1}^{m-2} P_j(it) z^j\Big) + O(|t|^{m+1}).$$

Since the right-hand side (without the remainder term) represents an analytic function near zero, the comparison of coefficients of powers of t immediately leads to (7.2).

The identity (6.5) suggests introducing special operators defined on the space V_s of all s-times differentiable functions $v\colon \mathbf{R} \to \mathbf{C}$, $s \geq 2$, such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$.

Definition 7.3. *Given $v \in V_s$, $s \geq 2$, and an integer $2 \leq m \leq s$, we put*

$$(T_m v)(t) = e^{-t^2/2}\Big(1 + \sum_{k=1}^{m-2} P_k(it)\Big), \quad t \in \mathbf{R},$$

where P_k are the polynomials from Definition 6.1 *based on the cumulants*

$$\gamma_k = \frac{d^k}{i^k\,dt^k}\,\log v(t)\big|_{t=0}, \qquad k = 3, \dots, m-2.$$

If $m = 2$, there are no cumulants and polynomials in the definition, so $(T_2 v)(t) = e^{-t^2/2}$ for any $v \in V_s$. If $m = 3$,

$$(T_3 v)(t) = e^{-t^2/2}\Big(1 + \frac{\gamma_3}{6}\,(it)^3\Big)$$

for any $v \in V_s$, $s \geq 3$ (where γ_3 may be an arbitrary complex number). If $m = 4$, then for any $v \in V_s$, $s \geq 4$,

$$(T_4 v)(t) = e^{-t^2/2}\Big(1 + \frac{\gamma_3}{6}\,(it)^3 + \frac{\gamma_4}{24}\,(it)^4 + \frac{\gamma_3^2}{72}\,(it)^6\Big).$$

Clearly, every $T_m v$ is an entire function and hence belongs to all V_s, $s \geq m$. This defines an operator $T_m\colon V_s \to V_s$ which turns out to be a projection operator.

Proposition 7.4. *We have $T_m T_m v = T_m v$ for any $v \in V_s$, $2 \leq m \leq s$. Moreover, $T_m v$ and v have identical derivatives at the origin up to order m.*

Proof. The statement is equivalent to the property that $e_m = T_m v$ and v have equal cumulants. Let $\widetilde{\gamma}_k$ and γ_k denote the cumulants of e_m and v, respectively ($3 \leq k \leq m$). By Definition 7.3,

$$\frac{\widetilde{\gamma}_k}{k!} \;=\; \frac{d^k}{i^k\,dt^k}\,\log e_m(t)\big|_{t=0} = \frac{d^k}{i^k\,dt^k}\,\log\Big(1 + \sum_{j=1}^{m-2} P_j(it)\Big)\Big|_{t=0}.$$

But the right-hand side equals $\frac{\gamma_k}{k!}$ according to Lemma 7.2 applied with $z = 1$.

Thus Proposition 7.4 is proved. $\square$

Note that $T_m v$ need not be a characteristic function, even if v is a characteristic function of some random variable. However, it always represents the Fourier–Stieltjes transform of a finite signed measure.

In the following we approximate v by its projections $T_m v$. Combining Proposition 7.4 with Proposition 2.2, we get:

Corollary 7.5. *Given $v \in V_s$, $2 \leq m \leq s$, as $t \to 0$,*

$$\frac{d^p}{dt^p}\big(v(t) - T_m v(t)\big) = o(|t|^{s-p}), \qquad p = 0, 1, \dots, m.$$

Finally, let us formulate an asymptotic property of the projection operators T_m for growing parameter m (although this will not be needed in the sequel).

Proposition 7.6. *Assume that $v(t)$ admits an analytic extension to the disc $|t| < \rho$, where it has no zeros, and $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Then $T_m v(t) \to v(t)$ as $m \to \infty$, i.e.,*

$$v(t) = e^{-t^2/2}\Big(1 + \sum_{k=1}^{\infty} P_k(it)\Big), \qquad |t| < \rho.$$

Moreover, the series is convergent absolutely.

If $v(t) = \mathbf{E}\, e^{itX}$ is the characteristic function of a random variable X, the assumptions of Proposition 7.6 are fulfilled, provided that $\mathbf{E}X = 0$, $\mathbf{E}\, X^2 = 1$, $\mathbf{E}\, e^{\rho|X|} < +\infty$ (that is, an exponential moment of order ρ is finite) and $v(t)$ does not vanish in the disc $|t| < \rho$.

