Bounds on the deficit in the logarithmic Sobolev inequality ☆

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A B S T R A C T

The deficit in the logarithmic Sobolev inequality for the Gaussian measure is considered and estimated by means of transport and information-theoretic distances.

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1. Introduction

Let γ denote the standard Gaussian measure on the Euclidean space $\mathbf{R}^n$, thus with density

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$$\frac{d\gamma(x)}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$$

with respect to the Lebesgue measure. (Here and in the sequel $|x|$ stands for the Euclidean norm of a vector $x \in \mathbf{R}^n$.) One of the basic results in the Gaussian Analysis is the celebrated logarithmic Sobolev inequality

$$\int f \log f \, d\gamma - \int f \, d\gamma \log \int f \, d\gamma \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma, \qquad (1.1)$$

holding true for all positive smooth functions f on $\mathbf{R}^n$ with gradient ∇f. In this explicit form it was obtained in the work of L. Gross [22], initiating fruitful investigations around logarithmic Sobolev inequalities and their applications in different fields. See e.g. a survey by M. Ledoux [24] and the books [25,1] for a comprehensive account of such activities up to the end of 90s. One should mention that in an equivalent form – as a relation between the Shannon entropy and the Fisher information, (1.1) goes back to the work by A.J. Stam [35], see [1, Chapter 10].

The inequality (1.1) is homogeneous in f, so the restriction $\int f \, d\gamma = 1$ does not lose generality. It is sharp in the sense that the equality is attained, namely for all $f(x) = e^{l(x)}$ with arbitrary affine functions l on $\mathbf{R}^n$ (in which case the measures $\mu = f\gamma$ are still Gaussian). It is nevertheless of a certain interest to realize how large the difference between both sides of (1.1) is. This problem has many interesting aspects. For example, as was shown by E. Carlen in [11], which was perhaps a first address of the sharpness problem, for $f = |u|^2$ with a smooth complex-valued u such that $\int |u|^2 \, d\gamma = 1$, (1.1) may be strengthened to

$$\int |u|^2 \log |u|^2 \, d\gamma + \int |Wu|^2 \log |Wu|^2 \, d\gamma \leq 2 \int |\nabla u|^2 \, d\gamma,$$

where W denotes the Wiener transform of u. That is, a certain non-trivial functional may be added to the left-hand side of (1.1).

One may naturally wonder how to bound from below the deficit in (1.1), that is, the quantity

$$\delta(f) = \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma - \left[\int f \log f \, d\gamma - \int f \, d\gamma \log \int f \, d\gamma \right],$$

in terms of more explicit, like distribution-dependent characteristics of f showing its closeness to the extremal functions e^l (when $\delta(f)$ is small). Recently, results of this type have been obtained by A. Cianchi, N. Fusco, F. Maggi and A. Pratelli [13] in their study of the closely related isoperimetric inequality for the Gaussian measure. The work by E. Mossel and J. Neeman [28] deals with dimension-free bounds for the deficit in one functional form of the Gaussian isoperimetric inequality appearing in [7]. See also the subsequent paper by R. Eldan [17] where almost tight two-sided robustness bounds have

been derived. In [19,34] the authors deal with quantitative Brunn–Minkowski inequality (which is related to the isoperimetric problem in Euclidean space), while bounds on the deficit in the Sobolev inequalities can be found in e.g. [20,16] and in the Gagliardo–Nirenberg–Sobolev inequality in [12] (see also the references therein for more on the literature).

As for (1.1), one may also want to involve distance-like quantities between the measures $\mu = f\gamma$ and γ. This approach looks even more natural, when the logarithmic Sobolev inequality is treated as the relation between classical information-theoretic distances as

$$D(X|Z) \leq \frac{1}{2} I(X|Z). \tag{1.2}$$

To clarify this inequality, let us recall standard notations and definitions. If random vectors X and Z in $\mathbf{R}^n$ have distributions μ and ν with densities p and q, and μ is absolutely continuous with respect to ν, the relative entropy of μ with respect to ν is defined by

$$D(X|Z) = D(\mu|\nu) = \int p(x) \log \frac{p(x)}{q(x)} \, dx.$$

Moreover, if p and q are smooth, one defines the relative Fisher information

$$I(X|Z) = I(\mu|\nu) = \int \left| \frac{\nabla p(x)}{p(x)} - \frac{\nabla q(x)}{q(x)} \right|^2 p(x) \, dx.$$

Both quantities are non-negative, and although non-symmetric in (μ, ν), they may be viewed as strong distances of μ to ν. This is already demonstrated by the well-known Pinsker inequality [30], connecting D with the total variation norm:

$$D(\mu|\nu) \geq \frac{1}{2} \|\mu - \nu\|_{\mathrm{TV}}^2.$$

In the sequel, we mainly consider the particular case where Z is standard normal, so that $\nu = \gamma$ in the above formulas. And in this case, as easy to see, for $d\mu = f \, d\gamma$ with $\int f \, d\gamma = 1$, the logarithmic Sobolev inequality (1.1) turns exactly into (1.2).

The aim of this note is to develop several lower bounds on the deficit in this inequality, $\frac{1}{2} I(X|Z) - D(X|Z)$, by involving also transport metrics such as the quadratic Kantorovich distance (see e.g. [38])

$$W_2(X, Z) = W_2(\mu, \gamma) = \inf_{\pi} \left(\iint |x - z|^2 \, d\pi(x, z) \right)^{1/2}$$

(where the infimum runs over all probability measures on $\mathbf{R}^n \times \mathbf{R}^n$ with marginals μ and γ). More generally, one may consider the optimal transport cost

$$\mathcal{T}(X, Z) = \mathcal{T}(\mu, \gamma) = \inf_{\pi} \iint c(x - z) \, d\pi(x, z)$$

for various "cost" functions $c(x - z)$.

The metric W_2 is of weak type in the sense that it metrizes the weak topology in the space of probability measures on $\mathbf{R}^n$ (under proper moment constraints). It may be connected with the relative entropy by virtue of M. Talagrand's transport-entropy inequality

$$W_2(X, Z)^2 \leq 2D(X|Z), \tag{1.3}$$

cf. [36]. In view of (1.2), this also gives an a priori weaker transport-Fisher information inequality

$$W_2(X, Z) \leq \sqrt{I(X|Z)}. \tag{1.4}$$

In formulations below, we use the non-negative convex function

$$\Delta(t) = t - \log(1 + t), \quad t > -1,$$

and denote by Z a random vector in $\mathbf{R}^n$ with the standard normal law.

Theorem 1.1. *For any random vector X in $\mathbf{R}^n$ with a smooth density, such that $I(X|Z)$ is finite,*

$$I(X|Z) - 2D(X|Z) \geq n\Delta\left(\frac{I(X)}{n} - 1\right). \tag{1.5}$$

Moreover,

$$I(X|Z) - 2D(X|Z) \geq \left(\sqrt{I(X|Z)} - W_2(X, Z)\right)^2 \\ + n\Delta\left(\frac{W_2(X, Z)}{\sqrt{I(X|Z)}}\left(\frac{I(X)}{n} - 1\right)\right). \tag{1.6}$$

As is common,

$$I(X) = \int \frac{|\nabla p(x)|^2}{p(x)} \, dx$$

stands for the usual (non-relative) Fisher information. Thus, (1.5)–(1.6) represent certain sharpenings of the logarithmic Sobolev inequality. The lower bounds of the deficit in (1.5) and (1.6) are not simply comparable. However, in the next section, we recall that (1.5) is a self-improvement of the logarithmic Sobolev inequality that obviously follows from (1.6).

An interesting feature of the bound (1.6) is that, by removing the last term in it, we arrive at the Gaussian case in the so-called HWI inequality due to F. Otto and C. Villani [29],

$$D(X|Z) \leq W_2(X, Z)\sqrt{I(X|Z)} - \frac{1}{2}W_2^2(X, Z). \tag{1.7}$$

As for (1.5), its main point is that, when $\mathbf{E}|X|^2 \leq n$, then necessarily $I(X) \geq n$, and moreover, one can use the lower bound

$$\frac{1}{n}I(X) - 1 = \frac{1}{n}I(X|Z) - \frac{1}{n}\mathbf{E}|X|^2 + 1 \geq \frac{1}{n}I(X|Z).$$

Since $\Delta(t)$ is increasing for $t \geq 0$, (1.5) is then simplified to

$$I(X|Z) - 2D(X|Z) \geq n\Delta\left(\frac{1}{n}I(X|Z)\right). \tag{1.8}$$

In fact, this estimate is rather elementary in that it surprisingly follows from the logarithmic Sobolev inequality itself by virtue of rescaling (as will be explained later on). Here, let us only stress that the right-hand side of (1.8) can further be bounded from below. For example, by (1.2)–(1.3), we have

$$I(X|Z) - 2D(X|Z) \geq n\Delta\left(\frac{2}{n}D(X, Z)\right) \geq n\Delta\left(\frac{1}{n}W_2^2(X, Z)\right).$$

But, $\frac{1}{n}W_2^2(X, Z) \leq \frac{1}{n}\mathbf{E}|X - Z|^2 \leq 4$, and using $\Delta(t) \sim \frac{t^2}{2}$ for small t, the above yields a simpler bound.

