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Non-Uniform Bounds in Local Limit Theorems in Case of Fractional Moments. II

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Abstract—Edgeworth-type expansions for convolutions of probability densities and powers of the characteristic functions with non-uniform error terms are established for i.i.d. random variables with finite (fractional) moments of order $s \geq 2$, where s may be noninteger.

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This paper is a continuation of [1]. We continue the numeration of equations and theorems (as well as propositions and lemmas) and keep all definitions and notations as in the first part. Let us recall some of them.

We say that a complex-valued function v on the real line is s-times differentiable ($s \geq 2$), if it has continuous derivatives up to order $m = [s]$, and for any point t_0, as $t \to t_0$,

$$v^{(m)}(t) = v^{(m)}(t_0) + o(|t - t_0|^{s-m}).$$

In the sequel, we will always assume that $v(0) = 1$, $v'(0) = 0$, and $v''(0) = -1$.

Define the cumulants of v by

$$\gamma_k = \frac{d^k}{i^k\, dt^k}\, \log v(t)\big|_{t=0}, \qquad k = 1, \dots, m.$$

In particular, $\gamma_1 = 0$, $\gamma_2 = 1$.

The associated polynomials P_k are defined in case $m \geq 3$ for integers $1 \leq k \leq m-2$ by

$$P_k(t) \ = \sum_{p_1 + 2p_2 + \dots + kp_k = k} \frac{1}{p_1! \dots p_k!} \left(\frac{\gamma_3}{3!}\right)^{p_1} \dots \left(\frac{\gamma_{k+2}}{(k+2)!}\right)^{p_k} t^{k + 2(p_1 + \dots + p_k)},$$

where the summation is extended over all non-negative integer solutions $(p_1, \dots, p_k)$ to the equation $p_1 + 2p_2 + \dots + kp_k = k$.

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10. PROOF OF THEOREM 1.3

Let $v(t)$ be s-times differentiable, $s \geq 2$, such that $v(0) = 1$, $v'(0) = 0$, $v''(0) = -1$. Note that v is not vanishing in some interval containing the origin.

Let us consider the family of the functions

$$u_m(t, z) = e^{-t^2/2} \left(1 + \sum_{k=1}^{m-2} P_k(it) z^k\right),$$

where $m = [s]$, and the polynomials P_k are based on the cumulants $\gamma_3, \ldots, \gamma_m$ of v. In order to approximate the powers $v_n(t) = v(t/\sqrt{n})^n$, one uses the values $z = 1/\sqrt{n}$, leading to the approximating functions

$$u_m(t) \ = \ u_m(t, n^{-1/2}) \ = \ e^{-t^2/2} \left(1 + \sum_{k=1}^{m-2} P_k(it)\, n^{-k/2}\right).$$

On the other hand, when $z = 1$, we deal with the projection operators T_m, i.e., with the functions

$$e_m(t) = u_m(t, 1) = e^{-t^2/2} \left(1 + \sum_{k=1}^{m-2} P_k(it)\right).$$

Theorem 1.3 is a particular case of the following more general proposition.

Proposition 10.1. *For all $p = 0, 1 \ldots, m$, and all $|t| \leq cn^{1/6}$,*

$$\frac{d^p}{dt^p}\,(v_n(t) - u_m(t)) = n\,\frac{d^p}{dt^p}\left[\left(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right) e^{-t^2/2}\right] + r_n \tag{10.1}$$

with

$$|r_n| \leq \left(1 + |t|^{4m^2}\right) e^{-t^2/2} \left(\frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}}\right). \tag{10.2}$$

Here C, c and ε_n are some positive constants such that $\varepsilon_n \to 0$ as $n \to \infty$.

It is worthwhile noting that Proposition 9.1 and thus relation (1.8) can be obtained on the basis of (10.1)–(10.2) as well using the property that $v(t)$ and $e_m(t)$ have equal derivatives up to order m and both are s-times differentiable. However, we have chosen a different road of proof and will derive Proposition 10.1 by virtue of Proposition 9.1 (its second part).

In order to show how it applies, apply the binomial formula to obtain that

$$\begin{aligned} v_n(t) - u_m(t) \ = \ \sigma_{n1} + \sigma_{n2} + \sigma_{n3} = & e_m\Big(\frac{t}{\sqrt{n}}\Big)^n - u_m(t) \\ & + n\left[v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right] e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-1} \\ & + \sum_{k=2}^{n} C_n^k \left[v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right]^k e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}. \end{aligned}$$

Thus, σ_{n2} is almost the term which appears on the right-hand side of (10.1), provided that $e_m(\frac{t}{\sqrt{n}})^{n-1}$ is replaced with the characteristic function of the standard normal law.

The first term $\sigma_{n1} = e_m(\frac{t}{\sqrt{n}})^n - u_m(t)$ is of the same nature as $v_n(t) - u_m(t)$, assuming that e_m plays the role of v. At this point, let us recall that, by Proposition 7.4, $T_m e_m = e_m$, and moreover, that e_m generates the same polynomials P_k as v. Hence Proposition 9.1, being applied to e_m in place of v with $z = 1/\sqrt{n}$, provides the bound on the derivatives of $e_m(\frac{t}{\sqrt{n}})^n - u_m(t)$. Since e_m is analytic, the second assertion (9.2) of Proposition 9.1 is more accurate. Namely, if $e_m(t)$ is not vanishing in the interval $|t| \leq c$ (which is true with some constant $c > 0$ depending on the cumulants only), it gives:

Lemma 10.2. *For all $p = 0, 1, \ldots, m$ and all $|t| \leq cn^{1/6}$,*

$$\big|\sigma_{n1}^{(p)}(t)\big| \leq A\big(1 + |t|^{2m^2}\big)\, e^{-t^2/2}\, n^{-(m-1)/2}, \tag{10.3}$$

where c and A are some positive constants, depending on the cumulants $\gamma_3, \ldots, \gamma_m$.

Moreover, using a similar argument one can estimate the derivatives of the functions $e_m(t)^k$, which appear both in σ_{n2} and σ_{n3}. Apply (9.2) with $v = e_m$ and $z = 1/\sqrt{k}$ to get

$$\frac{d^p}{dt^p}\, e_m\Big(\frac{t}{\sqrt{k}}\Big)^k = \frac{d^p}{dt^p}\, e^{-t^2/2} \left(1 + \sum_{j=1}^{m-2} \frac{P_j(it)}{k^{j/2}}\right) + A\big(1 + |t|^{2m^2}\big) e^{-t^2/2}\, k^{-(m-1)/2}, \tag{10.4}$$

where $A = A_k(t)$ is a bounded quantity in the interval $|t| \leq ck^{1/6}$. Putting $\alpha = \sqrt{\frac{k}{n}}$ and replacing the variable t with αt, we obtain

$$\frac{d^p}{dt^p}\, e_m\Big(\frac{t}{\sqrt{n}}\Big)^k = \frac{d^p}{dt^p}\, e^{-\alpha^2 t^2/2} \left(1 + \sum_{j=1}^{m-2} \frac{P_j(i\alpha t)}{k^{j/2}}\right) + B\big(1 + |t|^{2m^2}\big) e^{-\alpha^2 t^2/2}, \tag{10.5}$$

where now $B = B_k(t)$ is bounded in $|t| \leq cn^{1/6}$. Every P_j is a polynomial of degree at most $3j \leq 3m$, so all its derivatives of order up to m can be bounded by $C(1 + |t|)^{3m}$ on the whole real line. Hence, using $\alpha \leq 1$ and $k \geq 1$, the same is true for the polynomial in the large bracket of (10.5). Using the Leibnitz rule, it then follows from (10.5) that:

Lemma 10.3. *For all $p = 0, 1, \ldots, m$, $k = 0, 1, \ldots, n$, in the interval $|t| \leq cn^{1/6}$*

$$\left|\frac{d^p}{dt^p}\, e_m\Big(\frac{t}{\sqrt{n}}\Big)^k\right| \leq C\big(1 + |t|^{2m^2}\big) e^{-kt^2/(2n)}$$

with some positive constants c and C.