Proof. By assumption, $\log v(t)$ is analytic in the disc $|t| < \rho$, so it is representable as the sum of the absolutely convergent power series

$$\log v(t) = \sum_{k=3}^{\infty} \frac{\gamma_k}{k!}\,(it)^k, \qquad |t| < \rho. \tag{7.3}$$

Hence, starting with (6.5) with $z = 1$ and letting there $m \to \infty$, it is sufficient to show that

$$\sum_{k=m-1}^{\infty} |a_k(it)| \to 0$$

(note that a_k's also depend on m).

Rewrite the representation (6.4) as

$$a_k(t) \;=\; \sum_{p_1+2p_2+\dots+(m-2)p_{m-2}=k} \frac{1}{p_1!\dots p_{m-2}!}\left(\frac{\gamma_3\, t^3}{3!}\right)^{p_1}\dots\left(\frac{\gamma_m\, t^m}{m!}\right)^{p_{m-2}},$$

which implies that

$$|a_k(t)| \;\le\; \sum_{p_1+2p_2+\dots+(m-2)p_{m-2}=k} \frac{1}{p_1!\dots p_{m-2}!}\left(\frac{|\gamma_3|\,|t|^3}{3!}\right)^{p_1}\dots\left(\frac{|\gamma_m|\,|t|^m}{m!}\right)^{p_{m-2}}.$$

Here the right-hand side may be bounded by the quantity

$$b_k(t) \;=\; \sum_{p_1+2p_2+3p_3+\dots=k}\ \prod_{r=1}^{\infty} \frac{1}{p_r!}\left(\frac{|\gamma_{r+2}|\,|t|^{r+2}}{(r+2)!}\right)^{p_r},$$

which does not depend on m. After summation over all $k \ge 1$ (thus removing any constraint on p_r), we get $\sum_{k=1}^{\infty} b_k(t) = e^{C(|t|)} - 1$, where $C(a) = \sum_{k=3}^{\infty} \frac{|\gamma_k|}{k!}\, a^k$. But $C(|t|) < +\infty$ for all $|t| < \rho$ in view of the absolute convergence of the series (7.3). Hence in this case

$$\sum_{k=m-1}^{\infty} |a_k(it)| \le \sum_{k=m-1}^{\infty} b_k(t) \to 0 \quad \text{as } m \to \infty.$$

With similar arguments, we also obtain that $\sum_{k=1}^{\infty} |P_k(it)| < +\infty$ for $|t| < \rho$, in view of Definition 6.1. Thus Proposition 7.6 is proved. $\square$

8. BOUNDS OF P_k AND THEIR DERIVATIVES

We will need a bound similar to the one in Lemma 7.1 which extends to large values of t and involving derivatives of the polynomials a_k and P_k.

To this aim, we start with arbitrary complex numbers $\gamma_3, \dots, \gamma_m$, $m \geq 3$ (which may be interpreted as cumulants of a given function v) and return to the representation (6.2),

$$w_z(t) = e^{W_z(t)} = 1 + \sum_{k=1}^{\infty} a_k(it)\, z^k, \qquad t, z \in \mathbf{C}, \tag{8.1}$$

where

$$W_z(t) = \sum_{k=1}^{m-2} \frac{\gamma_{k+2}}{(k+2)!}\, (it)^{k+2}\, z^k, \tag{8.2}$$

and where the polynomials a_k are described in (6.4). By the very definition, $a_k = P_k$ as long as $k \leq m-2$.

As we have already noticed, the sum in (8.1) is absolutely convergent and therefore represents an entire function with respect to z for any fixed t. It is also clear that the series may be termwise differentiated, so that

$$w_z^{(p)}(t) = \sum_{k=1}^{\infty} i^p\, a_k^{(p)}(it)\, z^k, \qquad p \geq 1, \tag{8.3}$$

which is absolutely convergent as well.