Corollary 1.2. *For any random vector X in $\mathbf{R}^n$ with a smooth density and such that $\mathbf{E}|X|^2 \leq n$, we have*

$$I(X|Z) - 2D(X|Z) \geq \frac{c}{n}W_2^4(X, Z), \tag{1.9}$$

up to an absolute constant $c > 0$.

Remark. Dimensional refinements of the HWI inequality (1.7) similar to (1.6) were recently considered by several authors. For instance, F.-Y. Wang obtained in [39] some HWI type inequalities involving the dimension and the quadratic Kantorovich distance under the assumption that the reference measure enjoys some curvature dimension condition $CD(-K, N)$ with $K \geq 0$ and $N \geq 0$ (see [3] for the definition). See also the recent paper [18] for dimensional variants of the HWI inequality in an abstract metric space framework. The standard Gaussian measure does not enter directly the framework of [39] (or [18]), but we believe that it might be possible to use similar semigroup arguments

to derive (1.6). In the same spirit, D. Bakry, F. Bolley and I. Gentil [2] used semigroup techniques to prove a dimensional reinforcement of Talagrand's transport-entropy inequality. Finally, let us mention that G. Toscani also obtained in [37] a bound similar to (1.9) but involving the total variation distance instead of the W_2 distance.

Returning to (1.9), we note that, after a certain recentering of X, one may give some refinement over this bound, especially when $D(X|Z)$ is small. Given a random vector X in $\mathbf{R}^n$ with finite absolute moment, define the recentered random vector $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_n)$ by putting $\bar{X}_1 = X_1 - \mathbf{E}X_1$ and

$$\bar{X}_k = X_k - \mathbf{E}\left(X_k | X_1, \ldots, X_{k-1}\right), \quad k \geq 2,$$

where we use standard notations for the conditional expectations.

Theorem 1.3. *For any random vector X in $\mathbf{R}^n$ with a smooth density, such that $I(X|Z)$ is finite, the deficit in (1.2) satisfies*

$$\frac{1}{2} I(X|Z) - D(X|Z) \geq c \frac{\mathcal{T}^2(\bar{X}, Z)}{D(\bar{X}|Z)}. \tag{1.10}$$

Here the optimal transport cost $\mathcal{T}$ corresponds to the cost function $\Delta(|x - z|)$, c is a positive absolute constant and one uses the convention $0/0 = 0$ in the right hand side.

In particular, in dimension one, if a random vector X has mean zero, we get that

$$\frac{1}{2} I(X|Z) - D(X|Z) \geq c \frac{\mathcal{T}^2(X, Z)}{D(X|Z)}. \tag{1.11}$$

The bound (1.10) allows one to recognize the cases of equality in (1.2) – this is only possible when the random vector X is a translation of the standard random vector Z (an observation of E. Carlen [11] who used a different proof). The argument is sketched in Appendix C.

It is worthwhile noting that the transport cost $\mathcal{T}$ of Theorem 1.3 already appeared in the literature, cf. e.g. [8] or [5]. In particular, it was shown in [8] that this transport cost can be used to give an alternative representation of the Poincaré inequality. In fact, it may be connected with the classical Kantorovich transport distance W_1 based on the cost function $c(x, z) = |x - z|$. More precisely, due to the convexity of Δ, there are simple bounds

$$W_1(X, Z) \geq \mathcal{T}(X, Z) \geq \Delta\big(W_1(X, Z)\big) \sim \min\big\{W_1(X, Z), W_1^2(X, Z)\big\}.$$

Hence, if $D(\bar{X}|Z) \leq 1$, then according to (1.3), $W_1^2(X, Z) \leq W_2^2(X, Z) \leq 2$, and (1.10) is simplified to

$$\frac{1}{2}I(X|Z) - D(X|Z) \geq c'\frac{W_1^4(\bar{X}, Z)}{D(\bar{X}|Z)}, \tag{1.12}$$

for some other absolute constant c'.

In connection with such bounds, let us mention a recent preprint by E. Indrei and D. Marcon [23], which we learned about while the current work was in progress. For a C^2-smooth function V on $\mathbf{R}^n$, let us denote by $V''(x)$ the matrix of second partial derivatives of V at the point x. We use comparison of symmetric matrices in the usual matrix sense and denote by I_n the identity $n \times n$ matrix.

It is proved in [23] (Theorem 1.1 and Corollary 1.2) that, if a random vector X on $\mathbf{R}^n$ has a smooth density $p = e^{-V}$ satisfying $\varepsilon \mathrm{I}_n \leq V'' \leq M\mathrm{I}_n$ $(0 < \varepsilon < M)$, then

$$\frac{1}{2}I(X|Z) - D(X|Z) \geq cW_2^2(X - \mathbf{E}X, Z) \tag{1.13}$$

with some constants $c = c(\varepsilon, M)$. In certain cases it is somewhat stronger than (1.11). We will show that a slight adaptation of our proof of (1.11) leads to a bound similar to (1.13).

Theorem 1.4. *Let X be a random vector in $\mathbf{R}^n$ with a smooth density $p = e^{-V}$ with respect to Lebesgue measure such that $V'' \geq \varepsilon \mathrm{I}_n$, for some $\varepsilon > 0$. Then, the deficit in (1.2) satisfies*

$$\frac{1}{2}I(X|Z) - D(X|Z) \geq c\min(1, \varepsilon)W_2^2(\bar{X}, Z), \tag{1.14}$$

for some absolute constant c.

Note that Theorem 1.4 holds under less restrictive assumptions on p than the result from [23]. In particular, in dimension 1, we see that the constant c in (1.13) can be taken independent on M. In higher dimensions however, it is not clear how to compare $W_2(\bar{X}, Z)$ and $W_2(X - \mathbf{E}X, Z)$ in general. One favorable case is, for instance, when the distribution of X is unconditional (i.e., when its density p satisfies $p(x) = p(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$, for all $x \in \mathbf{R}^n$ and all $\varepsilon_i = \pm 1$). In this case, $\mathbf{E}X = 0$ and $\bar{X} = X$, and thus (1.14) reduces to (1.13) with a constant c independent on M.

Let us mention that in Theorem 1.3 of [23], the assumption $V'' \leq M\mathrm{I}_n$ can be relaxed into an integrability condition of the form $\int \|V''\|^r \, dx \leq M$, for some $r > 1$, but only at the expense of a constant c depending on the dimension n and of an exponent greater than 2 in the right-hand side of (1.13).

Finally, let us conclude this introduction by showing optimality of the bounds (1.11), (1.12), (1.14) for mean zero Gaussian random vectors with variance close to 1. An easy calculation shows that, if Z is a standard Gaussian random vector in $\mathbf{R}^n$, then for any $\sigma > 0$,

$$D(\sigma Z|Z) = \frac{n}{2}\big((\sigma^2 - 1) - 2\log\sigma\big), \qquad I(\sigma Z|Z) = n\sigma^2\left(\frac{1}{\sigma^2} - 1\right)^2,$$

so that

$$\frac{1}{2}I(X|Z) - D(X|Z) = \frac{n}{2}\left(\frac{1}{\sigma^2} - 1 + 2\log\sigma\right) \sim n(\sigma - 1)^2, \quad \text{as } \sigma \to 1.$$

On the other hand,

$$W_2^2(\sigma Z, Z) = n(\sigma - 1)^2, \qquad W_1(\sigma Z, Z) = |\sigma - 1|\mathbf{E}|Z| \simeq |\sigma - 1|\sqrt{n},$$

and thus the three quantities $W_2^2(\sigma Z, Z)$, $\mathcal{T}^2(\sigma Z, Z)/D(\sigma Z|Z)$ and $W_1^4(\sigma Z, Z)/D(\sigma Z|Z)$ are all of the same order $n(\sigma - 1)^2$, when σ goes to 1.

The paper is organized in the following way. In Section 2 we recall Stam's formulation of the logarithmic Sobolev inequality in the form of an "isoperimetric inequality for entropies" and discuss the involved improved variants of (1.1). Theorem 1.1 is proved in Section 3. In Section 4 we consider sharpened transport-entropy inequalities in dimension one, which are used to derive bounds on the deficit like those in (1.11)–(1.14). For general dimensions Theorems 1.3 and 1.4 are proved in Section 5. For the reader's convenience and so as to get a more self-contained exposition, we move to Appendices A, B and C several known results and arguments.

