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KLS-type isoperimetric bounds for log-concave probability measures

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Abstract The paper considers geometric lower bounds on the isoperimetric constant for logarithmically concave probability measures, extending and refining some results by Kannan et al. (Discret Comput Geom 13:541–559, 1995).

Keywords Isoperimetric inequalities · Logarithmically concave distributions · Geometric functional inequalities

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1 Introduction

Let μ be a log-concave probability measure on $\mathbf{R}^n$ with density f, thus satisfying

$$f((1-t)x+ty) \geq f(x)^{1-t} f(y)^t, \quad \text{for all } x, y \in \mathbf{R}^n,\ t \in (0,1). \tag{1.1}$$

We consider an optimal value $h = h(\mu)$, called an isoperimetric constant of μ, in the isoperimetric-type inequality

$$\mu^+(A) \geq h \min\{\mu(A), 1-\mu(A)\}. \tag{1.2}$$

Here, A is an arbitrary Borel subset of $\mathbf{R}^n$ of measure $\mu(A)$ with μ-perimeter

$$\mu^+(A) = \liminf_{\varepsilon \downarrow 0} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon},$$

where $A_\varepsilon = \{x \in \mathbf{R}^n : |x-a| < \varepsilon,\ \text{for some } a \in A\}$ denotes an open ε-neighborhood of A with respect to the Euclidean distance.

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The geometric characteristic h is related to a number of interesting analytic inequalities such as the Poincaré-type inequality

$$\int |\nabla u|^2 \, \mathrm{d}\mu \,\geq\, \lambda_1 \int |u|^2 \, \mathrm{d}\mu$$

(which is required to hold for all smooth bounded functions u on $\mathbf{R}^n$ with $\int u \, \mathrm{d}\mu = 0$). In general, the optimal value λ_1, also called the spectral gap, satisfies $\lambda_1 \geq \frac{h^2}{4}$. This relation goes back to the work by J. Cheeger in the framework of Riemannian manifolds [10] and to earlier works by V. G. Maz'ya (cf. [13,18,19]). The problem on bounding these two quantities from below has a long story. Here, we specialize to the class of log-concave probability measures, in which case λ_1 and h are equivalent ($\lambda_1 \leq 36\, h^2$, cf. [16,20]).

As an important particular case, one may consider a uniform distribution $\mu = \mu_K$ in a given convex body K in $\mathbf{R}^n$ (i.e., the normalized Lebesgue measure on K). In this situation, Kannan, Lovász and Simonovits proposed the following geometric bound on the isoperimetric constant. With K, they associate the function

$$\chi_K(x) = \max\{|h| : x + h \in K \text{ and } x - h \in K\}, \qquad x \in \mathbf{R}^n, \tag{1.3}$$

which expresses one half of the length of the longest interval inside K with center at x (Note that $\chi_K = 0$ outside K). Equivalently, in terms of the diameter,

$$\chi_K(x) = \frac{1}{2}\, \mathrm{diam}((K - x) \cap (x - K)).$$

It is shown in [15] that any convex body K admits an isoperimetric inequality

$$\mu^+(A) \geq \frac{1}{\int \chi_K \, \mathrm{d}\mu_K}\, \mu(A)(1 - \mu(A))$$

which thus provides the bound

$$h(\mu_K)^{-1} \leq 2 \int \chi_K \, \mathrm{d}\mu_K. \tag{1.4}$$

Some sharpening of this result were considered in [8], where it is shown that

$$\mu^+(A) \geq \frac{c}{\int \chi_K \, \mathrm{d}\mu_K}\, (\mu(A)(1 - \mu(A)))^{(n-1)/n}$$

where $c > 0$ is an absolute constant.

The purpose of the present note is to extend the KLS bound (1.4) to general log-concave probability measures. If μ on $\mathbf{R}^n$ has density f satisfying (1.1), define $\chi_{f,\delta}(x)$ with a parameter $0 < \delta < 1$ to be the supremum over all $|h|$ such that

$$\sqrt{f(x+h) f(x-h)} > \delta f(x). \tag{1.5}$$

Note that, by the hypothesis (1.1), we have an opposite inequality with $\delta = 1$, i.e.,

$$\sqrt{f(x+h) f(x-h)} \leq f(x). \tag{1.6}$$

Hence, in some sense, $\chi_{f,\delta}(x)$ measures the strength of log-concavity of f at a given point x.