The particular case $k = n - 1$ in (10.4) should be investigated in more detail by replacing (10.5) with a more accurate relation, namely

$$\frac{d^p}{dt^p}\, e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-1} = \frac{d^p}{dt^p}\, e^{-\alpha^2 t^2/2} \left(1 + \frac{1}{\sqrt{n-1}} \sum_{j=1}^{m-2} \frac{P_j(i\alpha t)}{(n-1)^{(j-1)/2}}\right) + \frac{B\big(1 + |t|^{2m^2}\big)}{\sqrt{n-1}}\, e^{-\alpha^2 t^2/2}$$

(assuming $n \geq 2$). Repeating the same argument concerning the growth of the polynomials P_j and their derivatives, and noting that, for $\alpha = \sqrt{\frac{n-1}{n}}$, we have

$$\left|\frac{d^p}{dt^p}\, e^{-\alpha^2 t^2/2} - \frac{d^p}{dt^p}\, e^{-t^2/2}\right| \leq \frac{C}{n}\,\big(1 + |t|^{p+2}\big)\, e^{-t^2/2}, \qquad |t| \leq \sqrt{n},$$

we arrive at the following bound (which also holds in the missing case $n = 1$):

Lemma 10.4. *For all $p = 0, 1, \ldots, m$, in the interval $|t| \leq cn^{1/6}$*

$$\left|\frac{d^p}{dt^p}\, e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-1} - \frac{d^p}{dt^p}\, e^{-t^2/2}\right| \leq \frac{C}{\sqrt{n}}\,\big(1 + |t|^{2m^2}\big)\, e^{-t^2/2}$$

with some positive constants c and C.

Finally, let us bound the derivatives of $y(t) = v(t) - e_m(t)$ and of its powers, which appear both in σ_{n2} and σ_{n3} as well. To this aim, one may appeal to Corollary 7.5, giving, as $t \to 0$,

$$y^{(r)}(t) = o\big(|t|^{s-r}\big) \quad \text{for any} \quad r = 0, \ldots, m. \tag{10.6}$$

In particular, $y(t)^k = o(|t|^{sk})$ for any $k \geq 1$. If $p \geq 1$, by the chain rule (cf. (2.3)), the pth derivative of $y(t)^k$ represents a linear combination of the terms

$$b(t) = y(t)^{k-(k_1+\dots+k_p)}\,(y'(t))^{k_1}\dots(y^{(p)}(t))^{k_p}$$

over all integer tuples $(k_1,\dots,k_p)$ such that $k_1 + 2k_2 + \dots + pk_p = p$ and $k_1 + \dots + k_p \leq k$ ($k_j \geq 0$). By (10.6), we have $b(t) = o(|t|^S)$, where

$$S = s\big(k - (k_1 + \dots + k_p)\big) + \sum_{r=1}^{p}(s-r)k_r = sk - p.$$

Hence $\frac{d^p}{dt^p}\,\big(v(t) - e_m(t)\big)^k = o\big(|t|^{sk-p}\big)$. Since $sk - p \leq sm$ for $1 \leq k \leq m$, we obtain:

Lemma 10.5. *Let $0 < \alpha < \frac{1}{2}$ and $c > 0$ be given. For some $\varepsilon_n \to 0$, for all $p = 0, 1, \dots, m$ and $k = 1, \dots, m$, we have, uniformly in the interval $|t| \leq cn^{\alpha}$,*

$$\left|\frac{d^p}{dt^p}\left(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right)^k\right| \leq \varepsilon_n\big(1 + |t|^{sm}\big)\,n^{-sk/2}.$$

Proof of Proposition 10.1. Using Lemmas 10.4 and 10.5 (with $k = 1$), we see that in σ_{n2} one may replace the term $e_m(\frac{t}{\sqrt{n}})^{n-1}$ with $e^{-t^2/2}$ at the expense of an error not exceeding

$$n \cdot \varepsilon_n\big(1 + |t|^{sm}\big)\,n^{-s/2} \cdot \frac{C}{\sqrt{n}}\,\big(1 + |t|^{2m^2}\big)\,e^{-t^2/2} \leq \frac{\varepsilon_n'}{n^{(s-1)/2}}\,\big(1 + |t|^{4m^2}\big)\,e^{-t^2/2}, \tag{10.7}$$

where $\varepsilon_n' \to 0$. The same is true for the first m derivatives of σ_{n2}. □

Now consider the products $y_k(t) = \big(v(\frac{t}{\sqrt{n}}) - e_m(\frac{t}{\sqrt{n}})\big)^k\,e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}$ appearing in σ_{n3} with $2 \leq k \leq n$. Writing

$$y_k^{(p)}(t) = \sum_{j=0}^{p} C_p^j\,\frac{d^j}{dt^j}\left(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\right)^k\,\frac{d^{p-j}}{dt^{p-j}}\,e_m\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}$$

and combining Lemmas 10.3 and 10.5 (which give estimates that are independent of j), we get

$$\begin{aligned}\big|y_k^{(p)}(t)\big| &\leq 2^p \cdot \varepsilon_n\big(1 + |t|^{sm}\big)\,n^{-sk/2} \cdot C\big(1 + |t|^{2m^2}\big)\,e^{-(n-k)t^2/(2n)}\\ &\leq \frac{\varepsilon_n''}{n^{sk/2}}\,\big(1 + |t|^{4m^2}\big)\,e^{-(n-k)t^2/(2n)},\end{aligned}$$

where $\varepsilon_n'' \to 0$. Therefore,

$$\begin{aligned}|\sigma_{n3}^{(p)}(t)| &\leq \sum_{k=2}^{n} C_n^k\,\big|y_k^{(p)}(t)\big| \leq \varepsilon_n''\,\big(1 + |t|^{4m^2}\big)\sum_{k=2}^{n} C_n^k\frac{1}{n^{sk/2}}\;e^{-(n-k)t^2/(2n)}\\ &= \varepsilon_n''\,\big(1 + |t|^{4m^2}\big)\,e^{-t^2/2}\,\Big(\big(1 + n^{-s/2}\,e^{t^2/2n}\big)^n - 1 - n^{-(s-2)/2}\,e^{t^2/2n}\Big).\end{aligned} \tag{10.8}$$

For $s > 2$, we have $\delta_n = n^{-s/2}\,e^{t^2/2n} = o(1/n)$ uniformly in the interval $|t| \leq cn^{1/6}$. So,

$$(1 + \delta_n)^n = e^{n\log(1+\delta_n)} = e^{n(\delta_n + O(\delta_n^2))} = 1 + n\delta_n + \frac{1}{2}\,(n\delta_n)^2 + nO\big(\delta_n^2\big).$$

Hence, for all n large enough, the expression in the large brackets in (10.8) does not exceed

$$\frac{1}{2}\,(n\delta_n)^2 + O\big(\delta_n^2\big) \leq \frac{1}{n^{s-2}} + O\Big(\frac{1}{n^{s-1}}\Big).$$

It remains to compare this bound with (10.7) and (10.3), and then we arrive at (10.2).

Finally, in the case $s = 2$, the expression in the large brackets in (10.8) is uniformly bounded in $|t| \leq cn^{1/6}$. Thus Proposition 10.1 is proved.

11. LIOUVILLE FRACTIONAL INTEGRALS AND DERIVATIVES

In this section we recall basic definitions and some results on Liouville fractional integrals and derivatives, and refer to [4], [3] for proofs and a more detailed exposition. At the end of the section we also formulate some special estimates for such operators. The proof of Proposition 11.3 below is rather routine and is therefore postponed to the next section.

Let α denote a real number with $0 < \alpha < 1$, and let $y = y(t)$ denote a (measurable) function defined for $t > 0$. The Liouville left- and right-sided fractional integrals on the positive half-axis $\mathbf{R}^+ = (0, +\infty)$ of order α are defined by

$$(I^\alpha_{0+} y)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{y(t)\,dt}{(x-t)^{1-\alpha}}, \qquad (I^\alpha_{-} y)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \frac{y(t)\,dt}{(t-x)^{1-\alpha}} \quad (x > 0).$$

The equalities are understood in the usual way (as Lebesgue integrals), if y is sufficiently "nice". According to a theorem by Hardy and Littlewood, I^α_{0+} and I^α_- are extended and act as bounded linear operators from $L^p(\mathbf{R}^+)$ to $L^q(\mathbf{R}^+)$, where $1 \le p, q \le +\infty$, if and only if $p < \frac{1}{\alpha}$ and $q = \frac{p}{1-\alpha p}$. They represent particular cases of the so-called Liouville (or Riemann–Liouville) fractional calculus operators.

The Liouville left- and right-sided fractional derivatives on the positive half-axis are defined by

$$(D^\alpha_{0+} y)(x) = \frac{d}{dx}\,(I^{1-\alpha}_{0+} y)(x), \qquad (D^\alpha_{-} y)(x) = \frac{d}{dx}\,(I^{1-\alpha}_{-} y)(x) \quad (x > 0).$$

The equalities are valid for sufficiently "nice" functions, including the class $C_0^\infty(\mathbf{R}^+)$ of all infinitely differentiable functions on $\mathbf{R}^+$ with a compact support (which can be used to approximate functions from larger spaces).

For example, for any complex number λ such that $\mathrm{Re}(\lambda) > 0$,

$$(I^\alpha_- e^{-\lambda t})(x) = \lambda^{-\alpha}\, e^{-\lambda x}, \qquad (D^\alpha_- e^{-\lambda t})(x) = \lambda^{\alpha}\, e^{-\lambda x}, \tag{11.1}$$

where the principal value of the power functions is used.