In order to bound a_k and its derivatives, we use the quantity

$$C = \sum_{k=3}^{m} |\gamma_k|.$$

One natural approach (which is however different from the one in [5]) is based on the application of Cauchy's integral formula

$$a_k(it) = \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{w_z(t)}{z^{k+1}}\, dz$$

with a suitably chosen parameter $\rho > 0$. In view of (8.3), there is a more general identity, involving the derivatives,

$$i^p\, a_k^{(p)}(it) = \frac{1}{2\pi i} \int\limits_{|z|=\rho} \frac{w_z^{(p)}(t)}{z^{k+1}}\, dz, \qquad p = 0, 1, 2 \dots \tag{8.4}$$

Lemma 8.1. *Let $|tz| \leq 2$ and $|t^3 z| \leq 2$, and let $0 \leq p \leq m$ be an integer. Then*

$$|W_z^{(p)}(t)| \leq 2^{m-2} C\, |t|^{-p} \min\{1, |t|^2\}. \tag{8.5}$$

Indeed, by definition (8.2),

$$W_z^{(p)}(t) = \sum_{q=\max(p,3)}^{m} \frac{\gamma_q\, i^q}{(q-p)!}\, t^{q-p} z^{q-2}.$$

But $|t^{q-p}\, z^{q-2}| = |tz|^{q-3}\, |t^3 z|\, |t|^{-p} \leq 2^{m-2}\, |t|^{-p}$ whenever $3 \leq q \leq m$. Hence

$$|W_z^{(p)}(t)| \leq 2^{m-2} C |t|^{-p}.$$

On the other hand, just using $|z| \leq 2/|t|$, we get

$$|t^{q-p}\, z^{q-2}| \leq |t|^{q-p}\, \frac{2^{q-2}}{|t|^{q-2}} = 2^{q-2} |t|^{2-p} \leq 2^{m-2} |t|^{2-p}.$$

This gives an improvement over the previous estimate in case $|t| \leq 1$ and proves (8.5).

Lemma 8.2. *For all integers $k \geq 1$, $0 \leq p \leq m$, and all complex t,*

$$|a_k^{(p)}(it)| \leq C_{m,p}\, |t|^{-p} \min\{1, |t|^2\} \left(\frac{\max\{|t|, |t|^3\}}{2}\right)^k$$

with constants $C_{m,p} = (4^m (1+C))^p\, e^{2^m C}$.

Proof. Given $t \neq 0$, we choose in (8.4) the radius

$$\rho = \frac{2}{\max\{|t|, |t|^3\}}.$$

Hence on the circle $|z| = \rho$, both $|tz| \leq 2$ and $|t^3 z| \leq 2$ are fulfilled, thus inequality (8.5) may be applied. In particular, $|W_z(t)| \leq 2^{m-2}C$, and from (8.4) with $p = 0$ we get the desired estimate

$$|a_k(it)| \leq \frac{1}{\rho^k}\, e^{2^{m-2}C}.$$

Next, by the formula (2.3) of Lemma 2.4, for all $p \geq 1$,

$$w_z^{(p)}(t) = p!\, w_z(t) \sum \prod_{r=1}^{p} \frac{1}{k_r!} \left(\frac{W_z^{(r)}(t)}{r!}\right)^{k_r},$$

where the summation is taken over all nonnegative integer solutions $(k_1, \dots, k_p)$ to the equation $k_1 + 2k_2 + \dots + pk_p = p$. Hence using (8.5), given that $|z| = \rho$, we arrive at

$$|w_z^{(p)}(t)| \ \leq \ e^{2^{m-2}C} |t|^{-p}\, p! \sum \prod_{r=1}^{p} \frac{1}{k_r!} \left(\frac{2^{m-2}C\ \min\{1, |t|^2\}}{r!}\right)^{k_r}.$$

Since necessarily $1 \leq k_1 + \dots + pk_p \leq p$, the product may be bounded by the product of $\min\{1, |t|^2\}$ (in the first power) and $2^{m-2}C$ (replaced by $2^{m-2}(1+C)$), and raised to power p. This leads to

$$|w_z^{(p)}(t)| \ \leq \ e^{2^{m-2}C} \left(2^{m-2}(1+C)\right)^p |t|^{-p} \min\{1, |t|^2\}\, B_p, \tag{8.6}$$

where $B_p = p! \sum \prod_{r=1}^{p} \frac{1}{k_r!} (\frac{1}{r!})^{k_r}$. This constant can also be described by virtue of the same formula (2.3) applied with $z = e^y$ and $y(s) = e^s$, in which case it reads

$$\frac{d^p}{ds^p}\, e^{e^s} = e^{e^s}\, p! \sum \prod_{r=1}^{p} \frac{1}{k_r!} \left(\frac{e^s}{r!}\right)^{k_r}.$$