Further comments on the definition of $\chi_{f,\delta}(x)$ will be given in the next section. Note however that in the convex body case, when $\mu = \mu_K$ with density $f(x) = \frac{1}{\mathrm{vol}_n(K)}\, 1_K(x)$, we recover the original quantity above, namely,

$$\chi_{f,\delta}(x) = \chi_K(x), \quad \text{for all } 0 < \delta < 1.$$

Our main observation about the new geometric characteristic is the following:

Theorem 1.1 *For any probability measure μ on $\mathbf{R}^n$ with a log-concave density f, and for any $0 < \delta < 1$, we have*

$$h(\mu)^{-1} \leq \frac{C}{1-\delta} \int \chi_{f,\delta}\, \mathrm{d}\mu \tag{1.7}$$

where C is an absolute constant. One may take $C = 64$.

We will comment on examples and applications later on (cf. Sect. 5). Here, let us only mention that in certain cases (1.7) provides a sharp estimate with respect to the dimension n. For example, for the standard Gaussian and other rotationally invariant measures, normalized by the condition $\int |x|^2\, \mathrm{d}\mu(x) = n$, both sides of (1.7) are of order 1 (with a fixed value of δ). One may also consider the class of uniformly convex log-concave measures, in which case the functions $\chi_{f,\delta}$ are necessarily bounded on the whole space.

In fact, the bound (1.7) admits a further self-improvement by reducing the integration on the right-hand side to a smaller region whose measure is however not small. On this step, one may apply recent stability results due to E. Milman. In particular, it is shown in [21] that, if a log-concave probability measure ν is absolutely continuous with respect to the given log-concave probability measure μ on $\mathbf{R}^n$, and if these measures are close in total variation so that

$$\|\nu - \mu\|_{\mathrm{TV}} = \sup_A |\nu(A) - \mu(A)| \leq \alpha < 1,$$

then $h(\mu)^{-1} \leq C_\alpha h(\nu)^{-1}$ up to some constant depending on α, only. As we will see, the above condition with a universal value of α is always fulfilled for the normalized restriction ν of μ to the Euclidean ball B_R with center at the origin and of radius

$$R = \int |x|\, \mathrm{d}\mu(x).$$

As a result, this leads, to:

Corollary 1.2 *For any probability measure μ on $\mathbf{R}^n$ with a log-concave density f, and for any $0 < \delta < 1$, we have*

$$h(\mu)^{-1} \leq C_\delta \int_{|x|\leq R} \chi_{f\cdot 1_{B_R},\delta}\, \mathrm{d}\mu \tag{1.8}$$

where the constant depends on δ, only.

In particular, since $\chi_{f\cdot 1_{B_R},\delta}(x) \leq \chi_{B_R}(x) = \sqrt{R^2 - |x|^2}$ (cf. Lemma 2.3 below), the estimate (1.8) implies

$$h(\mu)^{-1} \leq C_\delta \int_{|x|\leq R} \sqrt{R^2 - |x|^2}\, \mathrm{d}\mu(x) \leq C_\delta \left(\int_{|x|\leq R} (R^2 - |x|^2)\, \mathrm{d}\mu(x)\right)^{1/2}.$$

By another application of Cauchy's inequality (and taking, e.g., $\delta = \frac{1}{2}$), we get with some universal constant C

$$h(\mu)^{-1} \leq C\, \mathrm{Var}(|X|^2)^{1/4} \tag{1.9}$$

where X is a random vector in $\mathbf{R}^n$ distributed according to μ. This bound was obtained in [5] as a possible sharpening of another estimate by Kannan, Lovász and Simonovits,

$$h(\mu)^{-1} \leq C\, \mathbf{E}\,|X| = CR \tag{1.10}$$

(cf. also [2]). It is interesting that the two alternative bounds, (1.4) and (1.10), are characterized in [15] as non-comparable. Now, we see that the χ-estimate given in Corollary 1.2 unites and refines all these results including (1.9), as well.

2 Reduction to dimension one

The proof of Theorem 1.1 and Corollary 1.2 uses the localization lemma of Lovász and Simonovits [17], see also [11,12] for further developments. We state it below in an equivalent form.