We cite two standard facts about these operators (see [3], p. 75 and p. 83).

Proposition 11.1. *For all sufficiently "good" functions y on $\mathbf{R}^+$,*

$$(D^\alpha_{0+} I^\alpha_{0+} y)(x) = y(x), \qquad (D^\alpha_- I^\alpha_- y)(x) = y(x).$$

The equalities are extended to the space $L^1(\mathbf{R}^+)$. Moreover, if additionally $y(x) = o(x^\alpha)$ for $x \to 0$, then

$$(I^\alpha_{0+} D^\alpha_{0+} y)(x) = y(x).$$

Define the linear spaces $I^\alpha_{0+}(L^p(\mathbf{R}^+))$ and $I^\alpha_-(L^p(\mathbf{R}^+))$ as the images of $L^p(\mathbf{R}^+)$ under the operators I^α_{0+} and I^α_-, respectively.

Proposition 11.2. *For all sufficiently "good" functions f and g on $\mathbf{R}^+$,*

$$\int_0^{+\infty} f(x)\,(D^\alpha_{0+} g)(x)\,dx = \int_0^{+\infty} g(x)\,(D^\alpha_- f)(x)\,dx. \tag{11.2}$$

The equality may be extended to all $f \in I^\alpha_-(L^p(\mathbf{R}^+))$ and $g \in I^\alpha_{0+}(L^q(\mathbf{R}^+))$ with $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.

This is a formula for fractional integration by parts.

Now, let V be a function of bounded variation on the real line, also viewed as a finite measure, and denote by $|V|$ its variation (as a measure). Define the Fourier–Stieltjes transform

$$\hat{V}(x) = \int_{-\infty}^{+\infty} e^{itx}\, dV(t), \quad x \in \mathbf{R}.$$

For our purposes, the following proposition will play a crucial role in the study of the local limit theorem with fractional moments.

Proposition 11.3. *Let $g(x) = \hat{V}(x)\, h(x)$, where $h(x)$ is a continuously differentiable function on the real line such that, for a given integer $m \geq 0$ and $0 < \alpha < 1$, as $|x| \to \infty$,*

$$|h(x)| + |h'(x)| = O\big(|x|^{-(2+m+\alpha)}\big).$$

If $\int_{-\infty}^{+\infty} |t|^{m+\alpha}\, d|V|(t) < +\infty$ and $\hat{V}^{(k)}(0) = 0$ for all $k = 0, \dots, m$, then $(D^{\alpha}_{0+} g)(x)$ exists for all $x > 0$ and satisfies with some constant C, independent of x,

$$\big|(D^{\alpha}_{0+} g)(x)\big| \leq \frac{C}{(1+x)^{1+\alpha}} \int_{-\infty}^{+\infty} \min\big\{|u|, |u|^{\alpha'}\big\}\, |u|^m\, |V|(du). \tag{11.3}$$

Here $\alpha' = \alpha$ in case $m = 0$, and $\alpha' = 0$ in case $m \geq 1$. In addition, for all t real,

$$\int_0^{+\infty} e^{itx}\, (D^{\alpha}_{0+} g)(x)\, dx = (-it)^{\alpha} \int_0^{+\infty} e^{itx}\, g(x)\, dx. \tag{11.4}$$

More precisely, the Gaussian function $h(x) = e^{-x^2/2}$ and its derivatives will only be needed in this proposition to obtain the desired decay for the inverse Fourier–Stieltjes transform.

Proposition 11.4. *For all functions V and h as in Proposition 11.3, for all t real,*

$$\left| \int_{-\infty}^{+\infty} e^{itx}\, \hat{V}(zx)\, h(x)\, dx \right| \leq \frac{|z|^{m+\alpha}}{(1+|t|)^{\alpha}}\, \varepsilon(z), \tag{11.5}$$

where $\varepsilon(z)$ is bounded in $|z| \leq 1$ and satisfies $\varepsilon(z) \to 0$ as $z \to 0$.

Proof. Let $0 < z \leq 1$. We apply Proposition 11.3 with the function $V_z(u) = V(u/z)$ in place of V in which case $\hat{V}_z(x) = \hat{V}(zx)$. Then, for the function $g_z(x) = \hat{V}(zx)\, h(x)$, the fractional derivative $(D^{\alpha}_{0+} g_z)(x)$ exists for all $x > 0$ and satisfies (11.3).

In order to unite both cases, use $|u|^{\alpha'} \leq |u|^{\alpha}$ for $|u| \geq 1$ and write (11.3) in a slightly weaker form

$$\big|(D^{\alpha}_{0+} g_z)(x)\big| \leq \frac{C z^m}{(1+x)^{1+\alpha}}\, \delta(z), \tag{11.6}$$

where

$$\delta(z) = \int_{-\infty}^{+\infty} \min\big\{|zu|, |zu|^{\alpha}\big\}\, |u|^m\, |V|(du).$$

This integral is finite and behaves like $o(z^{\alpha})$ as $z \to 0$. Indeed, split it into the two integrals in terms of the (finite positive) measure $W(du) = |u|^m\, |V|(du)$ as

$$\delta(z) = z^{\alpha} I_0(z) + z^{\alpha} I_1(z) = z^{\alpha} \int_{|u| \leq 1/z} z^{1-\alpha} |u|\, dW(u) + z^{\alpha} \int_{|u| > 1/z} |u|^{\alpha}\, dW(u). \tag{11.7}$$

By the moment assumption on V, we have $\int |u|^\alpha \, dW(u) < +\infty$, so $I_1(z) \to 0$ as $z \to 0$. As for the first integral, note that $z^{1-\alpha}|u| \le |u|^\alpha$ in the region $|u| \le 1/z$. Hence the functions $f_z(u) = z^{1-\alpha}|u| \, 1_{\{|u| \le 1/z\}}$ have an integrable majorant $f(u) = |u|^\alpha$ with respect to W. Since also $f_z(u) \to 0$ as $z \to 0$, one may apply the Lebesgue dominated convergence theorem, which gives $I_0(z) = \int f_z \, dW \to 0$. Thus, from (11.6)–(11.7),

$$\left|(D^\alpha_{0+} g_z)(x)\right| \le \frac{C z^{m+\alpha}}{(1+x)^{1+\alpha}} \, \varepsilon(z), \tag{11.8}$$

where $\varepsilon(z) = \frac{\delta(z)}{z^\alpha} \to 0$ as $z \to 0$ and $\sup_{0<z\le 1} \varepsilon(z) < +\infty$.

Now, using the bound (11.8) in (11.4), we get $|\int_0^{+\infty} e^{itx} \, g_z(x) \, dx| \le \frac{C z^{m+\alpha}}{\alpha |t|^\alpha} \, \varepsilon(z)$. Obviously, a similar inequality will hold as well when integrating over the negative half-axis. Therefore,

$$\left|\int_{-\infty}^{+\infty} e^{itx} \, g_z(x) \, dx\right| \le \frac{2C z^{m+\alpha}}{\alpha |t|^\alpha} \, \varepsilon(z).$$

This estimate implies (11.5) in case of large values of $|t|$, say, when $|t| \ge 1$. The remaining range $|t| \le 1$ can be treated by straightforward arguments.

By the assumption on the decay of h, its Fourier transform $\hat{h}(t) = \int_{-\infty}^{+\infty} e^{itx} \, h(x) \, dx$ is well defined, bounded, and has bounded continuous derivatives up to order $m+1$. Introduce Taylor's approximation for $\hat{h}$ up to order m at a given point t, i.e., the function

$$(S_m \hat{h})(t, u) = \sum_{k=0}^m \frac{\hat{h}^{(k)}(t)}{k!} \, u^k, \qquad u \in \mathbf{R}.$$

From Taylor's theorem it follows that

$$\left|\hat{h}(t+u) - (S_m \hat{h})(t, u)\right| \le M \min\left\{|u|^m, |u|^{m+1}\right\} \tag{11.9}$$

with some constant M independent of t. Now write

$$\hat{g}_z(t) \equiv \int_{-\infty}^{+\infty} e^{itx} \, g_z(x) \, dx = \int_{-\infty}^{+\infty} e^{itx} \, \hat{V}(zx) \, h(x) \, dx = \int_{-\infty}^{+\infty} \hat{h}(t + zu) \, dV(u).$$

The assumption $\hat{V}(0) = \dots = \hat{V}^{(m)}(0) = 0$ implies that $\int_{-\infty}^{+\infty} (S_m \hat{h})(t, u) \, dV(u) = 0$ for all t. Therefore, using (11.9), we finally get

$$\begin{aligned} |\hat{g}_z(t)| &= \left|\int_{-\infty}^{+\infty} \left(\hat{h}(t+zu) - (S_m \hat{h})(t, zu)\right) dV(u)\right| \\ &\le M \int_{-\infty}^{+\infty} \min\left\{|zu|^m, |zu|^{m+1}\right\} d|V|(u) = o(z^{m+\alpha}). \end{aligned}$$

Note that the last relation has been already discussed in the previous step.