One should apply this formula at $s = 0$, thus we consider the functions $b_p(s) = (e^{e^s})^{(p)}$, $s \geq 0$, and their values $b_p = b_p(0) = B_p/e$. The recursive identity $b_{p+1}(s) = (e^s\, e^{e^s})^{(p)} = e^s \sum_{r=0}^{p} C_p^r\, b_r(s)$ implies that the sequence $r \to b_r$ is nondecreasing and $b_{p+1} \leq 2^p\, b_p$. Therefore,

$$b_p \leq 2^{p-1} b_{p-1} \leq 2^{p-1} 2^{p-2} b_{p-2} \leq \dots \leq 2^{p-1} 2^{p-2} \dots 2^0\, b_0 = 2^{p(p-1)/2}\, e.$$

Hence $B_p \leq 2^{p(p-1)/2}$, and together with (8.6) this gives the estimate

$$|w_z^{(p)}(t)| \ \leq \ e^{2^{m-2}C} \left(2^{(m-2)+(p-1)/2}(1+C)\right)^p |t|^{-p} \min\{1, |t|^2\}.$$

It remains to apply (8.4) and simplify the constant. Thus Lemma 8.2 is proved. □

Now, fix an integer $p = 0, 1 \dots, m$, and assume that $|z| \leq \frac{\rho}{2} = \frac{1}{\max\{|t|, |t|^3\}}$, that is, $|tz| \leq 1$ and $|t^3 z| \leq 1$. By Lemma 8.2,

$$\sum_{k=m-1}^{\infty} |a_k^{(p)}(it)\, z^k| \leq C_{m,p}\, |t|^{-p} \min\{1, |t|^2\} \sum_{k=m-1}^{\infty} \frac{|z|^k}{\rho^k}$$

$$\leq 2\, C_{m,p}\, |t|^{-p} \min\{1, |t|^2\}\, \frac{|z|^{m-1}}{\rho^{m-1}}$$
$$\leq C_{m,p}\, |t|^{-p} \min\{1, |t|^2\}\, \big(\max\{|t|, |t|^3\}\big)^{m-1}\, |z|^{m-1}.$$

To simplify the dependence on t, note that in case $|t| \leq 1$,

$$|t|^{-p} \min\{1, |t|^2\}\, \big(\max\{|t|, |t|^3\}\big)^{m-1} = |t|^{(m+1)-p},$$

while the left expression is equal to $|t|^{3(m-1)-p}$ in case $|t| \geq 1$.

Also note that the condition $|tz| \leq 1$ is fulfilled automatically, as long as $|t^3 z| \leq 1$ and $|z| \leq 1$. Therefore, recalling also that $P_k = a_k$ for $k \leq m-2$, we obtain:

Proposition 8.3. *If* $0 < |z| \leq 1$ *and* $|t^3 z| \leq 1$*, then for all* $p = 0, 1 \dots, m$*,*

$$\frac{d^p}{dt^p}\, e^{W_z(t)} = \frac{d^p}{dt^p} \left(1 + \sum_{k=1}^{m-2} P_k(it)\, z^k\right) + A\big(|t|^{m+1-p} + |t|^{3(m-1)-p}\big)|z|^{m-1},$$

where $|A| \leq C_{m,p} = (4^m(1+C))^p\, e^{2^m C}$.

9. EDGEWORTH-TYPE EXPANSION FOR THE FUNCTIONS $v(tz)^{1/z^2}$

Assume that $v(t)$ is s-times differentiable, $s \geq 2$, and not vanishing for $|t| \leq c$ ($c > 0$), and such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. For

$$v_z(t) = v(tz)^{1/z^2}$$

define the approximating functions

$$u_m(t) = u_m(t, z) = e^{-t^2/2} \left(1 + \sum_{k=1}^{m-2} P_k(it)\, z^k\right), \qquad m = [s],$$

where the polynomials P_k are based on the cumulants $\gamma_3, \dots, \gamma_m$ of v. Put $m'(s) = s - p$,

$$m''(s) = 3(m-2) + \max\{s + p, (s-1)p\}.$$

In particular, $m'(s) = s$ and $m''(s) = s + 3(m-2)$ in case $p = 0$. Note that $m''(s) \leq 2m^2$ in all admissible cases.

In this section, relation (1.8) is established in the following more general form.