2. Self-improvement of the logarithmic Sobolev inequality

To start with, let us return to the history and remind the reader Stam's information-theoretic formulation of the logarithmic Sobolev inequality. As a base for the derivation, one may take (1.2) and rewrite it in terms of the Fisher information $I(X)$ and the (Shannon) entropy

$$h(X) = -\int p(x)\log p(x)\, dx,$$

where X is a random vector in $\mathbf{R}^n$ with density p. Here the integral is well-defined, as long as X has finite second moment. Introduce also the entropy power

$$N(X) = \exp\big\{2h(X)/n\big\},$$

which is a homogeneous functional of order 2. The basic connections between the relative and non-relative information quantities are given by

$$D(X|Z) = h(Z) - h(X), \qquad I(X|Z) = I(X) - I(Z),$$

where Z has a normal distribution, and provided that $\mathbf{E}|X|^2 = \mathbf{E}|Z|^2$.

More generally, assuming that Z is standard normal and $\mathbf{E}|X|^2 < \infty$, the first above equality should be replaced with

$$D(X|Z) = -h(X) + \mathbf{E}\left(\frac{n}{2}\log(2\pi) + \frac{|X|^2}{2}\right),$$

while, as was mentioned before, under mild regularity assumptions on p,

$$I(X|Z) = I(X) + \mathbf{E}|X|^2 - 2n.$$

Inserting these expressions into the inequality (1.2), the second moment is canceled, and (1.2) becomes

$$I(X) + 2h(X) \geq 2n + n\log(2\pi).$$

However, this inequality is not homogeneous in X. So, one may apply it to λX in place of X with arbitrary $\lambda > 0$ and then optimize. The function

$$v(\lambda) = I(\lambda X) + 2h(\lambda X) = \frac{I(X)}{\lambda^2} + n\log\lambda^2 + 2h(X)$$

is minimized for $\lambda^2 = I(X)/n$, and at this point the inequality becomes:

Theorem 2.1. *(See [35].) If a random vector X in $\mathbf{R}^n$ has a smooth density and finite second moment, then*

$$I(X)\frac{N(X)}{2\pi e} \geq n. \tag{2.1}$$

This relation was first obtained by Stam and is sometimes referred to as the isoperimetric inequality for entropies, cf. e.g. [15]. Stam's original argument is based on the general entropy power inequality

$$N(X + Y) \geq N(X) + N(Y), \tag{2.2}$$

which holds for all independent random vectors X and Y in $\mathbf{R}^n$ with finite second moments (so that the involved entropies do exist, cf. also [6,27]). Then, (2.1) can be obtained by taking $Y = \sqrt{t}Z$ with Z having a standard normal law and combining (2.2) with the de Bruijn identity

$$\frac{d}{dt}h(X + \sqrt{t}Z) = \frac{1}{2}I(X + \sqrt{t}Z) \quad (t > 0). \tag{2.3}$$

Note that in the derivation (1.2) $\Rightarrow$ (2.1) the argument may easily be reversed, so these inequalities are in fact equivalent (as noticed by E. Carlen [11]). On the other

hand, the isoperimetric inequality for entropies can be viewed as a certain sharpening of (1.1)–(1.2). Indeed, let us rewrite (2.1) explicitly as

$$\int p(x) \log p(x)\, dx \le \frac{n}{2} \log\left(\frac{1}{2\pi e n} \int \frac{|\nabla p(x)|^2}{p(x)}\, dx\right). \qquad (2.4)$$

It is also called an optimal Euclidean logarithmic Sobolev inequality; cf. [4] for a detail discussion including deep connections with dimensional lower estimates on heat kernel measures. In terms of the density $f(x) = \sqrt{2\pi}e^{x^2/2}p(x)$ of X with respect to γ we have

$$\int p(x) \log p(x)\, dx = \frac{n}{2} \log\frac{1}{2\pi} - \frac{1}{2}\int |x|^2 f(x)\, d\gamma(x) + \int f \log f\, d\gamma,$$

while

$$\int \frac{|\nabla p(x)|^2}{p(x)}\, dx = \int \frac{|\nabla f(x)|^2}{f(x)}\, d\gamma(x) - \int |x|^2 f(x)\, d\gamma(x) + 2n.$$

Inserting these two equalities in (2.4), we arrive at the following reformulation of Theorem 2.1.

Corollary 2.2. *For any positive smooth function f on $\mathbf{R}^n$ such that $\int f\, d\gamma = 1$, putting $b = \frac{1}{n}\int |x|^2 f(x)\, d\gamma(x)$, we have*

$$\int f \log f\, d\gamma \le \frac{n}{2} \log\left(\frac{1}{n}\int \frac{|\nabla f|^2}{f}\, d\gamma + (2 - b)\right) + \frac{n}{2}(b - 1), \qquad (2.5)$$

which is exactly (1.5). In particular, if $b \le 1$,

$$\int f \log f\, d\gamma \le \frac{n}{2} \log\left(\frac{1}{n}\int \frac{|\nabla f|^2}{f}\, d\gamma + 1\right). \qquad (2.6)$$

An application of $\log t \le t - 1$ on the right-hand side of (2.5) returns us to the original logarithmic Sobolev inequality (1.1). It is in this sense that inequality (2.5) is stronger, although it was derived from (1.1). The point of self-improvement is that the log-value of

$$I = \int \frac{|\nabla f|^2}{f}\, d\gamma$$

may be much smaller than the integral itself. This can be used, for example, in bounding the deficit $\delta(f)$ in (1.1). Indeed, when $b \le 1$, (2.6) yields

$$2\delta(f) \ge I - n \log\left(\frac{1}{n}I + 1\right).$$

That is, using again the function $\Delta(t) = t - \log(t + 1)$, we have

$$2\delta(f) \geq n\Delta\left(\frac{1}{n}\int \frac{|\nabla f|^2}{f}\, d\gamma\right).$$

But this is exactly the information-theoretic bound (1.8), mentioned in Section 1 as a direct consequence of (1.5).

As the function Δ naturally appears in many related inequalities, let us collect together a few elementary bounds that will be needed in the sequel.

Lemma 2.3. *We have:*

a) $\Delta(ct) \geq \min(c, c^2)\Delta(t)$, *whenever* $c, t \geq 0$;

b) $\Delta(t) \geq \frac{1}{2}t^2$, *for all* $-1 < t \leq 0$;

c) $\Delta(t) \geq \frac{\Delta(a)}{a^2}t^2$, *for all* $0 \leq t \leq a$ $(a > 0)$;

d) $(1 - \log 2)\min\{t, t^2\} \leq \Delta(t) \leq t$, *for all* $t \geq 0$.

Moreover, for any random variable $\xi \geq 0$,

$$(1 - \log 2)\min\big\{\mathbf{E}\xi, (\mathbf{E}\xi)^2\big\} \leq \mathbf{E}\Delta(\xi) \leq \mathbf{E}\xi.$$

Proof. *a)* In case $0 \leq c \leq 1$, the required inequality follows from the representation

$$\Delta(ct) = \int\limits_0^{ct} \Delta'(s)\, ds = \int\limits_0^{ct} \frac{s}{1 + s}\, ds = c^2 \int\limits_0^t \frac{u}{1 + cu}\, du.$$

In case $c \geq 1$, it becomes $\log(1 + ct) \leq c\log(1 + t)$, which is obvious.

b) This bound immediately follows from the Taylor expansion for the function $-\log(1 - s)$.

c) It is easy to check that the function $\Delta(\sqrt{x})$ is concave in $x \geq 0$. Hence, the optimal value of the constant c in $\Delta(t) \geq ct^2$ on the interval $[0, a]$ corresponds to the endpoint $t = a$.

d) For $t \geq 1$, the first inequality becomes $ct \leq t - \log(1 + t)$, where $c = 1 - \log 2$. Both sides are equal at $t = 1$, and we have inequality for the derivatives at this point. Hence, it holds for all $t \geq 1$. For the interval $0 \leq t \leq 1$, the inequality $\Delta(t) \geq ct^2$ is given in c).

Finally, an application of Jensen's inequality with the convex function Δ together with $\Delta(\xi) \leq \xi$ leads to the last bounds of the lemma. $\square$

3. HWI inequality and its sharpening

We now turn to the remarkable HWI inequality of F. Otto and C. Villani and state it in full generality. Assume that the probability measure ν on $\mathbf{R}^n$ has density

$$\frac{d\nu(x)}{dx} = e^{-V(x)}$$

with a twice continuously differentiable $V : \mathbf{R}^n \to \mathbf{R}$.

Theorem 3.1. *(See [29].) Assume that $V''(x) \geq \kappa I_n$ for all $x \in \mathbf{R}^n$ with some $\kappa \in \mathbf{R}$. Then, for any probability measure μ on $\mathbf{R}^n$ with finite second moment,*

$$D(\mu|\nu) \leq W_2(\mu|\nu)\sqrt{I(\mu|\nu)} - \frac{\kappa}{2}W_2^2(\mu, \nu). \tag{3.1}$$

We refer to [14] for a simple proof of (3.1) based on optimal transport.