Thus Proposition 11.4 is proved. □

Remark 11.5. The second part of the above proof also covers the limit case $\alpha = 0$ of the inequality (11.5), which may be written as

$$\left|\int_{-\infty}^{+\infty} e^{itx} \, \hat{V}(zx) \, h(x) \, dx\right| \le |z|^m \, \varepsilon(z). \tag{11.10}$$

More precisely, for this assertion we only need the assumptions $\int_{-\infty}^{+\infty} |u|^m \, |V|(du) < +\infty$, $\hat{V}(0) = \dots = \hat{V}^{(m)}(0) = 0$, and $\int_{-\infty}^{+\infty} |x|^{m+1} \, |h(x)| \, dx < +\infty$. Proposition 11.3 is irrelevant in this case.

12. FOURIER TRANSFORMS AND FRACTIONAL DERIVATIVES

In this section we give the proof of Proposition 11.3. By its very definition,

$$\Gamma(\alpha)\,(D_{0+}^{\alpha} g)(x) = \frac{d}{dx} \int_0^x \frac{h(t)}{(x-t)^{\alpha}}\, \hat{V}(t)\, dt, \tag{12.1}$$

provided that the derivative exists (where $0 < \alpha < 1$).

Given $m \geq 0$ integer, introduce the function of the real variable

$$\eta_m(t) = e^{it} - \sum_{k=0}^{m} \frac{(it)^k}{k!}.$$

The assumptions on V imply that the first m moments of the measure V are vanishing, i.e., $\int_{-\infty}^{\infty} u^k \, dV(u) = 0$ for $k = 0, \dots, m$. Hence

$$\hat{V}(t) = \int_{-\infty}^{\infty} \eta_m(tu)\, dV(u).$$

Respectively, changing the variable in (12.1) and applying Fubini's theorem, one may write

$$\begin{aligned}\Gamma(\alpha)\,(D_{0+}^{\alpha} g)(x) &= \frac{d}{dx} \int_0^x \frac{h(t)}{(x-t)^{\alpha}} \left[\int_{-\infty}^{+\infty} \eta_m(tu)\, dV(u) \right] dt \\ &= \frac{d}{dx} \int_{-\infty}^{+\infty} \left[\int_0^x h(x-t)\, \eta_m\big((x-t)u\big)\, \frac{dt}{t^{\alpha}} \right] dV(u).\end{aligned}$$

We intend to move the differentiation inside the outer integral. To justify this step, consider the derivative with respect to the inner integral,

$$I(x,u) = \frac{d}{dx} \int_0^x h(x-t)\, \eta_m\big((x-t)u\big)\, \frac{dt}{t^{\alpha}}.$$

Lemma 12.1. *Let $h(x)$ denote a continuously differentiable function on the real line such that $|h(x)| + |h'(x)| = O(x^{-(2+m+\alpha)})$ as $|x| \to \infty$. Then, for all $u \in \mathbf{R}$ and $x > 0$,*

$$|I(x,u)| \leq C(1+x)^{-(1+\alpha)} \, \min\{|u|,\, |u|^{\alpha}\} \qquad (m = 0),$$
$$|I(x,u)| \leq C(1+x)^{-(1+\alpha)} \, \min\{|u|^m, |u|^{m+1}\} \qquad (m \geq 1)$$

with some constant C depending on h and α only.

Proof. Put $\xi_u(t) = h(t)\eta_m(ut)$, so that

$$I(x,u) = \frac{d}{dx} \int_0^x \xi_u(x-t)\, \frac{dt}{t^{\alpha}}.$$

In this case we may interchange differentiation and integration. Thus, using

$$\frac{d}{dx}\, \xi_u(x-t) = -\frac{d}{dt}\, \xi_u(x-t)$$

together with $\eta_m(0) = 0$, we can write

$$I(x,u) = -\int_0^x \xi_u'(x-t)\,\frac{dt}{t^\alpha}.$$

Assume that $x \geq 1$ and split the integration domain into two regions such that

$$I(x,u) = I_0(x,u) + I_1(x,u) = -\int_0^1 \xi_u'(x-t)\,\frac{dt}{t^\alpha} - \int_1^x \xi_u'(x-t)\,\frac{dt}{t^\alpha}.$$

The integral I_1.

Integrating by parts, we have

$$I_1(x,u) = \alpha \int_1^x \xi_u(x-t)\,\frac{dt}{t^{1+\alpha}} - \xi_u(x-1). \tag{12.2}$$

To analyze this integral, we use the elementary bound

$$|\eta_m(t)| \leq 4 \min\left\{|t|^m, |t|^{m+1}\right\}, \qquad t \in \mathbf{R}. \tag{12.3}$$

Indeed, from Taylor's formula it follows that $|\eta_m(t)| \leq \frac{|t|^{m+1}}{(m+1)!}$. This settles (12.3) in case $|t| \leq 1$. In the other case $|t| \geq 1$, just write

$$|\eta_m(t)| \leq 1 + \sum_{k=0}^m \frac{|t|^k}{k!} \leq |t|^m \left(1 + \sum_{k=0}^m \frac{1}{k!}\right) < (1+e)\,|t|^m,$$

thus proving (12.3).

This bound implies that

$$|\xi_u(x-t)| \leq 4\,|h(x-t)| \min\left\{|(x-t)u|^m, |(x-t)u|^{m+1}\right\}. \tag{12.4}$$

By the assumption on h, we have $|h(x-1)| \leq Cx^{-(2+m+\alpha)}$, so, by (12.4),

$$|\xi_u(x-1)| \leq Cx^{-(1+\alpha)} \min\left\{|u|^m, |u|^{m+1}\right\} \tag{12.5}$$

with some constant C.

In the region $1 \leq t \leq x_1 = \frac{1+x}{2}$, we use the bound $|h(x-t)| \leq Cx^{-(2+m+\alpha)}$ with a constant independent of t and x. Hence, by (12.4), in this region

$$|\xi_u(x-t)| \leq Cx^{-(1+\alpha)} \min\left\{|u|^m, |u|^{m+1}\right\}$$

and

$$\int_1^{x_1} |\xi_u(x-t)|\,\frac{dt}{t^{1+\alpha}} \leq Cx^{-(1+\alpha)} \min\left\{|u|^m, |u|^{m+1}\right\}.$$

In the second region $x_1 \leq t \leq x$, just use $\frac{1}{t^{1+\alpha}} \leq Cx^{-(1+\alpha)}$. Then, by (12.4),

$$\begin{aligned}\int_{x_1}^x |\xi_u(x-t)|\,\frac{dt}{t^{1+\alpha}} &\leq 4Cx^{-(1+\alpha)}\,|u|^m \int_{x_1}^x |h(x-t)|\,(x-t)^m\,dt \\ &= 4Cx^{-(1+\alpha)}\,|u|^m \int_0^{(x-1)/2} |h(t)|\,t^m\,dt \;\leq\; C'x^{-(1+\alpha)}\,|u|^m,\end{aligned}$$

since the last integral is uniformly bounded. Similarly, again by (12.4),

$$\int_{x_1}^{x} |\xi_u(x-t)| \, \frac{dt}{t^{1+\alpha}} \leq 4Cx^{-(1+\alpha)} \, |u|^{m+1} \int_{x_1}^{x} |h(x-t)| \, (x-t)^{m+1} \, dt$$
$$\leq C' x^{-(1+\alpha)} \, |u|^{m+1}.$$

Collecting the bounds for the two regions, we get

$$\int_{1}^{x} |\xi_u(x-t)| \, \frac{dt}{t^{1+\alpha}} \leq Cx^{-(1+\alpha)} \min\left\{|u|^m, |u|^{m+1}\right\}$$

with some constant C. Applying it together with (12.5) in (12.2) we arrive at

$$I_1(x,u) \leq Cx^{-(1+\alpha)} \min\left\{|u|^m, |u|^{m+1}\right\}. \tag{12.6}$$

The integral I_0.

Now, let us turn to the integral $I_0(x,u) = -\int_0^1 \xi_u'(x-t) \, \frac{dt}{t^\alpha}$. After differentiation and using the identity $\eta_m' = i\eta_{m-1}$ (with the convention that $\eta_{-1}(t) = e^{it}$) one may represent it as $I_0 = I_{0,1} + I_{0,2}$, where

$$I_{0,1}(x,u) = -iu \int_0^1 \eta_{m-1}((x-t)u) \, h(x-t) \, \frac{dt}{t^\alpha},$$

$$I_{0,2}(x,u) = -\int_0^1 \eta_m((x-t)u) \, h'(x-t) \, \frac{dt}{t^\alpha}.$$

By (12.3), since $x - t \leq x$,

$$|I_{0,2}(x,u)| \leq 4 \int_0^1 |h'(x-t)| \, \min\left\{|xu|^m, |xu|^{m+1}\right\} \frac{dt}{t^\alpha}.$$

Using the assumption $h'(x) = O\big(x^{-(2+m+\alpha)}\big)$, we get

$$|I_{0,2}(x,u)| \leq Cx^{-(1+\alpha)} \, \min\left\{|u|^m, |u|^{m+1}\right\} \tag{12.7}$$

with some constant C.