Lemma 2.1 ([17]) *Suppose that u_1 and u_2 are continuous integrable functions defined on an open convex set $E \subset \mathbf{R}^n$ and such that, for any compact segment $\Delta \subset E$ and for any positive affine function l on Δ,*

$$\int_\Delta u_1\, l^{n-1} \le 0, \qquad \int_\Delta u_2\, l^{n-1} \le 0. \tag{2.1}$$

Then,

$$\int_E u_1 \le 0, \qquad \int_E u_2 \le 0. \tag{2.2}$$

The one-dimensional integrals in (2.1) are taken with respect to the Lebesgue measure on Δ, while the integrals in (2.2) are n-dimensional.

This lemma allows one to reduce various multidimensional integral relations to the corresponding one-dimensional relations with additional weights of the type l^{n-1}. In some reductions, the next lemma, which appears in a more general form as Corollary 2.2 in [15], is more convenient.

Lemma 2.2 ([15]) *Suppose that u_i, $i = 1, 2, 3, 4$ are nonnegative continuous functions defined on an open convex set $E \subset \mathbf{R}^n$ and such that, for any compact segment $\Delta \subset E$ and for any positive affine function l on Δ,*

$$\int_\Delta u_1\, l^{n-1} \int_\Delta u_2\, l^{n-1} \le \int_\Delta u_3\, l^{n-1} \int_\Delta u_4\, l^{n-1}. \tag{2.3}$$

Then,

$$\int_E u_1 \int_E u_2 \le \int_E u_3 \int_E u_4. \tag{2.4}$$

Lemmas 2.1 and 2.2 remain to hold for many discontinuous functions u_i, as well, like the indicator functions of open or closed sets in the space.

If a log-concave function f is defined on a convex set $E \subset \mathbf{R}^n$, we always extend it to be zero outside E. The definition (1.5) is applied with the convention that $\chi_{f,\delta} = 0$ outside the supporting set $E_f = \{f > 0\}$. More precisely, for $x \in E_f$,

$$\chi_{f,\delta}(x) = \sup\left\{|h| : \, x \pm h \in E_f, \ \sqrt{f(x+h)f(x-h)} > \delta f(x)\right\}.$$

One may always assume that E_f is relatively open (i.e., open in the minimal affine subspace of $\mathbf{R}^n$ containing E_f). In that case, the function

$$\rho(x,h) = \frac{\sqrt{f(x+h)f(x-h)}}{f(x)}$$

is continuous on the relatively open set $\{(x, h) \in \mathbf{R}^n \times \mathbf{R}^n : x \pm h \in E_f\}$, so that any set of the form

$$\{x \in E_f : \chi_{f,\delta}(x) > r\} = \bigcup_{|h|>r} \{x \in E_f : x \pm h \in E_f,\ \rho(x, h) > \delta\}$$

is open in E_f. This means that the function $\chi_{f,\delta}$ is lower semi-continuous on E_f and therefore Borel measurable on $\mathbf{R}^n$.

We will need two basic properties of the functions $\chi_{f,\delta}$ which are collected in the next lemma.

Lemma 2.3 *Let f be a log-concave function defined on a convex set $E \subset \mathbf{R}^n$, and let* $0 < \delta < 1$.

(*a*) *If F is a convex subset of E, then $\chi_{f|_F,\delta} \le \chi_{f,\delta}$ on F.*
(*b*) *If g is a positive log-concave function defined on a convex subset F of E, then $\chi_{fg,\delta} \le \chi_{f,\delta}$ on F.*

In fact, the first property follows from the second one with the particular log-concave function $g = 1_F$. To prove b), let $x \in F$ be fixed. Applying the definition and the inequality (1.6) with g, we get

$$\begin{aligned}\chi_{fg,\delta}(x) &= \sup\left\{|h| : x \pm h \in F,\ \sqrt{f(x+h)f(x-h)g(x+h)g(x-h)} > \delta f(x)g(x)\right\}\\ &\le \sup\left\{|h| : x \pm h \in F,\ \sqrt{f(x+h)f(x-h)} > \delta f(x)\right\}\\ &\le \chi_{f,\delta}(x).\end{aligned}$$

It is also worthwile to emphasize the homogeneity property of the functional $X \to \int \chi_{f,\delta}\, \mathrm{d}\mu$ with respect to X, where μ is the distribution of X.