This inequality connects together all three important distances: the relative entropy (which sometimes is denoted by H), the relative Fisher information I, and the quadratic transport distance W_2. It may equivalently be written as

$$D(\mu|\nu) \leq \frac{1}{2\varepsilon}I(\mu|\nu) + \frac{\varepsilon - \kappa}{2}W_2^2(\mu, \nu) \tag{3.2}$$

with an arbitrary $\varepsilon > 0$. Taking here $\varepsilon = \kappa$, one gets

$$D(\mu|\nu) \leq \frac{1}{2\kappa}I(\mu|\nu). \tag{3.3}$$

If $\nu = \gamma$, we arrive in (3.3) at the logarithmic Sobolev inequality (1.1) for the Gaussian measure, and thus the HWI inequality represents its certain refinement. In particular, (3.1) may potentially be used in the study of the deficit in (1.1), as is pointed in Theorem 1.1.

In the proof of the latter, we will use two results. The following lemma, reversing the transport-entropy inequality, may be found in the survey by Raginsky and Sason [33], Lemma 15. It is due to Y. Wu [40] who used it to prove a weak version of the Gaussian HWI inequality (without the curvature term $-\frac{1}{2}W_2^2(X, Z)$ appearing in (1.7)). The proof of Lemma 3.2 is reproduced in Appendix A.

For a random vector X in $\mathbf{R}^n$ with finite second moment, put

$$X_t = X + \sqrt{t}Z \quad (t \geq 0),$$

where Z is a standard normal random vector in $\mathbf{R}^n$, independent of X.

Lemma 3.2. *(See [40].) Given random vectors X and Y in $\mathbf{R}^n$ with finite second moments, for all $t > 0$,*

$$D(X_t|Y_t) \leq \frac{1}{2t}W_2^2(X, Y).$$

We will also need a convexity property of the Fisher information in the form of the Fisher information inequality. As a full analog of the entropy power inequality (2.2), it was apparently first mentioned by Stam [35].

Lemma 3.3. *Given independent random vectors X and Y in $\mathbf{R}^n$ with smooth densities,*

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}. \tag{3.4}$$

Proof of Theorem 1.1. Let Z be standard normal, and let the distribution of X not be a translation of γ (in which case both sides of (1.5) and of (1.6) are vanishing).

We recall that, if Y is a normal random vector with mean zero and covariance matrix $\sigma^2 I_n$, then

$$D(X|Y) = h(Y) - h(X) + \frac{1}{2\sigma^2}\big(\mathbf{E}|X|^2 - \mathbf{E}|Y|^2\big).$$

In particular,

$$D(X|Z) = h(Z) - h(X) + \frac{1}{2}\big(\mathbf{E}|X|^2 - \mathbf{E}|Z|^2\big),$$

where $\mathbf{E}|Z|^2 = n$. Using de-Bruijn's identity (2.3), $\frac{d}{dt}h(X_t) = \frac{1}{2}I(X_t)$, we therefore obtain that, for all $t > 0$,

$$
\begin{aligned}
D(X_t|Z_t) &= h(Z_t) - h(X_t) + \frac{1}{2(1+t)}\big(\mathbf{E}|X_t|^2 - \mathbf{E}|Z_t|^2\big) \\
&= h(Z_t) - h(X_t) + \frac{1}{2(1+t)}\big(\mathbf{E}|X|^2 - \mathbf{E}|Z|^2\big) \\
&= \big(h(Z) - h(X)\big) + \frac{1}{2}\int_0^t \big(I(Z_\tau) - I(X_\tau)\big)\,d\tau + \frac{1}{2(1+t)}\big(\mathbf{E}|X|^2 - \mathbf{E}|Z|^2\big) \\
&= D(X|Z) + \frac{1}{2}\int_0^t \big(I(Z_\tau) - I(X_\tau)\big)\,d\tau - \frac{t}{2(1+t)}\big(\mathbf{E}|X|^2 - \mathbf{E}|Z|^2\big).
\end{aligned}
$$

Equivalently,

$$
D(X|Z) = D(X_t|Z_t) + \frac{1}{2}\int_0^t \big(I(X_\tau) - I(Z_\tau)\big)\,d\tau
$$

$$
+ \frac{t}{2(1+t)}\big(\mathbf{E}|X|^2 - \mathbf{E}|Z|^2\big). \tag{3.5}
$$

In order to estimate from above the last integral, we apply Lemma 3.3 to the couple $(X, \sqrt{\tau}Z)$, which gives

$$I(X_\tau) \leq \frac{1}{\frac{1}{I(X)} + \frac{1}{I(\sqrt{\tau}Z)}} = \frac{nI(X)}{n + \tau I(X)}.$$

Inserting also $I(Z_\tau) = \frac{n}{1+\tau}$, we get

$$\int_0^t \big(I(X_\tau) - I(Z_\tau)\big)\, d\tau \le \int_0^t \left(\frac{nI(X)}{n + \tau I(X)} - \frac{n}{1+\tau}\right) d\tau$$

$$= \frac{n}{2} \log \frac{n + tI(X)}{n(1+t)}.$$

Thus, from (3.5),

$$D(X|Z) \le D(X_t|Z_t) + \frac{n}{2} \log \frac{n + tI(X)}{n(1+t)} + \frac{t}{2(1+t)} \big(\mathbf{E}|X|^2 - n\big).$$

Furthermore, an application of Lemma 3.2 together with the identity

$$\mathbf{E}|X|^2 - n = I(X|Z) - I(X) + n$$

yields

$$D(X|Z) \le \frac{1}{2t} W_2^2(X, Z) + \frac{n}{2} \log \frac{n + tI(X)}{n(1+t)}$$

$$+ \frac{t}{2(1+t)} \big(I(X|Z) - I(X) + n\big). \tag{3.6}$$

As t goes to infinity in (3.6), we get in the limit

$$D(X|Z) \le \frac{1}{2} I(X|Z) - \frac{n}{2} \Delta \left(\frac{I(X)}{n} - 1\right),$$

which is exactly the required inequality (1.5) of Theorem 1.1.

As for (1.6), let us restate (3.6) as the property that the deficit $I(X|Z) - 2D(X|Z)$ is bounded from below by

$$I(X|Z) - \frac{1}{t} W_2^2(X, Z) - n \log \frac{n + tI(X)}{n(1+t)} - \frac{t}{1+t} \big(I(X|Z) - I(X) + n\big). \tag{3.7}$$

Assuming that X is not normal, we end the proof by choosing the value

$$t = \frac{W_2(X, Z)}{\sqrt{I(X|Z)} - W_2(X, Z)}, \tag{3.8}$$

which is well-defined and positive. Indeed, by the assumption that $I(X|Z)$ is finite, $W_2(X, Z)$ is finite as well (according to the inequality (1.4), for example). Moreover, the case where $\sqrt{I(X|Z)} = W_2(X, Z)$ is impossible, since then $2D(X|Z) = I(X|Z)$. But the

latter is only possible, when the distribution of X represents a translation of γ, by the result of E. Carlen on the equality cases in (1.1) (cf. also Appendix C).

Putting for short $W = W_2(X, Z)$, $I = I(X|Z)$, $I_0 = I(X)$, we finally note that the expression (3.7) with the value of t specified in (3.8) turns into

$$I - W(\sqrt{I} - W) - n \log \frac{1 + \frac{W}{\sqrt{I}-W}\frac{I_0}{n}}{\frac{\sqrt{I}}{\sqrt{I}-W}} - \frac{W}{\sqrt{I}}(I - I_0 + n)$$

$$= (\sqrt{I} - W)^2 - n \log\left(1 + \frac{W}{\sqrt{I}}\left(\frac{I_0}{n} - 1\right)\right) + n\frac{W}{\sqrt{I}}\left(\frac{I_0}{n} - 1\right)$$

$$= (\sqrt{I} - W)^2 + n\Delta\left(\frac{W}{\sqrt{I}}\left(\frac{I_0}{n} - 1\right)\right). \qquad \square$$

4. Sharpened transport-entropy inequalities on the line

Nowadays, Talagrand's transport-entropy inequality (1.2),

$$\frac{1}{2}W_2^2(\mu, \gamma) \le D(\mu|\gamma), \tag{4.1}$$

has many proofs (cf. e.g. [9]). In the one dimensional case it admits the following refinement, which is due to F. Barthe and A. Kolesnikov.

Theorem 4.1. *(See [5].) For any probability measure μ on the real line with finite second moment, having the mean or median at the origin,*

$$\frac{1}{2}W_2^2(\mu, \gamma) + \frac{1}{4}\mathcal{T}'(\mu, \gamma) \le D(\mu|\gamma), \tag{4.2}$$

where the optimal transport cost $\mathcal{T}'$ is based on the cost function $c'(x - z) = \Delta(\frac{|x-z|}{\sqrt{2\pi}})$.

It is also shown in [5] that the constant $\frac{1}{4}$ may be replaced with 1 under the median assumption. Anyhow, the deficit in (4.1) can be bounded in terms of the transport distance $\mathcal{T}$ which represents a slight weakening of W_2 (since the function $\Delta(t) = t - \log(t + 1)$ is almost quadratic near zero).