As for the integral $I_{0,1}$, first rewrite it as $I_{0,1}(x,u) = -iu \int_0^1 h(x-t) \, d\zeta_{m-1}(t)$, where

$$\zeta_{m-1}(t) = \int_0^t \frac{\eta_{m-1}((x-w)u)}{w^\alpha} \, dw.$$

Integrating by parts one may represent it as $I_{0,1} = I_{0,1,1} + I_{0,1,2}$, where

$$I_{0,1,1}(x,u) = -iu \, h(x-1) \, \zeta_{m-1}(1),$$

$$I_{0,1,2}(x,u) = -iu \int_0^1 h'(x-t) \, \zeta_{m-1}(t) dt.$$

Claim. *For all $x \geq 1$, $t \in [0,1]$, we have $|\zeta_{-1}(t)| \leq C \min\{1, |u|^{\alpha-1}\}$ with some constant C, while in case $m \geq 1$,*

$$|\zeta_{m-1}(t)| \leq Cx^m \min\left\{|u|^{m-1}, \, |u|^m\right\}.$$

Proof. If $m = 0$, whenever $u \neq 0$,

$$|\zeta_{-1}(t)| = \left| \int_0^t e^{-iwu} \frac{dw}{w^\alpha} \right| = |u|^{\alpha-1} \left| \int_0^{tu} e^{-iw} \frac{dw}{w^\alpha} \right| \leq C\,|u|^{\alpha-1}.$$

On the other hand, $|\zeta_{-1}(t)| \leq \int_0^1 \frac{dw}{w^\alpha}$, so $|\zeta_{-1}(t)| \leq C \min\{1, |u|^{\alpha-1}\}$.

If $m \geq 1$, we just appeal to the estimate (12.4), which, for all $w \in (0,1)$ and $x \geq 1$, yields

$$\begin{aligned} |\eta_{m-1}((x-w)u)| &\leq 4 \min\left\{(x-w)^{m-1}\,|u|^{m-1},\ (x-w)^m\,|u|^m\right\} \\ &\leq 4x^m \min\left\{|u|^{m-1},\ |u|^m\right\}. \end{aligned}$$

It immediately implies the desired estimate. □

Continuation of the proof of Lemma 12.1. Now, by the assumption on h and using the claim in case $m = 0$, we obtain

$$|I_{0,1}(x,u)| \leq |I_{0,1,1}(x,u)| + |I_{0,1,2}(x,u)| \leq C\,x^{-(1+\alpha)} \min\left\{|u|, |u|^\alpha\right\}.$$

Similarly, in case $m \geq 1$,

$$\begin{aligned} |I_{0,1,1}(x,u)| + |I_{0,1,2}(x,u)| &\leq C|u|\left(|h(x-1)| + |h'(x-1)|\right) \cdot x^m \min\left\{|u|^{m-1},\ |u|^m\right\} \\ &\leq C\,x^{-(1+\alpha)} \min\left\{|u|^m,\ |u|^{m+1}\right\}. \end{aligned}$$

Thus

$$\begin{aligned} |I_{0,1}(x,u)| &\leq Cx^{-(1+\alpha)} \min\left\{|u|, |u|^\alpha\right\} \qquad (m = 0), \\ |I_{0,1}(x,u)| &\leq Cx^{-(1+\alpha)} \min\{|u|^m, |u|^{m+1}\} \qquad (m \geq 1). \end{aligned}$$

Taking into account (12.7) and (12.6), we arrive at similar bounds for $I_0(x,u)$ and $I(x,u)$, which are equivalent forms of the desired bounds in the lemma in case $x \geq 1$.

Finally, let us only note that the case $0 < x < 1$ may be treated in a similar manner (with simpler estimates). Thus Lemma 12.1 is proved. □

Proof of Proposition 11.3. Finally, we are prepared to justify the differentiation step. Define

$$\psi(x) = \int_{-\infty}^{+\infty} \left[\int_0^x \xi_u(x-t)\,\frac{dt}{t^\alpha} \right] dV(u),$$

where, as before, $\xi_u(t) = h(t)\eta_m(tu)$. Given $x > 0$ and $\varepsilon_n \to 0$ (with $\varepsilon_n \neq 0$, $x + \varepsilon_n > 0$), write

$$\frac{\psi(x+\varepsilon_n) - \psi(x)}{\varepsilon_n} = \int_{-\infty}^{+\infty} \left[\frac{1}{\varepsilon_n} \int_x^{x+\varepsilon_n} I(y,u)\,dy \right] dV(u).$$

By Lemma 12.1 in case $m = 0$, the expression in the square brackets is bounded in absolute value by

$$C\,\min\{|u|, |u|^\alpha\} \left| \frac{1}{\varepsilon_n} \int_x^{x+\varepsilon_n} (1+y)^{-(1+\alpha)}\,dy \right| \leq C\,\min\{|u|, |u|^\alpha\}.$$

On the right-hand side the function is integrable with respect to the measure $|V|$ according to the moment condition on the function V. A similar conclusion holds in case $m \geq 1$ (with appropriate modifications in the estimate). Therefore, one may apply the Lebesgue dominated convergence theorem, which gives

$$\psi'(x) = \lim_{n\to\infty} \frac{\psi(x+\varepsilon_n) - \psi(x)}{\varepsilon_n} = \int_{-\infty}^{+\infty} I(x,u)\,dV(u).$$

Thus, the fractional derivative

$$(D_{0+}^{\alpha} g)(x) = \frac{1}{\Gamma(\alpha)}\,\psi'(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} I(x,u)\,dV(u)$$

is well defined for all $x > 0$. Moreover, from Lemma 12.1 we also obtain that

$$|(D_{0+}^{\alpha} g)(x)| \leq C\,(1+x)^{-(1+\alpha)} \int_{-\infty}^{+\infty} \min\{|u|^{m_1}, |u|^{m_2}\}\,d|V|(u),$$

where $m_1 = 1$, $m_2 = \alpha$ in case $m = 0$, and $m_1 = m$, $m_2 = m+1$ in case $m \geq 1$.

This proves the first assertion and the inequality (11.3) in Proposition 11.3. For the second assertion, apply Proposition 11.1 and the formula (11.2) for the fractional integration by parts with the functions $f(x) = e^{-(\varepsilon - it)\,x}$ $(\varepsilon > 0)$ and use the second formula in (11.1) with $\lambda = \varepsilon - it$ for the fractional derivatives of f. Then we obtain

$$\int_0^{+\infty} e^{-(\varepsilon - it)\,x}\,(D_{0+}^{\alpha} g)(x)\,dx = (\varepsilon - it)^{\alpha} \int_0^{+\infty} e^{-(\varepsilon - it)\,x}\,g(x)\,dx.$$

Letting $\varepsilon \to 0$ and using the integrability of both g and $D_{0+}^{\alpha} g$ (due to (11.3)), we arrive in the limit at the required equality (11.4). Thus Proposition 11.3 is proved. $\square$

13. BINOMIAL DECOMPOSITION OF CONVOLUTIONS

We shall now treat the probability densities $\widetilde{\rho}_n$ in Theorem 1.2. The following procedure is known; a related approach has been used, e.g., in [5], [2] to study the central limit theorem with respect to the total variation distance.

Let $0 < c < 1$ be a prescribed number, $m = [s]$, and $n \geq m+2$.

Without loss of generality, one may assume that $n_0 = 1$, that is, $S_1 = X_1$ has a density, say, ρ, which may or may not be bounded. For definiteness, assume it is (essentially) unbounded, so that the integral

$$b = \int_{\{\rho(x) > M\}} \rho(x)\,dx$$

is positive for all $M > 0$. We choose M to be sufficiently large to satisfy, e.g., $0 < b < \frac{c}{2}$, which implies $2n^{m+1}\,b^{n-m-1} < c^n$ for all $n \geq n_1$ large enough.