Proposition 9.1. *Let* $s \geq 3$*. Given* z *real,* $0 < |z| \leq 1$*, in the interval* $|t^3 z| \leq c^3$*, for all* $p = 0, 1, \dots, m$*,*

$$\big|v_z^{(p)}(t) - u_m^{(p)}(t)\big| \leq \big(|t|^{m'} + |t|^{m''}\big)\, e^{-t^2/2}\, |z|^{s-2}\, \varepsilon(z), \tag{9.1}$$

where $\varepsilon(z) \to 0$ *as* $z \to 0$*. Moreover, if* $s \geq 2$ *and* $v(t)$ *has* $(m+1)$ *continuous derivatives, then with some constant* A *and with* m', m'' *corresponding to* $s = m+1$*,*

$$\big|v_z^{(p)}(t) - u_m^{(p)}(t)\big| \leq A\, \big(|t|^{m'} + |t|^{m''}\big)\, e^{-t^2/2}\, |z|^{m-1}. \tag{9.2}$$

We will refer to (9.1) and (9.2) as the scenarios 1 and 2, respectively. Note that in the second case, although v has cumulants up to order $m+1$, we require that γ_{m+1} does not participate in the definition of the polynomials P_k. In particular, the value $m = 2$ is covered in (9.2), and then $u_m(t) = e^{-t^2/2}$ (that is, P_1 is not present).

Proof. Without loss of generality, assume $c = 1$. Write $v_z(t) = e^{-t^2/2}\, w_z(t)\, e^{h_z(t)}$, where

$$w_z(t) = e^{W_z(t)}, \qquad W_z(t) = \sum_{k=3}^{m} \frac{\gamma_k}{k!}\,(it)^k z^{k-2},$$

$$\psi_z(t) = \frac{1}{2}\,t^2 + \frac{1}{z^2}\,\log v(tz) = \log\big(e^{t^2/2} v_z(t)\big), \qquad h_z(t) = \psi_z(t) - W_z(t).$$

By definition of a_k and P_k,

$$w_z(t) = 1 + \sum_{k=1}^{m-2} P_k(it)\, z^k + R_z(t), \qquad R_z(t) = \sum_{k=m-1}^{\infty} a_k(it)\, z^k.$$

Therefore,

$$v_z(t) = u_m(t)\, e^{h_z(t)} + R_z(t)\, g(t)\, e^{h_z(t)}, \qquad g(t) = e^{-t^2/2}.$$

Given $p = 0, 1 \dots, m$, we differentiate this representation according to the Leibnitz rule:

$$v_z^{(p)}(t) - u_m^{(p)}(t) = I_1 + I_2 + I_3 = u_m^{(p)}(t)\,(e^{h_z(t)} - 1) + \sum_{k=1}^{p} C_p^k\, u_m^{(p-k)}(t)\,(e^{h_z(t)})^{(k)}$$

$$+ \sum_{k=0}^{p} C_p^k\,(R_z(t)\, g(t))^{(k)}\,(e^{h_z(t)})^{(p-k)}, \tag{9.3}$$

where $C_p^k = \frac{p!}{k!(p-k)!}$ are the combinatorial coefficients. Note that when $p = 0$, the second term I_2 is vanishing.

Estimation of I_1.

In Corollary 4.2 it is shown that, if $|z| \le 1$ and $|t^3 z| \le 1$, the functions h_z and their derivatives are uniformly bounded and admit the bounds

$$|h_z^{(p)}(t)| \le |t|^{s-p} |z|^{s-2} \varepsilon_p(z), \qquad p = 0, 1, \dots, m, \tag{9.4}$$

where each $\varepsilon_p(z)$ is defined in $|z| \le 1$ and satisfies $\varepsilon_p(z) \to 0$ as $z \to 0$. Moreover, if v has $m+1$ continuous derivatives, then we have a sharper estimate

$$|h_z^{(p)}(t)| \le A_p\, |t|^{(m+1)-p}\, |z|^{m-1} \tag{9.5}$$

with some constants A_p. In particular, when $p = 0$, these bounds correspondingly give

$$|e^{h_z(t)} - 1| \le |t|^s |z|^{s-2} \varepsilon(z), \qquad |e^{h_z(t)} - 1| \le A_0\, |t|^{m+1}\, |z|^{m-1} \tag{9.6}$$

with some $\varepsilon(z) \to 0$ as $z \to 0$, and a constant A_0.