In [5], the reinforced transport inequality above was only stated for probability measures with median at 0, but the argument can be easily adapted to the mean zero case. For the sake of completeness, the proof of Theorem 4.1 is recalled in Appendix B. In order to work with the usual cost function $c(x - z) = \Delta(|x - z|)$, the inequality (4.2) will be modified to

$$\frac{1}{2}W_2^2(\mu, \gamma) + \frac{1}{8\pi}\mathcal{T}(\mu, \gamma) \le D(\mu|\gamma) \tag{4.3}$$

under the assumption that μ has mean zero. (Here we use the elementary inequality $\Delta(ct) \geq c^2 \Delta(t)$, for $0 \leq c \leq 1$, $t \geq 0$, cf. Lemma 2.3.)

As a natural complement to Theorem 4.1, it will be also shown in Appendix B that, under an additional log-concavity assumption on μ, the transport cost $\mathcal{T}$ in the inequalities (4.2)–(4.3) may be replaced with W_2^2. That is, the constant $\frac{1}{2}$ in (4.1) may be increased.

Theorem 4.2. *Suppose that the probability measure μ on the real line has a twice continuously differentiable density $\frac{d\mu}{dx}(x) = e^{-v(x)}$ such that, for a given $\varepsilon > 0$,*

$$v''(x) \geq \varepsilon, \quad x \in \mathbf{R}. \tag{4.4}$$

If μ has mean at the origin, then with some absolute constant $c > 0$ we have

$$\left(\frac{1}{2} + c\min\{1, \sqrt{\varepsilon}\} \right) W_2^2(\mu, \gamma) \leq D(\mu|\gamma). \tag{4.5}$$

Here, one may take $c = 1 - \log 2$.

Let us now explain how these refinements can be used in the problem of bounding the deficit in the one dimensional logarithmic Sobolev inequality. Returning to (4.3), we are going to combine this bound with the HWI inequality (3.1). Putting

$$W = W_2(\mu, \gamma), \qquad D = D(\mu|\gamma), \qquad I = I(\mu|\gamma),$$

we rewrite (3.1) as

$$I - 2D \geq (\sqrt{I} - W)^2.$$

On the other hand, applying the logarithmic Sobolev inequality $I \geq 2D$, (4.3) yields $I \geq W^2 + \frac{1}{4\pi}\mathcal{T}$, where $\mathcal{T} = \mathcal{T}(\mu, \gamma)$. Hence,

$$I - 2D \geq \left(\sqrt{W^2 + \frac{1}{4\pi}\mathcal{T}} - W \right)^2 = W^2 \left(\sqrt{1 + \frac{\mathcal{T}}{4\pi W^2}} - 1 \right)^2.$$

Here, by the very definition of the transport distance, one has $\mathcal{T} \leq W^2$, so $\varepsilon = \frac{\mathcal{T}}{4\pi W^2} \leq \frac{1}{4\pi}$. This implies that $\sqrt{1 + \varepsilon} - 1 \geq c\varepsilon$ with $c = 4\pi(\sqrt{1 + \frac{1}{4\pi}} - 1)$. Thus, up to a positive numerical constant,

$$D + c\frac{\mathcal{T}^2}{W^2} \leq \frac{1}{2}I. \tag{4.6}$$

In order to get a more flexible formulation, denote by μ_t the shift of the measure μ,

$$\mu_t(A) = \mu(A - t), \quad A \subset \mathbf{R} \text{ (Borel)},$$

which is the distribution of the random variable $X + t$ (with fixed $t \in \mathbf{R}$), when X has the distribution μ. As easy to verify,

$$D(\mu_t|\gamma) = D(\mu|\gamma) + \frac{t^2}{2} + t\mathbf{E}X,$$

$$\frac{1}{2}I(\mu_t|\gamma) = \frac{1}{2}I(\mu|\gamma) + \frac{t^2}{2} + t\mathbf{E}X.$$

Hence, the deficit

$$\delta(\mu) = \frac{1}{2}I(\mu|\gamma) - D(\mu|\gamma)$$

in the logarithmic Sobolev inequality (1.2) is translation invariant: $\delta(\mu_t) = \delta(\mu)$. Applying (4.6) to μ_t with $t = -\int x \, d\mu(x)$, so that μ_t would have mean zero, therefore yields:

Corollary 4.3. *For any non-Gaussian probability measure μ on the real line with finite second moment, up to an absolute constant $c > 0$,*

$$D(\mu|\gamma) + c\frac{\mathcal{T}^2(\mu_{-t}, \gamma)}{W_2^2(\mu_{-t}, \gamma)} \leq \frac{1}{2}I(\mu|\gamma), \tag{4.7}$$

where the optimal transport cost $\mathcal{T}$ is based on the cost function $\Delta(|x - z|)$, and where $t = \int x \, d\mu(x)$. In particular,

$$D(\mu|\gamma) + \frac{c}{2}\frac{\mathcal{T}^2(\mu_{-t}, \gamma)}{D(\mu_{-t}|\gamma)} \leq \frac{1}{2}I(\mu|\gamma). \tag{4.8}$$

Here the second inequality follows from the first one by using $W_2^2 \leq 2D$. It will be used in the next section to perform tensorisation for a multidimensional extension. Note that (4.8) may be derived directly from (4.3) with similar arguments. Indeed, one can write

$$I - 2D \geq (\sqrt{I} - W)^2 \geq (\sqrt{2D} - W)^2$$
$$= \frac{(2D - W^2)^2}{(\sqrt{2D} + W)^2} \geq \frac{(2D - W^2)^2}{(2\sqrt{2D})^2} \geq \frac{\mathcal{T}^2}{128\pi^2 D^2},$$

thus proving (4.8) with constant $c = 1/(128\pi^2)$.

Let us now turn to Theorem 4.2 with its additional hypothesis (4.4). Note that the property $v'' \geq 0$ describes the so-called log-concave probability distributions on the real line (with C^2-smooth densities), so (4.4) represents its certain quantitative strengthening. It is also equivalent to the property that X has a log-concave density with respect to the Gaussian measure with mean zero and variance ε.

Arguing as before, from (4.5) we have

$$I - 2D \geq W^2 \left(\sqrt{1 + c \min\{1, \sqrt{\varepsilon}\,\}} - 1 \right)^2.$$

Hence, we obtain:

Corollary 4.4. *Let μ be a probability measure on the real line with mean zero, and satisfying (4.4) with some $\varepsilon > 0$. Then, up to an absolute constant $c > 0$,*

$$D(\mu|\gamma) + c \min\{1, \varepsilon\} W_2^2(\mu, \gamma) \leq \frac{1}{2} I(\mu|\gamma), \tag{4.9}$$

5. Proof of Theorems 1.3 and 1.4

As the next step, it is natural to try to tensorize the inequality (4.8) so that to extend it to the multidimensional case.

If $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, denote by $x_{1:i}$ the subvector $(x_1, \ldots, x_i)$, $i = 1, \ldots, n$. Given a probability measure μ on $\mathbf{R}^n$, denote by μ_1 its projection to the first coordinate, i.e., $\mu_1(A) = \mu(A \times \mathbf{R}^{n-1})$ for Borel sets $A \subset \mathbf{R}$. For $i = 2, \ldots, n$, let $\mu_i(dx_i|x_{1:i-1})$ denote the conditional distribution of the i-th coordinate under μ knowing the first $i-1$ coordinates $x_1, \ldots, x_{i-1}$. Under mild regularity assumptions on μ, all these conditional measures are well-defined, and we have a general formula for the "full expectation"

$$\int h(x)\, d\mu(x) = \int h(x_1, \ldots, x_n) \mu_n(dx_n|x_{1:n-1}) \ldots \mu_2(dx_2|x_1) \mu_1(dx_1), \tag{5.1}$$

for any bounded measurable function h on $\mathbf{R}^n$. For example, it suffices to require that μ has a smooth positive density, which is polynomially decaying at infinity. Then we will say that μ is regular. In many inequalities, the regularity assumption is only technical for purposes of the proof, and may easily be omitted in the resulting formulations.

The distance functionals D, I, and $\mathcal{T}$ satisfy the following tensorisation relations with respect to product measures similarly to (5.1). To emphasize the dimension, we denote by γ_n the standard Gaussian measure on $\mathbf{R}^n$.

Lemma 5.1. *For any regular probability measure μ on $\mathbf{R}^n$ with finite second moment,*

$$D(\mu|\gamma_n) = D(\mu_1|\gamma_1) + \sum_{i=2}^{n} \int D\big(\mu_i(\cdot|x_{1:i-1})|\gamma_1\big)\, d\mu(x),$$

$$I(\mu|\gamma_n) \geq I(\mu_1|\gamma_1) + \sum_{i=2}^{n} \int I\big(\mu_i(\cdot|x_{1:i-1})|\gamma_1\big)\, d\mu(x),$$

$$\mathcal{T}(\mu, \gamma_n) \leq \mathcal{T}(\mu_1, \gamma_1) + \sum_{i=2}^{n} \int \mathcal{T}\big(\mu_i(\cdot|x_{1:i-1}), \gamma_1\big)\, d\mu(x).$$

Note that this statement remains to hold also for other product references measures ν^n on $\mathbf{R}^n$ in place of γ_n (with necessary regularity assumptions for the case of Fisher information).