Consider the decomposition

$$\rho(x) = ap(x) + bq(x),$$

where $a = 1 - b$ and $p(x)$, $q(x)$ are the normalized restrictions of ρ to the sets $\{\rho(x) \leq M\}$ and $\{\rho(x) > M\}$, respectively. Hence for the convolutions we have a binomial decomposition

$$\rho^{*n} = \sum_{k=0}^{n} C_n^k\,a^k b^{n-k}\,p^{*k} * q^{*(n-k)}.$$

Then split the above sum into two parts to get $\rho^{*n}(x) = p_n(x) + q_n(x)$, where

$$p_n = \sum_{k=m+2}^{n} C_n^k\,a^k b^{n-k}\,p^{*k} * q^{*(n-k)}, \qquad q_n = \sum_{k=0}^{m+1} C_n^k\,a^k b^{n-k}\,p^{*k} * q^{*(n-k)}.$$

Note that

$$\beta_n \equiv \int_{-\infty}^{+\infty} q_n(x)\,dx = \sum_{k=0}^{m+1} C_n^k\,a^k b^{n-k} \leq n^{m+1}\,b^{n-m-1} < \frac{c^n}{2} \qquad (n \geq n_1).$$

Finally define

$$\widetilde{\rho}_n(x) = \frac{\sqrt{n}}{1-\beta_n}\, p_n\big(x\sqrt{n}\big), \qquad \widetilde{v}_n(t) = \int_{-\infty}^{+\infty} e^{itx}\widetilde{\rho}_n(x)\,dx.$$

Let us recall that $\rho_n(x) = \sqrt{n}\,\rho^{*n}\big(x\sqrt{n}\big)$ has the characteristic function $v_n(t) = v(\frac{t}{\sqrt{n}})^n$, where v is the characteristic function of X_1. By construction, the densities $\widetilde{\rho}_n$ are bounded and provide a strong approximation for ρ_n. Namely, we immediately obtain:

Lemma 13.1. *For all $n \geq n_1$, $\int_{-\infty}^{+\infty} |\widetilde{\rho}_n(x) - \rho_n(x)|\,dx < c^n$. In particular, for all $t \in \mathbf{R}$,*

$$|\widetilde{v}_n(t) - v_n(t)| < c^n.$$

A similar inequality also holds for the first m derivatives of $\widetilde{v}_n$ and v_n with n large enough.

The last assertion of the lemma needs a more detailed explanation, which we postpone to the end of the section.

We will also need some integrability properties for $\widetilde{v}_n$ and their first m derivatives that are due to the boundedness of the probability density $p(x)$ and the finiteness of the mth absolute moment of X_1.

Lemma 13.2. *Let $\mathbf{E}\,|X_1|^m < +\infty$, $m \geq 2$. There exist positive constants A and σ, depending on X_1 and m, such that, for all $0 \leq T \leq \sqrt{n}$,*

$$\int_{\{|t|\geq T\}} |\widetilde{v}_n(t)|\,dt < A\,e^{-\sigma^2 T^2}. \tag{13.1}$$

A similar bound is also true for the first m derivatives of $\widetilde{v}_n$ (with arbitrary $n \geq m+2$).

Proof. Let $\hat{p}$, $\hat{q}$ denote the Fourier transforms of p and q, respectively. Then

$$\widetilde{v}_n(t) = \frac{1}{1-\beta_n} \sum_{k=m+2}^{n} C_n^k\, a^k b^{n-k}\, \hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^k\, \hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}. \tag{13.2}$$

By the Riemann–Lebesgue theorem,

$$\sup_{|t|\geq 1} |\hat{p}(t)| < \gamma, \quad \sup_{|t|\geq 1} |\hat{q}(t)| < \gamma \qquad (0 \leq \gamma < 1). \tag{13.3}$$

Hence, from (13.2), for all $|t| \geq \sqrt{n}$,

$$|\widetilde{v}_n(t)| < \frac{\gamma^{n-2}}{1-\beta_n} \Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^2 \sum_{k=m+2}^{n} C_n^k\, a^k b^{n-k} \leq \gamma^{n-2} \Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^2. \tag{13.4}$$

In addition, by the Plancherel theorem and using $p(x) \leq M/a$ we have

$$\int_{-\infty}^{+\infty} \Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^2 dt = 2\pi\sqrt{n} \int_{-\infty}^{+\infty} p(x)^2\,dt \leq \frac{2\pi M}{a}\,\sqrt{n}. \tag{13.5}$$

Therefore integrating the inequality (13.4) we get

$$\int_{\{|t|\geq\sqrt{n}\}} |\widetilde{v}_n(t)|\,dt < \frac{2\pi M}{a}\,\gamma^{n-2}\sqrt{n}. \tag{13.6}$$

On the other hand, since both p and q represent the densities of probability distributions with finite second moments, their characteristic functions near zero satisfy

$$|\hat{p}(t)| \leq e^{-\sigma^2 t^2}, \qquad |\hat{q}(t)| \leq e^{-\sigma^2 t^2} \qquad (|t| \leq 1) \tag{13.7}$$

with some constant $\sigma > 0$. Hence, for $|t| \le \sqrt{n}$, (13.2) gives the estimate $|\widetilde{v}_n(t)| \le e^{-\sigma^2 t^2}$ and

$$\int\limits_{\{T \le |t| \le \sqrt{n}\}} |\widetilde{v}_n(t)|\,dt \;\le \int\limits_{T \le |t| \le \sqrt{n}} e^{-\sigma^2 t^2}\,dt \;<\; \frac{1}{\sigma}\,e^{-\sigma^2 T^2}.$$

Together with (13.6) the latter yields

$$\int\limits_{\{|t| \ge T\}} |\widetilde{v}_n(t)|\,dt < \frac{1}{\sigma}\,e^{-\sigma^2 T^2} + \frac{2\pi M}{a}\,\gamma^{n-2}\sqrt{n}.$$

Finally, since for the values $0 \le T \le \sqrt{n}$ one always has $\gamma^{n-2}\sqrt{n} \le A_1\,e^{-\sigma_1^2 T^2}$ with some constants A_1 and $\sigma_1 > 0$, the desired bound (13.1) easily follows.

As for the derivatives, a bound of this type can be proved by similar arguments, so let us restrict ourselves to the basic case of the mth derivative (needed for the proof of Theorem 1.2).

The condition $\mathbf{E}|X_1|^m = \int |x|^m\,\rho(x)\,dx < +\infty$ implies a similar property for the densities $p(x)$ and $q(x)$. Hence $\hat{p}(t)$ and $\hat{q}(t)$ have continuous derivatives up to order m, bounded in absolute value by some common constant.

In view of (13.2), $\widetilde{v}_n^{(m)}(t)$ represents a linear combination of the terms

$$\frac{d^m}{dt^m}\left[\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^k\,\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}\right] = \sum_{r=0}^{m} C_m^r\,\frac{d^r}{dt^r}\left[\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^k\right]\frac{d^{m-r}}{dt^{m-r}}\left[\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}\right] \tag{13.8}$$

with integers $m+2 \le k \le n$. Given $0 \le r \le m$, by the chain rule (2.3), the rth derivative of the function $\hat{p}(\frac{t}{\sqrt{n}})^k$ represents a linear combination of the terms

$$n^{-r/2}\,\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^{k-(k_1+\dots+k_r)}\,\hat{p}'\Big(\frac{t}{\sqrt{n}}\Big)^{k_1}\dots\hat{p}^{(r)}\Big(\frac{t}{\sqrt{n}}\Big)^{k_r} \tag{13.9}$$

over all integer tuples $(k_1,\dots,k_r)$ such that $k_1 + 2k_2 + \dots + rk_r = r$ and $k_1 + \dots + k_r \le k$ ($k_j \ge 0$). Moreover, the coefficients in that linear combination do not depend on n, and the total number of such terms is bounded by a quantity that depends on m only.

Using $k_1 + \dots + k_r \le r \le m$ and the boundedness of the derivatives, the absolute value of the expression (13.9) as well as the sum of all such terms are bounded by $|\hat{p}(\frac{t}{\sqrt{n}})|^{k-m}$ up to a constant factor.

Similarly, the $(m-r)$th derivative of the function $\hat{q}(\frac{t}{\sqrt{n}})^{n-k}$ represents a linear combination of the terms

$$n^{-(m-r)/2}\,\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{(n-k)-(k_1+\dots+k_{m-r})}\,\hat{q}'\Big(\frac{t}{\sqrt{n}}\Big)^{k_1}\dots\hat{q}^{(m-r)}\Big(\frac{t}{\sqrt{n}}\Big)^{k_{m-r}} \tag{13.10}$$

over all integer tuples $(k_1,\dots,k_{m-r})$ such that $k_1 + 2k_2 + \dots + (m-r)k_{m-r} = m-r$ and $k_1 + \dots + k_{m-r} \le n-k$ ($k_j \ge 0$). Again, the coefficients in the linear combination do not depend on n, and the total number of such terms is bounded by a quantity depending on m only. Since $k_1 + \dots + k_{m-r} \le \min(n-k, m-r) \le \min(n-k, m)$, the absolute value of the expression (13.10) and the sum of all such terms are bounded by $|\hat{q}(\frac{t}{\sqrt{n}})|^{(n-k)-\min(n-k,m)}$ up to a constant factor.

