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MATHEMATICS

Localization for Infinite-Dimensional Hyperbolic Measures

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Following Borell [1], we say that a Radon probability measure μ on a locally convex space E is α-concave ($-\infty \le \alpha \le 1$) if it obeys the Brunn–Minkowski-type inequality

$$\mu((1-t)A + tB) \ge [(1-t)\mu(A)^{\alpha} + t\mu(B)^{\alpha}]^{1/\alpha} \quad (1)$$

for any nonempty compact sets $A, B \subset E$ and all $0 < t < 1$. Here, we use the standard notation for Minkowski weighted averaging of sets, i.e., $(1-t)A + tB = \{(1-t)a + tb\colon a \in A,\ b \in B\}$. The same definition is used for general (positive) measures, not necessarily probability.

In spite of the long history of study, α-concave measures remain very popular in probability theory and convex geometry. Even in the case $\alpha = 0$, where inequality (1) takes the form

$$\mu((1-t)A + tB) \ge \mu(A)^{1-t}\mu(B)^{t},$$

we come to the important class of logarithmically concave measures, which includes, in particular, all Gaussian measures. On Euclidean space $E = \mathbb{R}^n$, the class of logarithmically concave measures was first considered by Prékopa [2]; in the one-dimensional case, it was studied earlier by other authors [3, 4].

Inequality (1) becomes less restrictive as the parameter α decreases, and in the limit case $\alpha = -\infty$, we obtain the largest class, which is described by the inequality

$$\mu((1-t)A + tB) \ge \min\{\mu(A), \mu(B)\}.$$

Such measures are said to be convex (in Borell's terminology) or hyperbolic (this term was suggested later by V.D. Mil'man).

α-Concave probability measures arise as distributions of certain random processes, so that the dimensionless character of inequality (1) turns out to be very useful in studying various properties of processes with distribution μ. Moreover, many assertions depend only on the convexity parameter α; interesting examples related to Brownian motion are given in [1]. Borell has also investigated the most general properties of α-concave measures, such as the zero-one law and the integrability of norms.

A more detailed analysis of α-concave measures on finite-dimensional spaces, in particular, solving isoperimetric problems and studying Sobolev-type integro-differential inequalities and Khintchine-type inequalities for various functionals, has become possible after Lovász and Simonovits [5] had introduced the so-called localization lemma.

Theorem 1 [5]. *If lower semicontinuous integrable functions u, v: $\mathbb{R}^n \to \mathbb{R}$ satisfy the condition*

$$\int_{\mathbb{R}^n} u(x)dx > 0, \qquad \int_{\mathbb{R}^n} v(x)dx > 0,$$

then there always exist points $a, b \in \mathbb{R}^n$ and a positive affine function l on $(0, 1)$ *for which*

$$\int_0^1 u((1-t)a + tb)l(t)^{n-1}dt > 0,$$

$$\int_0^1 v((1-t)a + tb)l(t)^{n-1}dt > 0.$$

There also exist other versions of this important theorem ([6]; see also [7]). The approach of Lovász and Simonovits, developing ideas of Payne and Weinberger [8], is based on the notion of a needle, arising as a result of the localization of sets on which the integrals of the functions u and v remain positive (simultaneously). Later, Fradelizi and Guédon suggested an alternative approach based on a description of extremal compactly supported α-concave measures followed by the application of the well-known Krein–Milman theorem (see [9]).

The localization lemma is stated for the Lebesgue measure on $\mathbb{R}^n$ (as a conjecture) and for a measure with weight l^{n-1} on an interval $[a, b]$ (as a conclusion). In both cases, it deals with α-concave measures for

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$\alpha = \frac{1}{n}$. It turns out that Theorem 1 can be extended by fixing α on arbitrary (including infinite-dimensional) spaces.

Theorem 2. *Let* (E, μ) *be a complete locally convex space with finite* α*-concave measure. If lower semicontinuous* μ*-integrable functions u and* v *on E satisfy the condition*

$$\int_E u\,d\mu > 0, \quad \int_E v\,d\mu > 0,$$

then there exist points $a, b \in E$ *and a finite* α*-concave measure* ν *concentrated on* $\Delta = [a, b]$ *such that*

$$\int_\Delta u\,d\nu > 0, \quad \int_\Delta v\,d\nu > 0.$$

Theorem 1 is often applied to reduce various integral relations in $\mathbb{R}^n$ to similar inequalities on a straight line, although with an additional weight under the integral sign. By virtue of Theorem 2, if such a relation must cover the class of all α-concave measures on E, then the problem reduces to obtaining the required result for α-concave measures concentrated on straight lines in the space E.

By way of example, we give several inequalities for measures of sets arising as a result of applying a contraction-type operation:

$$A_\delta = \{x \in A\colon \forall \Delta \subset F(x \in \Delta \Rightarrow m_\Delta(A) \ge 1 - \delta)\},$$
$$\delta \in [0, 1].$$

Here, A is a Borel measurable subset of a closed convex set $F \subset E$, Δ is any segment inside F, and m_Δ is the normalized Lebesgue measure on Δ (a uniform distribution).

For example, if $F = E$ and A is the complement to a centrally symmetric open convex set $B \subset E$, then $A_\delta = E \backslash \left(\frac{2}{\delta} - 1\right) B$.