Lemma 2.4 *Let a random vector X in $\mathbf{R}^n$ have the distribution μ with a log-concave density f. For $\lambda > 0$, let μ_λ denotes the distribution and*

$$f_\lambda(x) = \lambda^{-n} f(x/\lambda), \qquad x \in \mathbf{R}^n,$$

the density of λX. Then $\chi_{f_\lambda,\delta}(\lambda x) = \lambda \chi_{f,\delta}(x/\lambda)$, for all $x \in \mathbf{R}^n$. In particular,

$$\int \chi_{f_\lambda,\delta}\, \mathrm{d}\mu_\lambda = \lambda \int \chi_{f,\delta}\, \mathrm{d}\mu.$$

The statement is straightforward and does not need a separate proof. It is not used in the proof of Theorem 1.1, but just shows that the inequality (1.7) is homogeneous with respect to dilations of μ.

Following [15], let us now describe how to reduce Theorem 1.1 to a similar assertion in dimension one (with an additional factor of 2). Let $E \subset \mathbf{R}^n$ be an open convex supporting set for the log-concave density f of μ. We are going to apply Lemma 2.2 to the functions of the form

$$u_1 = 1_A, \quad u_2 = 1_B, \quad u_3 = 1_C, \quad u_4(x) = \frac{c}{\varepsilon}\, \chi_{f,\delta}(x),$$

where A is an arbitrary closed subset of $\mathbf{R}^n$, $B = \mathbf{R}^n \setminus A_\varepsilon$, and $C = \mathbf{R}^n \setminus (A \cup B)$, with fixed constants $c > 0$ and $\varepsilon > 0$. Then, (2.4) turns into

$$\mu(A)\mu(B) \le \mu(C)\, \frac{c}{\varepsilon} \int \chi_{f,\delta}\, \mathrm{d}\mu, \tag{2.5}$$

and letting $\varepsilon \to 0$, we arrive at the isoperimetric inequality of the form

$$\mu(A)(1-\mu(A)) \le c\mu^+(A) \int \chi_{f,\delta}\,\mathrm{d}\mu. \tag{2.6}$$

Actually, (2.5) also follows from (2.6), cf. [8], Proposition 10.1, and consequently, the inequalities (2.5) and (2.6) are equivalent. It should be mentioned as well that, once (2.6) holds true in the class of all closed subsests of the space, it extends to all Borel sets in $\mathbf{R}^n$.

Now, for the chosen functions u_i, the one-dimensional inequality (2.3) is the same as (2.5),

$$\mu_l(A)\mu_l(B) \le \mu_l(C)\,\frac{c}{\varepsilon} \int \chi_{f,\delta}\,\mathrm{d}\mu_l, \tag{2.7}$$

but written for the probability measure μ_l on Δ with density

$$f_l(x) = \frac{\mathrm{d}\mu_l}{\mathrm{d}x} = \frac{1}{Z}\, f(x)l(x)^{n-1}, \qquad x \in \Delta.$$

Here $Z = \int_\Delta f(x)l^{n-1}(x)\,\mathrm{d}x$ is a normalizing constant, and $\mathrm{d}x$ denotes the Lebesgue measure on Δ. Likewise (2.6), it is equivalent to the isoperimetric inequality

$$\mu_l(A)(1-\mu_l(A)) \le c\mu_l^+(A) \int \chi_{f,\delta}\,\mathrm{d}\mu_l. \tag{2.8}$$

The density of μ_l is log-concave and has the form $f_l = fg$, where g is log-concave on $F = \Delta$. Hence, by Lemma 2.3, (2.8) will only be strengthened, if we replace $\chi_{f,\delta}$ by $\chi_{f_l,\delta}$. But the resulting stronger inequality

$$\mu_l(A)(1-\mu_l(A)) \le c\mu_l^+(A) \int \chi_{f_l,\delta}\,\mathrm{d}\mu_l \tag{2.9}$$

is again equivalent to

$$\mu_l(A)\mu_l(B) \le \mu_l(C)\,\frac{c}{\varepsilon} \int \chi_{f_l,\delta}\,\mathrm{d}\mu_l. \tag{2.10}$$

Thus, by Lemma 2.2, (2.5)–(2.6) will hold for μ with density f, as soon as the one-dimensional inequalities (2.9)–(2.10) hold true for μ_l with density f_l. In particular, (2.5)–(2.6) with a constant c will be true in the entire class of full-dimensional log-concave probability measures, as soon as they hold with the same constant in dimension one (since the assertion on the intervals $\Delta \subset E$ may be stated for intervals on the real line).