Thus (13.8) is bounded in absolute value by

$$C\left|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\right|^{k-m}\left|\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)\right|^{(n-k)-\min(n-k,m)} \tag{13.11}$$

with some constant C depending on X_1 and m only. It then follows from (13.2) that

$$\big|\widetilde{v}_n^{(m)}(t)\big| \;\le\; \frac{C}{1-\beta_n}\sum_{k=m+2}^{n} C_n^k\,a^k b^{n-k}\left|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\right|^{k-m}\left|\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)\right|^{(n-k)-\min(n-k,m)}. \tag{13.12}$$

Now, like in the previous step, using (13.3), for all $|t| \geq \sqrt{n}$, we get

$$\big|\widetilde{v}_n^{(m)}(t)\big| \leq \frac{C}{1-\beta_n}\,\Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^2 \sum_{k=m+2}^n C_n^k\, a^k b^{n-k}\, \gamma^{(n-m-2)-\min(n-k,m)} \leq C\,\gamma^{n-2m-2}\,\Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^2.$$

Integrating this inequality with the help of (13.5), we obtain that

$$\int\limits_{\{|t|\geq\sqrt{n}\}} |\widetilde{v}_n^{(m)}(t)|\,dt < \frac{2\pi MC}{a}\,\gamma^{n-2m-2}\sqrt{n}. \tag{13.13}$$

In addition, using (13.7) for $|t| \leq \sqrt{n}$, the product (13.11) is bounded by $C\,e^{-d\sigma^2 t^2}$, where

$$d = \frac{1}{n}\,\big((n-m)-\min(n-k,m)\big) \geq d' = \frac{1}{n}\,\big((n-m)-\min(n-m-2,m)\big).$$

If $n \geq 2m+2$, then $d' = \frac{n-2m}{n} \geq \frac{1}{m+1}$. In the other case $m+2 \leq n < 2m+2$, we also have $d' = \frac{2}{n} \geq \frac{1}{m+1}$. Hence (13.11) is bounded by $C\,e^{-\sigma^2 t^2/(m+1)}$, and we derive from (13.12)

$$\big|\widetilde{v}_n^{(m)}(t)\big| \;\leq\; \frac{C}{1-\beta_n} \sum_{k=m+2}^n C_n^k\, a^k b^{n-k}\, e^{-\sigma^2 t^2/(m+1)} \;\leq\; C\,e^{-\sigma^2 t^2/(m+1)}.$$

It remains to integrate this inequality to get

$$\int\limits_{\{T\leq|t|\leq\sqrt{n}\}} |\widetilde{v}_n^{(m)}(t)|\,dt < C \int\limits_{T\leq|t|\leq\sqrt{n}} e^{-\sigma^2 t^2/(m+1)}\,dt < \frac{C\sqrt{m+1}}{\sigma}\,e^{-\sigma^2 T^2/(m+1)}.$$

Together with (13.13), it yields the desired estimate (13.1). Thus Lemma 13.2 is proved. □

Remark 13.3. If the density ρ is bounded, the decomposition procedure is not needed, and then Lemma 13.2 should read as follows. Let $\mathbf{E}\,|X|^m < +\infty$, $m \geq 2$, for a random variable having a bounded density. There exist constants A and $\sigma > 0$ such that for all $n \geq 2$

$$\int\limits_{\{|t|\geq T\}} \Big|v\Big(\frac{t}{\sqrt{n}}\Big)\Big|^n\,dt < A\,e^{-\sigma^2 T^2}, \qquad 0 \leq T \leq \sqrt{n}, \tag{13.14}$$

where v is the characteristic function of X. A similar bound holds as well for the first m derivatives of $v(\frac{t}{\sqrt{n}})^n$ with arbitrary $n \geq m+2$.

Proof of Lemma 13.1. By the construction, for all $n \geq n_1$,

$$\int_{-\infty}^{+\infty} \big|\widetilde{\rho}_n(x) - \rho_n(x)\big|\,dx \leq 2\beta_n < c^n,$$

so $|\widetilde{v}_n(t) - v_n(t)| < c^n$ as well. In order to extend this inequality to the derivatives, recall the representation (13.2) to write

$$\widetilde{v}_n(t) - v_n(t) = \frac{\beta_n}{1-\beta_n}\,\Sigma_1 - \Sigma_2, \tag{13.15}$$

where

$$\Sigma_1 = \sum_{k=m+2}^n C_n^k\, a^k b^{n-k}\, \hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^k\, \hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}, \qquad \Sigma_2 = \sum_{k=0}^{m+1} C_n^k\, a^k b^{n-k}\, \hat{p}\Big(\frac{t}{\sqrt{n}}\Big)^k\, \hat{q}\Big(\frac{t}{\sqrt{n}}\Big)^{n-k}.$$

As before, we will only consider the case of the mth derivative.

It was shown in the proof of Lemma 13.2 that, given $m+2 \leq k \leq m$, the function $\hat{p}(\frac{t}{\sqrt{n}})^k\, \hat{q}\big(\frac{t}{\sqrt{n}}\big)^{n-k}$ has the mth derivative bounded in absolute value by the expression (13.11). So, it is bounded by a constant C depending on X_1 and m only. In the general case including the values $0 \leq k \leq m+1$ (13.11) should be replaced with

$$C\,\Big|\hat{p}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^{\max(k-m,0)}\,\Big|\hat{q}\Big(\frac{t}{\sqrt{n}}\Big)\Big|^{(n-k)-\min(n-k,m)},$$

which is also bounded by C. Therefore, from (13.15),

$$\big|\widetilde{v}_n^{(m)}(t) - v_n^{(m)}(t)\big| \leq \frac{C\beta_n}{1-\beta_n} \sum_{k=m+2}^{n} C_n^k\, a^k b^{n-k} + C\sum_{k=0}^{m+1} C_n^k\, a^k b^{n-k} = 2C\beta_n \,<\, c^n,$$

where the last inequality holds true for all n starting with a certain n_1.

Thus, Lemma 13.1 is proved. □

14. PROOF OF THEOREMS 1.1 AND 1.2

We are prepared to make the last step in the proof of Theorems 1.1 and 1.2. Recall that $s \geq 2$, $m = [s]$, and put $\alpha = s - m$.

Let $v(t)$ be the characteristic function of X_1 and $v_n(t) = v(\frac{t}{\sqrt{n}})^n$ be the characteristic function of S_n. We will assume that all S_n have densities ρ_n (since only minor modifications have to be done in the more general case, where S_n have densities for all n large enough).

If ρ_{n_0} and therefore all ρ_n with $n \geq n_0$ are (essentially) bounded for some n_0, then there is no need to use the binomial decomposition of the previous section, and we put $\widetilde{\rho}_n = \rho_n$. This case corresponds to Theorem 1.1. Otherwise, if ρ_n are unbounded for all $n \geq 1$, then the binomial decomposition is applied to $\rho = \rho_1$, and we obtain the modified densities $\widetilde{\rho}_n$ together with the associated characteristic functions $\widetilde{v}_n$, which we considered in the previous section. Thus, the requirement c) in Theorem 1.2 is met.

The inversion formula.

The characteristic functions $\widetilde{v}_n$ have continuous bounded derivatives up to order m, which are integrable according to inequality (13.1) of Lemma 13.2 or (13.14) of Remark 13.3. Hence, by the inversion formula,

$$(ix)^p\, \widetilde{\rho}_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\, \widetilde{v}_n^{(p)}(t)\,dt, \qquad p = 0,1,\dots,m.$$

By the construction, the approximating functions $\varphi_m(x) = \varphi(x) + \sum_{k=1}^{m-2} q_k(x)\, n^{-k/2}$, which appear in the relation (1.3), have the integrable Fourier transform

$$u_m(t) = e^{-t^2/2}\,\Big(1 + \sum_{k=1}^{m-2} P_k(it)\, n^{-k/2}\Big).$$

Consequently, for all $p = 0,1,\dots,m$,

$$(ix)^p\,\big(\widetilde{\rho}_n(x) - \varphi_m(x)\big) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\,\big(\widetilde{v}_n^{(p)}(t) - u_m^{(p)}(t)\big)\,dt. \tag{14.1}$$

Our task is thus to give proper upper bounds on the absolute value of these integrals in the particular cases $p = 0$ and $p = m$.