Applying the first two inequalities, we see that the deficit δ satisfies a similar property,

$$\delta(\mu) \geq \delta(\mu_1) + \sum_{i=2}^{n} \int \delta\big(\mu_i(\cdot|x_{1:i-1})\big) \, d\mu(x). \tag{5.2}$$

Proof of Lemma 5.1. The equality for the relative entropy is a straightforward calculation. We refer to Appendix A of [21] for a (general) tensorisation inequality for transport costs. Below, we sketch the proof of the inequality involving Fisher information.

Let μ be a regular probability measure on $\mathbf{R}^n$ admitting a smooth density f with respect to γ_n. Note that the first marginal $\tilde{\mu}$ of μ on the first $n-1$ coordinates has density $\tilde{f}(x_{1:n-1}) = \int f(x_{1:n-1}, x_n) \, \gamma(dx_n)$ and that $\mu_n(\cdot|x_{1:n-1})$ has density $f(x_n|x_{1:n-1}) = f(x_{1:n-1}, x_n)/\tilde{f}(x_{1:n-1})$. We have

$$
\begin{aligned}
I(\mu|\gamma_n) &= \sum_{i=1}^{n-1} \int \frac{(\partial_{x_i} f)^2}{f}(x) \, \gamma_n(dx) + \int \frac{(\partial_{x_n} f)^2}{f}(x) \, \gamma_n(dx) \\
&= \sum_{i=1}^{n-1} \int \left(\int \frac{(\partial_{x_i} f)^2}{f}(x_{1:n-1}, x_n) \, \gamma_1(dx_n) \right) \gamma_{n-1}(dx_{1:n-1}) \\
&\quad + \int I\big(\mu_n(\cdot|x_{1:n-1})|\gamma_1\big) \, \tilde{\mu}(dx_{1:n-1}) \\
&\geq \sum_{i=1}^{n-1} \int \frac{(\partial_{x_i} \tilde{f})^2}{\tilde{f}}(x_{1:n-1}) \, \gamma_{n-1}(dx_{1:n-1}) + \int I\big(\mu_n(\cdot|x_{1:n-1})|\gamma_1\big) \, \tilde{\mu}(dx_{1:n-1}) \\
&= I(\tilde{\mu}|\gamma_{n-1}) + \int I\big(\mu_n(\cdot|x_{1:n-1})|\gamma_1\big) \, d\mu(x),
\end{aligned}
$$

where the inequality holds by an application of Jensen's inequality with the function $\psi(u,v) = u^2/v$ which is convex on the upper half-plane $\mathbf{R} \times (0, \infty)$. The proof is completed by induction. $\square$

Proof of Theorem 1.3. Let us apply the one dimensional result (4.8) with constant $c = 1/(128\pi^2)$ in (5.2) to the measures μ_1 and $\mu_i(\cdot|x_{1:i-1})$. Put $t_1 = \int x_1 \, \mu_1(dx_1)$,

$$t_i(x) = t_i(x_1, \ldots, x_{i-1}) = \int x_i \, \mu_i(dx_i|x_{1:i-1}), \quad x = (x_1, \ldots, x_n) \in \mathbf{R}^n,$$

and denote by $\tilde{\mu}_i(\cdot|x_{1:i-1})$ the corresponding shift of $\mu_i(\cdot|x_{1:i-1})$ as in Corollary 4.3: $\tilde{\mu}_i(\cdot|x_{1:i-1}) = \mu_i(\cdot|x_{1:i-1})_{-t_i}$. Then we have

$$256\pi^2 \delta(\mu) \geq \frac{\mathcal{T}^2(\tilde{\mu}_1, \gamma_1)}{D(\tilde{\mu}_1|\gamma_1)} + \sum_{i=2}^{n} \int \frac{\mathcal{T}^2(\tilde{\mu}_i(\cdot|x_{1:i-1}), \gamma_1)}{D(\tilde{\mu}_i(\cdot|x_{1:i-1})|\gamma_1)} \, d\mu(x).$$

By Jensen's inequality with the convex function $\psi(u,v) = u^2/v$ $(u \in \mathbf{R}, v \geq 0)$,

$$256\pi^2\delta(\mu) \geq \frac{\mathcal{T}^2(\tilde{\mu}_1, \gamma_1)}{D(\tilde{\mu}_1|\gamma_1)} + \sum_{i=2}^{n} \frac{(\int \mathcal{T}(\tilde{\mu}_i(\cdot|x_{1:i-1}), \gamma_1) \, d\mu(x))^2}{\int D(\tilde{\mu}_i(\cdot|x_{1:i-1})|\gamma_1) \, d\mu(x)}$$

$$\geq \frac{(\mathcal{T}(\tilde{\mu}_1, \gamma_1) + \sum_{i=2}^{n} \int \mathcal{T}(\tilde{\mu}_i(\cdot|x_{1:i-1}), \gamma_1) \, d\mu(x))^2}{D(\tilde{\mu}_1|\gamma_1) + \sum_{i=2}^{n} \int D(\tilde{\mu}_i(\cdot|x_{1:i-1})|\gamma_1) \, d\mu(x)},$$

where the last bound comes from the inequality

$$\sum_{i=1}^{n} \psi(u_i, v_i) \geq \psi\left(\sum_{i=1}^{n} u_i, \sum_{i=1}^{n} v_i\right),$$

which is due to the convexity of ψ and its 1-homogeneity. Note that the first inequality could also be proved by using Cauchy–Schwarz inequality.

Now consider the map $T : \mathbf{R}^n \to \mathbf{R}^n$ defined for all $x \in \mathbf{R}^n$ by

$$T(x) = \big(x_1 - t_1, x_2 - t_2(x_1), \ldots, x_n - t_n(x_1, x_2, \ldots, x_{n-1})\big).$$

By definition, T pushes forward μ onto $\bar{\mu}$. The map T is invertible and its inverse $U = (u_1, \ldots, u_n)$ satisfies

$$u_1(x) = x_1 + t_1,$$
$$u_2(x) = x_2 + t_2\big(u_1(x)\big),$$
$$\vdots$$
$$u_i(x) = x_i + t_i\big(u_1(x), \ldots, u_{i-1}(x)\big),$$
$$\vdots$$
$$u_n(x) = x_n + t_n\big(u_1(x), \ldots, u_{n-1}(x)\big).$$

It is not difficult to check that $\bar{\mu}_1 = \tilde{\mu}_1$ and for all $i \geq 2$, $\bar{\mu}_i(\cdot|x_{1:i-1}) = \tilde{\mu}_i(\cdot|u_1(x), \ldots, u_{k-1}(x))$. Therefore, since U pushes forward $\bar{\mu}$ onto μ,

$$\mathcal{T}(\tilde{\mu}_1, \gamma_1) + \sum_{i=2}^{n} \int \mathcal{T}\big(\tilde{\mu}_i(\cdot|x_{1:i-1}), \gamma_1\big) \, d\mu(x)$$

$$= \mathcal{T}(\bar{\mu}_1, \gamma_1) + \sum_{i=2}^{n} \int \mathcal{T}\big(\tilde{\mu}_i(\cdot|u_1(x), \ldots, u_{i-1}(x)), \gamma_1\big) \, d\bar{\mu}(x)$$

$$= \mathcal{T}(\bar{\mu}_1, \gamma_1) + \sum_{i=2}^{n} \int \mathcal{T}\big(\bar{\mu}_i(\cdot|x_{1:i-1}), \gamma_1\big) \, d\bar{\mu}(x) \geq \mathcal{T}(\bar{\mu}, \gamma_n),$$

where we made use of Lemma 5.1 on the last step. The same with equality sign holds true for the D-functional. As a result, in terms of the recentered measure $\bar{\mu}$, we arrive at the following bound:

$$D(\mu|\gamma_n) + \frac{1}{256\pi^2} \frac{\mathcal{T}^2(\bar{\mu}, \gamma_n)}{D(\bar{\mu}|\gamma_n)} \leq \frac{1}{2} I(\mu|\gamma_n). \tag{5.3}$$

Thus, we have established in (5.3) the desired inequality (1.10) with constant $c = \frac{1}{256\pi^2}$. $\quad \square$

Remark 5.2. In order to relate the transport distance $\mathcal{T}$ to W_1, one may apply Lemma 2.3. Following the very definition of the transport distances, it implies that

$$(1 - \log 2) \min\{W_1(\mu, \nu), W_1^2(\mu, \nu)\} \leq \mathcal{T}(\mu, \nu) \leq W_1(\mu, \nu),$$

for all probability measures μ and ν on $\mathbf{R}^n$.