Theorem 3. *Let* (E, μ) *be a complete locally convex space with* α*-concave probability measure concentrated on a closed convex set F. For any Borel measurable set* $A \subset F$ *and any* $\delta \in [0, 1]$ *such that* $\mu^*(A_\delta) > 0$, *we have*

$$\mu(A) \ge [\delta \mu^*(A_\delta)^\alpha + (1 - \delta)]^{1/\alpha}. \tag{2}$$

Here, μ^* denotes the outer measure associated with μ.

In space $\mathbb{R}^n$, Theorem 3 was proved in [10, 11]. It generalizes a result of Nazarov et al. [12] for logarithmically concave measures. In this case, the sets A_δ are universally measurable, and inequality (2) takes the form

$$\mu(A) \ge \mu(A_\delta)^\delta.$$

For applications, it is useful to state Theorem 3 in terms of distributions of functionals on the space E. For each Borel measurable μ-almost everywhere finite function $f\colon E \to [-\infty, \infty]$ and any $0 < \varepsilon < 1$, we define the so-called modulus of regularity

$$\delta_f(\varepsilon) = \sup \text{mes}\{t \in (0, 1)\colon |f((1-t)x + ty)| \le \varepsilon |f(x)|\},$$

where the supremum is over all points $x, y \in E$ such that $|f(x)| < \infty$.

Corollary 1. *For any* $0 < \lambda < \text{esssup}|f|$ *and* $\varepsilon \in (0, 1)$,

$$\mu\{|f| > \lambda\varepsilon\} \ge [\delta\mu\{|f| \ge \lambda\}^\alpha + (1 - \delta)]^{1/\alpha}, \tag{3}$$

where $\delta = \delta_f(\varepsilon)$.

As mentioned in [13], the behavior of the quantity $\delta_f(\varepsilon)$ at small ε is responsible for the probabilities of both small and large deviations of f with respect to the measure μ (see also [10]).

For example, if f is a measurable μ-almost everywhere finite norm on E (not identically zero), then

$$\delta_f(\varepsilon) = \frac{2\varepsilon}{1 + \varepsilon}, \quad 0 < \varepsilon \le 1.$$

In this case, setting

$$B = \{x \in E\colon f(x) \le 1\},$$

we see that inequality (3) takes the form

$$1 - \mu(B) \ge \left[\frac{2}{r+1}(1 - \mu(rB))^\alpha + \frac{r-1}{r+1}\right]^{1/\alpha}, \tag{4}$$
$$r > 1.$$

According to Theorem 3, this inequality holds for any centrally symmetric Borel measurable convex set B in a complete locally convex space E endowed with a α-concave probability measure μ. In the case $\alpha = 0$, we obtain the exponential estimate

$$1 - \mu(rB) \le (1 - \mu(B))^{(r+1)/2}$$

for the class of all logarithmically concave measures. By using the localization lemma, this inequality was first proved by Lovász and Simonovits for balls and then extended by Guédon to arbitrary symmetric bodies in space $\mathbb{R}^n$ [5, 14].

For $\alpha < 0$, (4) leads to similar estimates with a polynomial decrease with respect to r.

It is also easy to derive estimates for small ball probabilities on the basis of (4). For example, under the condition $\mu(B) \le \frac{1}{2}$, we conclude that

$$\mu(\varepsilon B) \le C_\alpha \varepsilon \quad (0 \le \varepsilon \le 1)$$

with the constant $C_\alpha = \frac{2(2^{-\alpha} - 1)}{-\alpha}$ $(\alpha \le 0)$.

Finally, note that, for infinite-dimensional α-concave probability measures, the condition $\alpha \le 0$ is necessary [15].

REFERENCES

1. C. Borell, Ark. Math. **12**, 239–252 (1974).
2. A. Prekopa, Acta Sci. Math. (Szeged) **32**, 301–316 (1971).
3. I. A. Ibragimov, Teor. Veroyatn. Ee Primen. **1**, 283–288 (1956).

4. Yu. S. Davidovich, B. I. Korenblyum, and B. I. Khaset, Dokl. Akad. Nauk SSSR **185**, 1215–1218 (1969).
5. L. Lovász and M. Simonovits, Random Struct. Algor. **4** (4), 359–412 (1993).
6. R. Kannan, L. Lovász, and M. Simonovits, Discrete Comput. Geom. **13** (3/4), 541–559 (1995).
7. M. Gromov and V. D. Milman, Compos. Math. **62**, 263–282 (1987).
8. L. E. Payne and H. F. Weinberger, Arch. Ration. Mech. Anal. **5**, 286–292 (1960).
9. M. Fradelizi and O. Guedon, Discrete Comput. Geom. **31** (2), 327–335 (2004).
10. S. G. Bobkov and F. L. Nazarov, J. Math. Sci. (N.Y.) **152** (6), 826–839 (2008).
11. M. Fradelizi, Electron. J. Probab. **14** (71), 2068–2090 (2009).
12. F. Nazarov, M. Sodin, and A. Vol'berg, St. Petersb. Math. J. **14** (2), 351–366 (2003).
13. S. G. Bobkov, Adv. Math. Res. **2**, 151–175 (2003).
14. O. Guedon, Mathematika **46** (1), 165–173 (1999).
15. C. Borell, Math. Scand. **53** (1), 125–144 (1983).

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