What is rather standard, one should split the integration into two regions. Given $T_n \to +\infty$, $0 \leq T_n \leq \sqrt{n}$ (to be specified later on), let

$$I_{n,p} = \int_{|t|\leq T_n} e^{-itx}\,\big(\widetilde{v}_n^{(p)}(t) - u_m^{(p)}(t)\big)\,dt, \qquad J_{n,p} = \int_{|t|\geq T_n} e^{-itx}\,\big(\widetilde{v}_n^{(p)}(t) - u_m^{(p)}(t)\big)\,dt.$$

It should be clear that

$$\int\limits_{|t|\geq T_n} \left|u_m^{(p)}(t)\right| dt \leq Ae^{-\sigma^2 T_n^2}$$

with some positive constants A and σ depending on m. By Lemma 13.2 and Remark 13.3, we have a similar bound for $\widetilde{v}_n^{(p)}(t)$ whenever $n \geq m+2$, so

$$|J_{n,p}| \leq A\, e^{-\sigma^2 T_n^2}. \tag{14.2}$$

The integral $I_{n,p}$.

To treat this integral, we subtract and add $v_n^{(p)}(t)$ inside the integrand and apply Lemma 13.1 (the second part). Then it gives (using $T_n \leq \sqrt{n}$)

$$|I_{n,p}| \leq |I'_{n,p}| + c^n\sqrt{n},$$

where $0 < c < 1$ is the prescribed parameter in Theorem 1.2 and

$$I'_{n,p} = \int\limits_{|t|\leq T_n} e^{-itx}\big(v_n^{(p)}(t) - u_m^{(p)}(t)\big)\, dt.$$

Using (14.1)–(14.2), we obtain that, for all x,

$$|x|^p \left|\widetilde{\rho}_n(x) - \varphi_m(x)\right| \leq \frac{1}{2\pi}\, |I'_{n,p}| + A\, e^{-\sigma^2 T_n^2} + c^n\sqrt{n}, \tag{14.3}$$

up to some positive constants A and σ.

Proof of (1.4) in case $|x| \geq 1$. Note that for $|x| \leq 1$, the relation (1.4) follows from (1.3). As for the values $|x| \geq 1$, only the value $p = m$ is of interest in (14.3).

The integral $I'_{n,p}$ can be treated with the help of Theorem 1.3. It gives that in the interval $|t| \leq c_1 n^{1/6}$ with some constant $0 < c_1 \leq 1$ we have

$$v_n^{(p)}(t) - u_m^{(p)}(t) = n\,\frac{d^p}{dt^p}\bigg[\Big(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\Big)\, e^{-t^2/2}\bigg] + r_n,$$

where the remainder satisfies

$$|r_n| \leq e^{-t^2/4}\left(\frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}}\right).$$

Here C and ε_n are some positive constants such that $\varepsilon_n \to 0$ as $n \to \infty$. Hence assuming that $T_n \leq c_1 n^{1/6}$ and noting that $c^n\sqrt{n}$ will be absorbed by other remainder terms we get that

$$|x|^p \left|\widetilde{\rho}_n(x) - \varphi_m(x)\right| \leq \frac{n}{2\pi}\, |I''_{n,p}| + A\, e^{-\sigma^2 T_n^2} + \frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}}, \tag{14.4}$$

where

$$I''_{n,p} = \int\limits_{|t|\leq T_n} e^{-itx}\,\frac{d^p}{dt^p}\bigg[\Big(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\Big)\, e^{-t^2/2}\bigg]\, dt. \tag{14.5}$$

Now, one can differentiate inside the last integral, which will lead to the terms containing $e^{-t^2/2}$ up to polynomial factors (due to the property that v has m bounded derivatives). Hence integration in (14.5) may be extended to the whole real line at the expense of an error not exceeding $Ce^{-T_n^2/4}$. Hence (14.4) may be replaced with

$$|x|^p \left|\widetilde{\rho}_n(x) - \varphi_m(x)\right| \leq \frac{n}{2\pi}\, |I'''_{n,p}| + A\, e^{-\sigma^2 T_n^2} + \frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}}, \tag{14.6}$$

where

$$I'''_{n,p} = \int_{-\infty}^{+\infty} e^{-itx} \frac{d^p}{dt^p} \Big[\Big(v\Big(\frac{t}{\sqrt{n}}\Big) - e_m\Big(\frac{t}{\sqrt{n}}\Big)\Big) e^{-t^2/2} \Big] \, dt.$$

Letting $p = m$ and $w(t) = v(t) - e_m(t)$ and performing differentiation, rewrite the above integral as

$$I'''_{n,m} = \sum_{k=0}^{m} \frac{C_m^k}{n^{k/2}} \int_{-\infty}^{+\infty} e^{-itx}\, w^{(k)}\Big(\frac{t}{\sqrt{n}}\Big)\, h_{m-k}(t)\, dt, \tag{14.7}$$

where $h_{m-k}(t) = (e^{-t^2/2})^{(m-k)} = (-1)^{m-k} H_{m-k}(t)\, e^{-t^2/2}$.

Recall that $w(t) = \hat{V}(t)$ represents the Fourier transform of a finite signed measure, V, such that $\int_{-\infty}^{+\infty} |u|^{m+\alpha}\, |V|(du) < +\infty$ (where $|V|$ denotes the variation of V, treated as a positive finite measure). In addition, the first m derivatives of w are vanishing (cf. Section 7 and Proposition 7.4). Hence $w^{(k)}(t) = \hat{V}_k(t)$ represents the Fourier transform of a finite signed measure V_k such that $\int_{-\infty}^{+\infty} |u|^{(m-k)+\alpha}\, |V_k|(du) < +\infty$ and the first $m-k$ derivatives of w_k are vanishing. Therefore, we are in a position to apply Proposition 11.4 to the functions $\hat{V}_k(t)$ in place of $\hat{V}$, h_{m-k} in place of h, and with $m-k$ in place of the parameter m. Choosing $z = 1/\sqrt{n}$, the inequalities (11.5) and (11.10) (cf. Remark 11.5) give

$$\Big| \int_{-\infty}^{+\infty} e^{-itx}\, w^{(k)}\Big(\frac{t}{\sqrt{n}}\Big)\, h_{m-k}(t)\, dt \Big| \le \frac{\varepsilon_n}{n^{(m-k+\alpha)/2}}\, (1+|x|)^{-\alpha}$$

with some sequence $\varepsilon_n \to 0$ as $n \to \infty$. Applying this bound in (14.7) we obtain

$$|I'''_{n,m}| \le \frac{\varepsilon_n}{n^{s/2}}\, (1+|x|)^{-\alpha},$$

and then (14.6) with $p = m$ yields

$$|x|^s \left|\widetilde{\rho}_n(x) - \varphi_n(x)\right| \le \frac{\varepsilon_n}{n^{(s-2)/2}} + |x|^\alpha \bigg(A\, e^{-\sigma^2 T_n^2} + \frac{C}{n^{(m-1)/2}} + \frac{\varepsilon_n}{n^{s-2}} \bigg). \tag{14.8}$$

It remains to invoke information about the possible growth of T_n. But, as we have seen, one could choose T_n to be of order $n^{1/6}$ regardless of s. With this choice (14.8) leads to the announced inequality (1.4) of Theorem 1.2.

Proof of (1.3). Let us return to (14.3). The integral $I'_{n,p}$ can also be estimated by virtue of Proposition 9.1 and Propositions 5.1–5.2 (cf. Corollary 9.2). Namely, they give that

$$v_n^{(p)}(t) - u_m^{(p)}(t) = o(n^{-(s-2)/2})\, e^{-t^2/4}, \qquad p = 0, 1 \dots, m, \tag{14.9}$$

uniformly over all t in the intervals $|t| \le T_n$, where T_n are of order $n^{1/6}$ in case $s \ge 3$ and of order $n^{(s-2)/(2s)}$ in case $2 < s < 3$. If $s = 2$, we may only have $T_n \to +\infty$. Clearly, in all cases $I'_{n,p} = o(n^{-(s-2)/2})$, and (14.3) yields

$$|x|^p\, |\widetilde{\rho}_n(x) - \varphi_m(x)| \le o\big(n^{-(s-2)/2}\big) + A\, e^{-\sigma^2 T_n^2}. \tag{14.10}$$

It remains to apply this inequality with $p = 0$ and $p = m$.

Theorems 1.1–1.2 are thus proved. □

Remark. If $\mathbf{E}\,|X_1|^{m+1} < +\infty$, $m \ge 2$, but φ_m are constructed with the help of the same cumulants $\gamma_3, \dots, \gamma_m$ (like in the case $m \le s < m+1$), relation (1.3) for both Theorems 1.1 and 1.2 may be sharpened. Indeed, by Proposition 9.1 (second part), (14.9) should be replaced with a stronger relation

$$v_n^{(p)}(t) - u_m^{(p)}(t) = O(n^{-(m-1)/2})\, e^{-t^2/4}, \qquad p = 0, 1 \dots, m,$$

which holds uniformly in the intervals $|t| \le c_1 n^{1/6}$. Respectively, it provides a stronger version of (14.10), namely,

$$(1+|x|^m)\,|\widetilde{\rho}_n(x) - \varphi_m(x)| = O\big(n^{-(m-1)/2}\big).$$

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