The proof of Theorem 1.4 will make use of the classical Prékopa–Leindler theorem, which we state below.

Theorem 5.3. *(See [31,32,26].) For a number $t \in (0, 1)$, assume that measurable functions $f, g, h : \mathbf{R}^d \to \mathbf{R}$ satisfy*

$$h\big((1 - t)x + ty\big) \leq (1 - t)f(x) + tg(y), \quad \text{for all } x, y \in \mathbf{R}^d.$$

Then

$$\int e^{-h(z)} \, dz \geq \left(\int e^{-f(x)} \, dx \right)^{1-t} \left(\int e^{-g(y)} \, dy \right)^{t}.$$

Proof of Theorem 1.4. It is similar to the proof of Theorem 1.3. The main point is that, if μ has a smooth density $f = e^{-V}$ with respect to Lebesgue measure, with a V such that $V'' \geq \varepsilon I_n$ for some $\varepsilon > 0$, then the first marginal μ_1 has a density of the form e^{-v_1} with $v_1'' \geq \varepsilon$. Moreover, for each $i = 2, \ldots, n$ and all $x \in \mathbf{R}^n$, the one dimensional conditional probability $\mu_i(\cdot|x_{1:i-1})$ has a density $e^{-v_i(x_i|x_{1:i-1})}$ with $(\partial^2/\partial x_i^2)v_i(x_i|x_{1:i-1}) \geq \varepsilon$. Indeed, by definition of conditional probabilities,

$$v_i(x_i|x_{1:i-1}) = -\log\left(\int e^{-V(x_{1:i}, y_{i+1:n})} \, dy_{i+1} \cdots dy_n \right) + w(x_{1:i-1}),$$

where $w(x_{1:i-1}) = \log(\int e^{-V(x_{1:i-1}, y_{i:n})} \, dy_i dy_{i+1} \cdots dy_n)$ does not depend on x_i. Since $V'' \geq \varepsilon I_n$, for any $i = 2, \ldots, n$ and any $x \in \mathbf{R}^n$, the function

$$(y_i, y_{i+1}, \ldots, y_n) \mapsto V(x_{1:i-1}, y_i, \ldots, y_n) - \frac{\varepsilon}{2} y_i^2$$

is convex. Thus defining, for $t \in (0, 1)$, $x \in \mathbf{R}^n$ and $a_i, b_i \in \mathbf{R}$, the functions

$$f(y_{i+1}, \ldots, y_n) = V(x_{1:i-1}, a_i, y_{i+1:n}) - \frac{\varepsilon}{2} a_i^2,$$

$$g(y_{i+1}, \ldots, y_n) = V(x_{1:i-1}, b_i, y_{i+1:n}) - \frac{\varepsilon}{2} b_i^2,$$

$$h(y_{i+1}, \ldots, y_n) = V\big(x_{1:i-1}, (1-t)a_i + tb_i, y_{i+1:n}\big) - \frac{\varepsilon}{2} \big((1-t)a_i + tb_i\big)^2,$$

one sees that

$$h\big((1-t)y_{i+1:n} + tz_{i+1:n}\big) \leq (1-t)f(y_{i+1:n}) + tg(z_{i+1:n}), \quad \text{for all } y, z \in \mathbf{R}^n.$$

Therefore, applying Theorem 5.3 to the triple (f, g, h), one gets easily that

$$v_i\big((1-t)a_i + tb_i | x_{1:i-1}\big) \leq (1-t)v_i(a_i | x_{1:i-1}) + tv_i(b_i | x_{1:i-1}) - \frac{\varepsilon}{2} t(1-t)(a_i - b_i)^2.$$

Since v_i is smooth, this inequality is equivalent to $(\partial/\partial x_i)^2 v_i(x_i | x_{1:i-1}) \geq \varepsilon$. A similar conclusion holds for v_1. Therefore, μ_1 and the conditional probabilities $\mu_i(\cdot | x_{1:i-1})$ verify the assumption of Corollary 4.4. Thus, applying the tensorisation formula (5.2), we get

$$\delta(\mu) \geq c \min\{1, \varepsilon\} \left(W_2^2(\tilde{\mu}_1, \gamma_1) + \sum_{i=2}^{n} W_2^2\big(\tilde{\mu}_i(\cdot | x_{1:i-1}), \gamma_1\big) \right),$$

where, as before, $\tilde{\mu}_i(\cdot | x_{1:i-1})$ is the shift of $\mu_i(\cdot | x_{1:i-1})$ by its mean. Reasoning as in the proof of Theorem 1.3, we see that the quantity inside the brackets is bounded from below by $W_2^2(\bar{\mu}, \gamma_n)$. $\square$

Acknowledgments

Appendix A. The reversed transport-entropy inequality

Here we include a simple proof of the general inequality of Lemma 3.2,

$$D(X_t | Y_t) \leq \frac{1}{2t} W_2^2(X, Y), \quad t > 0,$$

where X and Y are random vectors in $\mathbf{R}^n$ with finite second moments.

We denote by p_U the density of a random vector U and by $p_{U|V=v}$ the conditional density of U knowing the value of a random vector $V = v$. Note that the regularized random vectors $X_t = X + \sqrt{t}Z$ have smooth densities.

By the chain rule formula for the relative entropy, one has

$$D(X, Y, X_t | X, Y, Y_t) = D(X_t | Y_t) + \int D(p_{X,Y|X_t=v} | p_{X,Y|Y_t=v}) p_{X_t}(v)\, dv,$$

and therefore

$$D(X, Y, X_t | X, Y, Y_t) \geq D(X_t | Y_t).$$

On the other hand, we also have

$$D(X, Y, X_t | X, Y, Y_t) = \iint D(p_{X_t|(X,Y)=(x,y)} | p_{Y_t|(X,Y)=(x,y)}) p_{X,Y}(x,y)\, dx dy.$$

Now observe that $p_{X_t|(X,Y)=(x,y)}$ is the density of a normal law with mean x and covariance matrix tI_n, and similarly for $p_{Y_t|(X,Y)=(x,y)}$. But

$$D(x + \sqrt{t}Z | y + \sqrt{t}Z) = \frac{|x-y|^2}{2t},$$

so

$$D(X, Y, X_t | X, Y, Y_t) = \frac{1}{2t} \iint |x-y|^2 p_{X,Y}(x,y)\, dx dy = \frac{1}{2t} W_2^2(X, Y),$$

where the last equality follows by an optimal choice for the coupling density of X and Y.

Appendix B. Reinforced transport-entropy inequalities

In this section, we explain how to derive Theorem 4.1 in the form (4.3).

Proof of Theorem 4.1. To derive the inequality (4.3) for probability measures with mean zero, we follow an argument of [5]. Let μ be a probability measure on $\mathbf{R}$ such that $D(\mu|\gamma)$ is finite and consider the monotone rearrangement map T transporting γ onto μ. It is defined by $T(x) = F_\mu^{-1} \circ F_\gamma(x)$, where $F_\mu(x) = \mu(-\infty, x]$ and $F_\gamma(x) = \gamma(-\infty, x]$ are the corresponding distribution functions, and $F_\mu^{-1}(t) = \inf\{x \in \mathbf{R} : F_\mu(x) \geq t\}$ is the generalized inverse of F_μ (defined for $0 < t < 1$). It is well known that T pushes forward γ on μ and achieves the minimal value in the optimal transport problem:

$$W_2^2(\mu, \gamma) = \int \left(T(x) - x\right)^2 d\gamma(x).$$

The starting point is the following inequality going back to Talagrand's paper [36] (see Eq. (2.5) of [36]):

$$D(\mu|\gamma) \geq \frac{1}{2}W_2^2(\mu,\gamma) + \int \big(T'(x) - 1 - \log T'(x)\big)\, d\gamma(x)$$

$$\geq \frac{1}{2}W_2^2(\mu,\gamma) + \int \Delta\big(|T'(x) - 1|\big)\, d\gamma(x), \tag{B.1}$$

where the second inequality comes from the fact that $\Delta(x) \geq \Delta(|x|)$ for all $x > -1$. On the other hand, γ is known to satisfy the Cheeger-type analytic inequality

$$\lambda \int |f - m(f)|\, d\gamma \leq \int |f'|\, d\gamma \tag{B.2}$$

with optimal constant $\lambda = \sqrt{\frac{2}{\pi}}$ (see e.g. Theorem 1.3 of [10]). Here, $f : \mathbf{R} \to \mathbf{R}$ may be an arbitrary locally Lipschitz function with Radon–Nikodym derivative f', and $m(f)$ denotes a median of f under γ. According to Theorem 3.1 of [10], (B.2) can be generalized as

$$\int L\big(f - m(f)\big)\, d\gamma \leq \int L\big(c_L f'/\lambda\big)\, d\gamma \tag{B.3}$$

with an arbitrary even convex function $L : \mathbf{R} \to [0, \infty)$, such that $L(0) = 0$, $L(t) > 0$ for $t > 0$, and

$$c_L = \sup_{t>0} \frac{tL'(t)}{L(t)} < \infty,$$

where $L'(t)$ may be understood as the right derivative at t.