The inequalities (2.6) and (2.9) are not of the Cheeger-type (1.2), but are equivalent to it within the universal factor of 2, in view of the relations $p(1-p) \le \min\{p, 1-p\} \le 2p(1-p)$ $(0 \le p \le 1)$. Hence, we may summarize this reduction in the following:

Corollary 2.5 *Given numbers $C > 0$ and $0 < \delta < 1$, assume that, for any probability measure μ on $\mathbf{R}$ with a log-concave density f,*

$$h(\mu)^{-1} \le C \int \chi_{f,\delta}\,\mathrm{d}\mu.$$

Then, the same inequality with constant $2C$ in place of C holds true for any probability measure μ on $\mathbf{R}^n$ with a log-concave density f.

3 Dimension one

Next, we consider the one-dimensional case in Theorem 1.1. Let μ be a probability measure on the real line with a log-concave density f supported on the interval (a, b), finite or not. In particular, f is bounded and continuous on that interval.

As a first step, we will bound from below the function $\chi_{f,\delta}$ near the median m of μ in terms of the maximum value $M = \sup_{a<x<b} f(x)$. Without the loss of generality, assume that $m = 0$, that is,

$$\mu(-\infty, 0] = \mu[0, \infty) = \frac{1}{2}.$$

Let $F(x) = \mu(a, x)$, $a < x < b$, denote the distribution function associated to μ, and let $F^{-1} : (0, 1) \to (a, b)$ be its inverse. The function $I(t) = f(F^{-1}(t))$ is concave on $(0, 1)$, so, for each fixed $t \in (0, 1)$, it admits linear bounds

$$I(s) - I(t) \geq -I(t)\,\frac{t-s}{t}, \qquad 0 < s \leq t,$$
$$I(s) - I(t) \geq -I(t)\,\frac{s-t}{1-t}, \qquad t \leq s < 1.$$

Equivalently, after the change $t = F(x)$, $s = F(y)$, we have

$$f(y) - f(x) \geq -f(x)\,\frac{F(x) - F(y)}{F(x)}, \qquad a < y \leq x < b,$$
$$f(y) - f(x) \geq -f(x)\,\frac{F(y) - F(x)}{1 - F(x)}, \qquad a < x \leq y < b.$$

On the other hand, $F(y) - F(x) \leq M(y - x)$ and $|F(x) - \frac{1}{2}| \leq M\,|x|$, so that

$$F(x) \geq \frac{1}{2} - M\,|x|, \qquad 1 - F(x) \geq \frac{1}{2} - M\,|x|.$$

This yields

$$f(y) - f(x) \geq -Mf(x)\,\frac{x-y}{\frac{1}{2} - M\,|x|}, \qquad a < y \leq x < b,$$
$$f(y) - f(x) \geq -Mf(x)\,\frac{y-x}{\frac{1}{2} - M\,|x|}, \qquad a < x \leq y < b.$$

In particular, if $|x| \leq \frac{1}{4M}$, in both cases we get

$$f(y) - f(x) \geq -4Mf(x)\,|y - x|, \qquad a < y < b. \tag{3.1}$$

For $y = x \pm h$, rewrite the above as $f(x \pm h) \geq f(x)(1 - 4M\,|h|)$. Hence, for any $h > 0$, such that $x \pm h \in (a, b)$,

$$\sqrt{f(x+h)f(x-h)} \geq f(x)(1 - 4Mh).$$

If $4M\,|h| < 1$, the condition $x \pm h \in (a, b)$ will be fulfilled automatically, since then $|x \pm h| < \frac{1}{2M}$ and therefore $\int_0^{|x|+h} f(y)\,dy < \frac{1}{2}$ and $\int_{-|x|-h}^0 f(y)\,dy < \frac{1}{2}$. (Alternatively, in order to avoid the verification that $x \pm h \in (a, b)$, one could assume that f is positive on the whole real line.)