On the other hand, since every P_k has degree $3k \le 3(m-2)$ and $|z| \le 1$, for all $p = 0, 1, \dots, m$, $m \ge 3$,

$$\left| \frac{d^p}{dt^p} \left(1 + \sum_{k=1}^{m-2} P_k(it)\, z^k \right) \right| \le C\big(1 + |t|^{3(m-2)-p}\big)$$

with some constant C depending on m, p and the cumulants γ_k's. Since also

$$|g^{(p)}(t)| = \left| \frac{d^p}{dt^p}\, e^{-t^2/2} \right| \le C_p\,(1 + |t|^p)\, e^{-t^2/2}, \tag{9.7}$$

we get, by the Leibnitz rule,

$$|u_m^{(p)}(t)| \le C\,\big(1 + |t|^{3(m-2)+p}\big)\, e^{-t^2/2}, \tag{9.8}$$

where we allow the constants depend on m, p and the cumulants $\gamma_3, \dots, \gamma_m$. For $m = 2$, $u_m(t) = e^{-t^2/2}$, so (9.8) holds in this case as well ($p = 0, 1, 2$). Combining this with (9.6), we correspondingly arrive at

$$|I_1| \le \left(|t|^s + |t|^{s+p+3(m-2)}\right) e^{-t^2/2}\, |z|^{s-2} \varepsilon(z), \tag{9.9}$$

$$|I_1| \le A \left(|t|^{m+1} + |t|^{p+(4m-5)}\right) e^{-t^2/2}\, |z|^{m-1} \tag{9.10}$$

with some constant A and $\varepsilon(z) \to 0$ as $z \to 0$. As a result, we obtain the desired bounds on the first term I_1 in (9.3) for both scenarios.

Estimation of I_2.

To treat the second term, or, more precisely, the products $u_m^{(p-k)}(t)\,(e^{h_z(t)})^{(k)}$, assume that $p \ge 1$. By formula (2.3), for any $k = 1, \dots, p$,

$$(e^{h_z(t)})^{(k)} = e^{h_z(t)}\, k! \sum \prod_{r=1}^{k} \frac{1}{p_r!} \left(\frac{h_z^{(r)}(t)}{r!}\right)^{p_r},$$

where the summation is performed over all nonnegative integer solutions $(p_1, \dots, p_k)$ to the equation $p_1 + 2p_2 + \dots + kp_k = k$. From (9.4)–(9.5) we get for the two scenarios

$$|h_z^{(r)}(t)|^{p_r} \le |t|^{(s-r)p_r} |z|^{(s-2)p_r} \varepsilon_r(z)^{p_r}, \qquad |h_z^{(r)}(t)|^{p_r} \le A_r^{p_r}\, |t|^{((m+1)-r)p_r}\, |z|^{(m-1)p_r}.$$

After multiplication of these inequalities over all $r = 1, \dots, k$ (separately in both scenarios), using $1 \le p_1 + \dots + p_k \le k$ together with

$$s - k \le \sum_{r=1}^{k} (s-r)p_r \le (s-1)k, \qquad (m+1) - k \le \sum_{r=1}^{k} ((m+1)-r)p_r \le mk,$$

we obtain that

$$\left|(e^{h_z(t)})^{(k)}\right| \le \left(|t|^{s-k} + |t|^{(s-1)k}\right)|z|^{s-2}\varepsilon(z), \tag{9.11}$$

$$\left|(e^{h_z(t)})^{(k)}\right| \le A \left(|t|^{(m+1)-k} + |t|^{mk}\right)|z|^{m-1} \tag{9.12}$$

with some constant A and $\varepsilon(z) \to 0$ as $z \to 0$. One may combine these bounds with (9.8), which immediately yields

$$\left|u_m^{(p-k)}(t)\,(e^{h_z(t)})^{(k)}\right| \le \left(|t|^{s-k} + |t|^{(s-1)k+(p-k)+3(m-2)}\right) e^{-t^2/2}\, |z|^{s-2}\, \varepsilon(z),$$

$$\left|u_m^{(p-k)}(t)\,(e^{h_z(t)})^{(k)}\right| \le A \left(|t|^{(m+1)-k} + |t|^{mk+(p-k)+3(m-2)}\right) e^{-t^2/2}\, |z|^{m-1}.$$

Since k varies from 1 to p, the right-hand sides can be made independent of k, and we arrive at the desired bounds on the second term I_2 in (9.3), needed for the values $p \ge 1$:

$$|I_2| \le \left(|t|^{s-p} + |t|^{(s-1)p+3(m-2)}\right) e^{-t^2/2}\, |z|^{s-2}\varepsilon(z), \tag{9.13}$$

$$|I_2| \le \left(|t|^{(m+1)-p} + |t|^{mp+3(m-2)}\right) e^{-t^2/2}\, |z|^{m-1}. \tag{9.14}$$

Estimation of I_3.