We apply (B.3) with $L(t) = \Delta(|t|) = |t| - \log(1 + |t|)$ in which case $c_L = 2$, so that

$$\int \Delta\big(|f - m(f)|\big)\, d\gamma \leq \int \Delta\big(2|f'|/\lambda\big)\, d\gamma. \tag{B.4}$$

It will be convenient to replace here the median with the mean $\gamma(f) = \int f\, d\gamma$. First observe that, by Jensen's inequality, (B.4) yields

$$\Delta\big(|\gamma(f) - m(f)|\big) \leq \int \Delta\big(2|f'|/\lambda\big)\, d\gamma. \tag{B.5}$$

Hence, using once more the convexity of Δ together with (B.4)–(B.5) for the function $2f$, we get

$$\int \Delta\big(|f - \gamma(f)|\big)\, d\gamma \leq \frac{1}{2}\int \Delta\big(2|f - m(f)|\big)\, d\gamma + \frac{1}{2}\Delta\big(2|\gamma(f) - m(f)|\big)$$

$$\leq \int \Delta\big(4|f'|/\lambda\big)\, d\gamma.$$

Equivalently,

$$\int \Delta(|f'|)\, d\gamma \geq \int \Delta\left(\frac{\lambda}{4}|f - \gamma(f)|\right) d\gamma.$$

To further simplify, one may use the lower bound a) of Lemma 2.3 which yields

$$\int \Delta(|f'|)\, d\gamma \geq \left(\frac{\lambda}{4}\right)^2 \int \Delta(|f - \gamma(f)|)\, d\gamma.$$

It remains to apply the latter with $f(x) = T(x) - x$ when estimating the last integral in (B.1). Since μ and γ have mean zero, this gives

$$D(\mu|\gamma) \geq \frac{1}{2}W_2^2(\mu, \gamma) + \frac{1}{8\pi}\int \Delta(T(x) - x)\, d\gamma(x),$$

and the last integral is certainly greater than (and actually equals to) $\mathcal{T}(\mu, \gamma)$.   □

Proof of Theorem 4.2. Let us return to the inequality (B.1), i.e.,

$$D(\mu|\gamma) \geq \frac{1}{2}W_2^2(\mu, \gamma) + \int \Delta(T'(x) - 1)\, d\gamma(x). \tag{B.6}$$

The basic assumption (4.4) ensures that T has a Lipschitz norm $\leq \frac{1}{\sqrt{\varepsilon}}$, so $T'(x) \leq \frac{1}{\sqrt{\varepsilon}}$. Using in (B.6) the lower quadratic bounds on Δ given in b) and c) of Lemma 2.3, we obtain that

$$D(\mu|\gamma) \geq \frac{1}{2}W_2^2(\mu, \gamma) + c(\varepsilon)\int \left(T'(x) - 1\right)^2 d\gamma(x), \tag{B.7}$$

where

$$c(\varepsilon) = \frac{1}{2}, \quad \text{for } \varepsilon \geq 1, \qquad c(\varepsilon) = \frac{\Delta(\frac{1}{\sqrt{\varepsilon}} - 1)}{(\frac{1}{\sqrt{\varepsilon}} - 1)^2}, \quad \text{for } 0 < \varepsilon < 1.$$

On the other hand, applying the Poincaré-type inequality for the Gaussian measure

$$\mathrm{Var}_\gamma(f) \leq \int f'^2\, d\gamma$$

with $f(x) = T(x) - x$, together with the assumption that $\int x\, d\mu(x) = \int T(x)\, d\gamma(x) = 0$, the last integral in (B.7) can be bounded from below by

$$\int \left(T(x) - x\right)^2 d\gamma(x) = W_2^2(\mu, \gamma).$$

It remains to use, for $0 < \varepsilon < 1$, the bound $\Delta(a) \geq (1 - \log 2)\min\{a, a^2\}$. The inequality (4.5) is proved.   □

Appendix C. Equality cases in the logarithmic Sobolev inequality for the standard Gaussian measure

In this last section, we show how Theorem 1.3 can be used to recover the following result by E. Carlen [11].

Theorem C.1. *(See [11].) Let μ be a probability measure on $\mathbf{R}^n$ such that $D(\mu|\gamma) < \infty$. We have*

$$D(\mu|\gamma) = \frac{1}{2}I(\mu|\gamma),$$

if and only if μ is a translation of γ.

In what follows, we denote by $\mathcal{S}_n$ the set of permutations of $\{1, \ldots, n\}$. If μ is a probability measure on $\mathbf{R}^n$, we denote by μ_σ its image under the permutation map

$$(x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

If μ has density f with respect to the standard n-dimensional Gaussian measure γ, then the density of μ_σ with respect to γ is given by

$$f_\sigma(x_1, \ldots, x_n) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

Obviously,

$$I(\mu_\sigma|\gamma) = I(\mu|\gamma) \quad \text{and} \quad D(\mu_\sigma|\gamma) = D(\mu|\gamma).$$

Hence, we have the following automatic improvement of Theorem 1.3.

Theorem C.2. *Let X be a random vector in $\mathbf{R}^n$ with law μ. Then,*

$$D(\mu|\gamma) + c \max_{\sigma \in \mathcal{S}_n} \frac{\mathcal{T}^2(\overline{\mu_\sigma}, \gamma)}{D(\overline{\mu_\sigma}|\gamma)} \leq \frac{1}{2}I(\mu|\gamma),$$

where $\overline{\mu_\sigma}$ is the law of the random vector Y^σ defined by

$$Y_i^\sigma = X_{\sigma(i)} - \mathbf{E}(X_{\sigma(i)}|X_{\sigma(1)}, \ldots, X_{\sigma(i-1)}).$$

Proof of Theorem C.1. To avoid complicated notations, we will restrict ourselves to the dimension $n = 2$. We may assume that μ has a smooth density p with respect to the Lebesgue measure such that $D(\mu|\gamma) = \frac{1}{2}I(\mu|\gamma) < \infty$. Necessarily, μ has a finite second moment, and moreover, $\overline{\mu_\sigma} = \gamma$, for all $\sigma \in \mathcal{S}_2$, i.e., for $\sigma = \mathrm{id} = (12)$ and $\sigma = (21)$.

For a random vector X with law μ, put $m_1 = \mathbf{E}X_1$, $m_2 = \mathbf{E}X_2$, $a(X_1) = \mathbf{E}(X_2|X_1)$ and $b(X_2) = \mathbf{E}(X_1|X_2)$. The probability measure $\gamma = \overline{\mu_{\mathrm{id}}}$ represents the image of μ under the map $(x_1, x_2) \mapsto (x_1 - m_1, x_2 - a(x_1))$. It then easily follows that

$$p(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1 - m_1)^2 - \frac{1}{2}\big(x_2 - a(x_1)\big)^2 \right)$$

for almost all $(x_1, x_2) \in \mathbf{R}^2$. Since also $\gamma = \overline{\mu_{(2,1)}}$, the same reasoning yields

$$p(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_2 - m_2)^2 - \frac{1}{2}\big(x_1 - b(x_2)\big)^2 \right),$$

for almost all $(x_1, x_2) \in \mathbf{R}^2$. Therefore, for almost all $(x_1, x_2) \in \mathbf{R}^2$, it holds

$$(x_1 - m_1)^2 + \big(x_2 - a(x_1)\big)^2 = (x_2 - m_2)^2 + \big(x_1 - b(x_2)\big)^2.$$

Let us denote by A the set of all couples (x_1, x_2) for which there is equality, and for $x_1 \in \mathbf{R}$, let $A_{x_1} = \{x_2 \in \mathbf{R} : (x_1, x_2) \in A\}$ denote the corresponding section of A. By Fubini's theorem,

$$0 = \big|\mathbf{R}^2 \setminus A\big| = \int_{-\infty}^{\infty} \big|\mathbf{R} \setminus A_{x_1}\big| \, dx_1,$$

where $|\cdot|$ stands for the Lebesgue measure of a set in the corresponding dimension. Hence, for almost all x_1, the set $\mathbf{R} \setminus A_{x_1}$ is of Lebesgue measure 0. For any such x_1,

$$2x_2\big(m_2 - a(x_1)\big) + a(x_1)^2 - m_2^2 + (x_1 - m_1)^2 \geq 0, \quad \forall x_2 \in A_{x_1}.$$

Thus, $a(x_1) = m_2$ (otherwise letting $x_2 \to \pm\infty$ would lead to a contradiction). This proves that $a = m_2$ almost everywhere, and therefore, the random vector $(X_1 - \mathbf{E}X_1, X_2 - \mathbf{E}X_2)$ is standard Gaussian. But this means that μ is a translation of γ. $\quad\square$

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