In order to estimate $\chi_{f,\delta}(x)$, it remains to solve $1 - 4Mh = \delta$, giving $h = \frac{1-\delta}{4M}$. Thus, we arrive at:

Lemma 3.1 *Let μ be a probability measure on the real line with median at the origin, having a log-concave density f, and let $M = \sup f$. Then, for any $0 < \delta < 1$,*

$$\chi_{f,\delta}(x) \geq \frac{1-\delta}{4M}, \quad \textit{for all } |x| \leq \frac{1}{4M}.$$

Next, we need to integrate this inequality over the measure μ. This leads to

$$\int \chi_{f,\delta}\, \mathrm{d}\mu \geq \frac{1-\delta}{4M} \int_{-\frac{1}{4M}}^{\frac{1}{4M}} f(x)\, \mathrm{d}x.$$

By (3.1), applied to $x = 0$ and with x replacing y, we have $f(x) \geq f(0)(1 - 4M|x|)$, so,

$$\int_{-\frac{1}{4M}}^{\frac{1}{4M}} f(x)\, \mathrm{d}x \geq f(0) \int_{-\frac{1}{4M}}^{\frac{1}{4M}} (1 - 4M|x|)\, \mathrm{d}x = \frac{f(0)}{4M}.$$

Hence,

$$\int \chi_{f,\delta}\, \mathrm{d}\mu \geq \frac{f(0)}{16M^2}\,(1-\delta).$$

But, by the concavity of I,

$$M = \sup_{0<t<1} I(t) \leq 2I\left(\frac{1}{2}\right) = 2f(0).$$

Thus, we obtain an integral bound

$$\int \chi_{f,\delta}\, \mathrm{d}\mu \geq \frac{1}{64\, f(0)}\,(1-\delta).$$

Finally, it is known that $h(\mu) = 2f(0)$, cf. [2]. Hence, dropping the median assumption, we may conclude:

Lemma 3.2 *Let μ be a probability measure on the real line with a log-concave density f. Then, for any $0 < \delta < 1$,*

$$h(\mu)^{-1} \leq \frac{32}{1-\delta} \int \chi_{f,\delta}\, \mathrm{d}\mu.$$

Proof of Theorem 1.1 Combine Lemma 3.2 with Corollary 2.5. □

4 Concentration in the ball

In order to turn to Corollary 1.2, we need the following assertion which might be of an independent interest.

Proposition 4.1 *If a random vector X in $\mathbf{R}^n$ has a log-concave distribution, then*

$$\mathbf{P}\{|X| \leq \mathbf{E}\,|X|\} \geq \frac{1}{24\, e^2}. \tag{4.1}$$

Here, the numerical constant is apparently not optimal (but it is dimension-free).

Proof The localization lemma (Lemma 2.1) allows us to reduce the statement to the case where the distribution of X is supported on some line of the Euclidean space. Indeed, (4.1) may be rewritten as the property that, for all $a > 0$,

$$\mathbf{E}\,|X| \le a \ \Rightarrow\ \mathbf{P}\{|X| \le a\} \ge \frac{1}{24\,e^2}.$$

In terms of the log-concave density of X, say f, this assertion is the same as

$$\int (|x| - a)\, f(x)\,\mathrm{d}x \le 0 \ \Rightarrow\ \int \left(1_{\{|x|\le a\}} - \frac{1}{24\,e^2}\right) f(x)\,\mathrm{d}x \ge 0.$$

If this was not true (with a fixed value of a), then we would have

$$\int (|x| - a)\, f(x)\,\mathrm{d}x \le 0, \qquad \int \left(1_{\{|x|\le a\}} - \frac{1}{24\,e^2}\right) f(x)\,\mathrm{d}x < 0.$$

Hence, by Lemma 2.1, for some compact segment $\Delta \subset E_f$ and some positive affine function l on Δ,

$$\int_\Delta (|x| - a)\, f(x)\, l(x)^{n-1}\,\mathrm{d}x \le 0, \qquad \int_\Delta \left(1_{\{|x|\le a\}} - \frac{1}{24\,e^2}\right) f(x)\, l(x)^{n-1}\,\mathrm{d}x < 0.$$

But this would contradict to the validity of (4.1) for a random vector X which takes values in Δ and has a density proportional to $f(x)\,l(x)^{n-1}$ with respect to the uniform measure on Δ. Since $f\,l^{n-1}$ is log-concave and defines a one-dimensional log-concave measure, the desired reduction has been achieved.