Now, let us turn to the third term, i.e., to the products $(R_z(t)\, g(t))^{(k)}\ (e^{h_z(t)})^{(p-k)}$. By Proposition 8.3, for all $p = 0, 1, \dots, m$,

$$\left|R_z^{(p)}(t)\right| \le C\left(|t|^{(m+1)-p} + |t|^{3(m-1)-p}\right)|z|^{m-1}.$$

Using (9.7) and the Leibnitz formula, the latter gives, for all $k = 0, \dots, p$,

$$\left|\left(R_z(t)\, g(t)\right)^{(k)}\right| \le C\left(|t|^{(m+1)-k} + |t|^{3(m-1)+k}\right) e^{-t^2/2}\, |z|^{m-1}. \tag{9.15}$$

Case $p = 0$. Then necessarily $k = 0$, and the above inequality yields

$$|I_3| \le C\left(|t|^{m+1} + |t|^{3(m-1)}\right) e^{-t^2/2}\, |z|^{m-1}. \tag{9.16}$$

It has only to be compared with (9.9)–(9.10). In the second scenario, one clearly obtains from (9.10) and (9.16) that

$$|v_z(t) - u_m(t)| \leq |I_1| + |I_3| \leq C\big(|t|^{m+1} + |t|^{4m-5}\big)\, e^{-t^2/2}\, |z|^{m-1}.$$

This proves (9.2) in case $p = 0$.

In the first scenario, just write in (9.16) $|z|^{m-1} = |z|^{s-2}\,\widetilde{\varepsilon}(z)$ with $\widetilde{\varepsilon}(z) \to 0$ as $z \to 0$. Together with (9.9) this leads to a similar estimate

$$|v_z(t) - u_m(t)| \leq |I_1| + |I_3| \leq C\big(|t|^{s} + |t|^{s+3(m-2)}\big)\, e^{-t^2/2}\, |z|^{s-2},$$

proving (9.1) in case $p = 0$. Thus, Proposition 9.1 is proved in this case.

Case $1 \leq p \leq m$. If $k = p$, the absolute value of the product

$$(R_z(t)\, g(t))^{(k)}\ (e^{h_z(t)})^{(p-k)} = (R_z(t)\, g(t))^{(p)}\ e^{h_z(t)}$$

may be estimated according to (9.15) by

$$C\big(|t|^{(m+1)-p} + |t|^{3(m-1)+p}\big)\, e^{-t^2/2}\, |z|^{m-1}. \tag{9.17}$$

As in the previous step, $|z|^{m-1}$ may be replaced here with $|z|^{s-2}\,\varepsilon(z)$.

If $0 \leq k \leq p-1$, by (9.11)–(9.12) for the two scenarios we have

$$\big|(e^{h_z(t)})^{(p-k)}\big| \leq \big(|t|^{s-(p-k)} + |t|^{(s-1)(p-k)}\big)|z|^{s-2}\varepsilon(z), \tag{9.18}$$

$$\big|(e^{h_z(t)})^{(p-k)}\big| \leq A\,\big(|t|^{(m+1)-(p-k)} + |t|^{m(p-k)}\big)|z|^{m-1}. \tag{9.19}$$

It remains to multiply these inequalities by (9.15). At this step we consider the two scenarios separately.