In the one-dimensional case, we may write

$$X = a + \xi\theta$$

where the vectors $a, \theta \in \mathbf{R}^n$ are orthogonal, $|\theta| = 1$, and where ξ is a random variable having a log-concave distribution on the real line. Then, (4.1) turns into the bound

$$\mathbf{P}\left\{\sqrt{\xi^2 + |a|^2} \le \mathbf{E}\sqrt{\xi^2 + |a|^2}\right\} \ge c \tag{4.2}$$

with the indicated constant c.

To further simplify, we first notice that the inequality $\sqrt{\xi^2 + |a|^2} \le \mathbf{E}\sqrt{\xi^2 + |a|^2}$ is getting weaker, as $|a|$ grows, and hence, the case $a = 0$ is the worst one. Indeed, this inequality may be rewritten as

$$\xi^2 \le \psi(t) \equiv \left(\mathbf{E}\sqrt{\xi^2 + t}\right)^2 - t, \qquad t = |a|^2.$$

We have $\psi(0) = \mathbf{E}\,|\xi|$, while for $t > 0$,

$$\psi'(t) = \mathbf{E}\sqrt{\xi^2 + t}\ \mathbf{E}\frac{1}{\sqrt{\xi^2 + t}} - 1 \ge 0,$$

due to Jensen's inequality. Hence, $\psi(t)$ is non-decreasing, proving the assertion.

Now, when $a = 0$, (4.2) becomes

$$\mathbf{P}\{|\xi| \le \mathbf{E}\,|\xi|\} \ge c, \tag{4.3}$$

which is exactly (4.1) in dimension one.

If, for example, $\xi \ge 0$, this statement is known, since we always have

$$\mathbf{P}\{\xi \le \mathbf{E}\,\xi\} \ge \frac{1}{e}, \tag{4.4}$$

cf. [2]. However, (4.3) is more delicate and requires some analysis similar to the one of the previous section.

Denote by μ the distribution of ξ. We use the same notations, such as f, F, I, m, and M, as in the proof of Lemma 2.3. To prove (4.3), we may assume that $\mathbf{E}\xi \le 0$. By the concavity of I,

$$I(t) \ge 2I(1/2)\ \min\{t, 1-t\}, \qquad 0 < t < 1.$$

Hence, after the change $t = F(x)$, we have

$$f(x) \ge 2f(m)\ \min\{F(x), 1-F(x)\}, \qquad a < x < b,$$

and after integration,

$$\mu(x_0, x_1) \ge 2f(m) \int_{x_0}^{x_1} \min\{F(x), 1-F(x)\}\,\mathrm{d}x, \qquad a \le x_0 \le x_1 \le b.$$

Note that the function $u(x) = \min\{F(x), 1-F(x)\}$ has the Lipschitz semi-norm $\|u\|_{\mathrm{Lip}} \le M$, so

$$u(x) \ge u(x_0) - M|x - x_0| \qquad a < x < b.$$

Hence,

$$\begin{aligned}\mathbf{P}\{x_0 \le \xi \le x_1\} &= \mu(x_0, x_1)\\ &\ge 2f(m) \int_{x_0}^{x_1} (u(x_0) - M(x - x_0))\ \mathrm{d}x\\ &= 2f(m)(x_1 - x_0)\ \left(u(x_0) - \frac{M}{2}\,(x_1 - x_0)\right).\end{aligned}$$

Here, we choose $x_0 = \mathbf{E}\xi$. Recall that, by (4.4), $F(x_0) \ge \frac{1}{e}$. An application of (4.4) to the random variable $-\xi$ gives a similar bound $1 - F(x_0) \ge \frac{1}{e}$, so that $u(x_0) \ge \frac{1}{e}$ and thus

$$\mathbf{P}\{\mathbf{E}\xi \le \xi \le x_1\} \ge 2f(m)(x_1 - \mathbf{E}\xi)\ \left(\frac{1}{e} - \frac{M}{2}\,(x_1 - \mathbf{E}\xi)\right).$$