Scenario 1 (Inequality (9.1)): When multiplying (9.15) by (9.18) and looking for the maximal power of $|t|$, notice that

$$(3(m-1) + k) + (s-1)(p-k)$$

is maximized for $k = 0$, and for this value it is equal to $3(m-1) + (s-1)p$. Therefore,

$$\big|\big(R_z(t)\, g(t)\big)^{(k)}\, (e^{h_z(t)})^{(p-k)}\big| \leq \big(|t|^{(m+1)+(s-p)} + |t|^{3(m-1)+(s-1)p}\big)\, e^{-t^2/2}\, |z|^{(m-1)+(s-2)}\varepsilon(z).$$

Now, comparing with (9.9), (9.13) and (9.17), we see that the smallest power of $|t|$ in these inequalities is $m' = s - p$. Hence we do not loose much by writing

$$\big|\big(R_z(t)\, g(t)\big)^{(k)}\, (e^{h_z(t)})^{(p-k)}\big| \leq \big(|t|^{m'} + |t|^{3(m-1)+(s-1)p}\big)\, e^{-t^2/2}\, |z|^{(m-1)+(s-2)}\varepsilon(z),$$

which holds for all $k = 0, 1, \dots, p$. To simplify, let us note that $|t|^{3(m-1)}|z|^{(m-1)} \leq 1$, which leads to

$$\big|\big(R_z(t)\, g(t)\big)^{(k)}\, (e^{h_z(t)})^{(p-k)}\big| \leq \big(|t|^{m'} + |t|^{(s-1)p}\big)\, e^{-t^2/2}\, |z|^{s-2}\varepsilon(z). \tag{9.20}$$

In addition, the largest power of $|t|$ in (9.9), (9.13), (9.17) and (9.20) is

$$m'' = 3(m-2) + \max\{s+p, (s-1)p\}.$$

Hence $I_3 \leq \big(|t|^{m'} + |t|^{m''}\big)\, e^{-t^2/2}\, |z|^{s-2}\varepsilon(z)$ and similarly for I_1 and I_2. This proves (9.1).

Scenario 2 (Inequality (9.2)): This case can be dealt with along the lines of scenario 1 by letting $s \to m+1$. Or, repeating the previous arguments, note that when multiplying (9.15) by (9.19), the expression $3(m-1) + k + m(p-k)$ is maximized for $k = 0$, and for this value it is equal to $3(m-1) + mp$. Therefore,

$$\big|\big(R_z(t)\, g(t)\big)^{(k)}\, (e^{h_z(t)})^{(p-k)}\big| \leq A\,\big(|t|^{2(m+1)-p} + |t|^{3(m-1)+mp}\big)\, e^{-t^2/2}\, |z|^{2(m-1)}.$$

In (9.10), (9.14) and (9.17) the smallest power of $|t|$ is $m' = (m+1) - p$. Hence

$$\big|\big(R_z(t)\,g(t)\big)^{(k)}\,(e^{h_z(t)})^{(p-k)}\big| \le A\left(|t|^{m'} + |t|^{3(m-1)+mp}\right) e^{-t^2/2}\,|z|^{2(m-1)}.$$

Again using $|t|^{3(m-1)}|z|^{m-1} \le 1$, we get

$$\big|\big(R_z(t)\,g(t)\big)^{(k)}\,(e^{h_z(t)})^{(p-k)}\big| \le \left(|t|^{m'} + |t|^{mp}\right) e^{-t^2/2}\,|z|^{m-1}. \tag{9.21}$$

In addition, the largest power of $|t|$ in (9.10), (9.14), (9.17) and (9.21) is

$$m'' = 3(m-2) + \max\{m+p+1, mp\}.$$

Hence $I_3 \le A\big(|t|^{m'} + |t|^{m''}\big)\,e^{-t^2/2}\,|z|^{m-1}$ and similarly for I_1 and I_2 in case $p \ge 1$. $\square$

This proves (9.2) and Proposition 9.1.

One may combine Proposition 9.1 (first part) with Propositions 5.1–5.2 for the case $2 \le s < 3$, if we do not care about polynomial factors in front of $e^{-t^2/2}$.

Corollary 9.2. *There is a function $T_z \to +\infty$ as $z \to 0$ $(0 \le |z| \le 1)$ such that in the interval $|t| \le T_z$, for all $p = 0, 1, \dots, m$, $m = [s]$,*

$$\big|v_z^{(p)}(t) - u_m^{(p)}(t)\big| \le \varepsilon(z)|z|^{s-2}\,e^{-t^2/4}$$

with $\varepsilon(z) \to 0$. Moreover, up to some constant $c > 0$, one can choose $T_z = c\,|z|^{-1/3}$ in case $s \ge 3$ and $T_z = c\,|z|^{-(s-2)/s}$ in case $2 < s < 3$.

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