Here we put

$$x_1 = \mathbf{E}\xi + \alpha\,\mathbf{E}\,|\xi - \mathbf{E}\xi|$$

with a parameter $0 < \alpha \le \frac{1}{2}$. Since $\mathbf{E}\xi \le 0$ is assumed, it follows that $x_1 \le \mathbf{E}\,|\xi|$. Thus, using $x_1 - x_0 = \alpha\,\mathbf{E}\,|\xi - \mathbf{E}\xi|$, we get

$$\mathbf{P}\{\mathbf{E}\xi \le \xi \le \mathbf{E}\,|\xi|\} \ \ge\ 2\alpha f(m)\,\mathbf{E}\,|\xi - \mathbf{E}\xi|\ \left(\frac{1}{e} - \frac{M\alpha}{2}\,\mathbf{E}\,|\xi - \mathbf{E}\xi|\right). \tag{4.5}$$

To simplify, let η be an independent copy of ξ. We have, again by the concavity of I,

$$\begin{aligned}\mathbf{E}\,|\xi - \mathbf{E}\xi|\ \le\ \mathbf{E}\,|\xi - \eta| &= \int_0^1\int_0^1 |F^{-1}(t) - F^{-1}(s)|\,\mathrm{d}t\mathrm{d}s\\ &= \int_0^1\int_0^1 \left|\int_t^s \frac{du}{I(u)}\right|\,\mathrm{d}t\mathrm{d}s\end{aligned}$$

$$\leq \int_0^1 \int_0^1 \left| \int_t^s \frac{du}{2I(\frac{1}{2})\,\min\{u, 1-u\}} \right| \mathrm{d}t\mathrm{d}s$$
$$= \frac{1}{2I(\frac{1}{2})}\,\mathbf{E}\,|\xi' - \eta'| \;=\; \frac{1}{2f(m)}\,\mathbf{E}\,|\xi' - \eta'|,$$

where ξ' and η' are independent and have a two-sided exponential distribution, with density $\frac{1}{2}\,e^{-|x|}$. To further bound, just use

$$\mathbf{E}\,|\xi' - \eta'| \leq \sqrt{\mathbf{E}\,(\xi' - \eta')^2} = \sqrt{2\,\mathrm{Var}(\xi')} = 2.$$

Hence, $\mathbf{E}\,|\xi - \mathbf{E}\xi| \leq \frac{1}{f(m)}$. In addition, as we have already mentioned, $M \leq 2f(m)$, implying $M\,\mathbf{E}\,|\xi - \mathbf{E}\xi| \leq 2$. Hence, from (4.5)

$$\mathbf{P}\,\{\mathbf{E}\xi \leq \xi \leq \mathbf{E}\,|\xi|\} \;\geq\; 2\alpha f(m)\,\mathbf{E}\,|\xi - \mathbf{E}\xi|\,\left(\frac{1}{e} - \alpha\right).$$

The expression on the right-hand side is maximized at $\alpha = \frac{1}{2e}$, which gives

$$\mathbf{P}\,\{\mathbf{E}\xi \leq \xi \leq \mathbf{E}\,|\xi|\} \;\geq\; \frac{1}{2e^2}\,f(m)\,\mathbf{E}\,|\xi - \mathbf{E}\xi|. \tag{4.6}$$

Now, $\mathbf{E}\,|\xi - \eta| \leq 2\,\mathbf{E}\,|\xi - \mathbf{E}\xi|$, while, since $I(t) \leq M \leq 2f(m)$,

$$\mathbf{E}\,|\xi - \eta| = 2\int_a^b F(x)(1 - F(x))\,\mathrm{d}x$$
$$= 2\int_0^1 \frac{t(1-t)}{I(t)}\,\mathrm{d}t$$
$$\geq \frac{2}{M}\int_0^1 t(1-t)\,\mathrm{d}t \;=\; \frac{1}{3M} \;\geq\; \frac{1}{6f(m)}.$$

Hence, $\mathbf{E}\,|\xi - \mathbf{E}\xi| \geq \frac{1}{12\,f(m)}$, so the right-hand side expression in (4.6) is greater than or equal to $\frac{1}{2e^2}\,\frac{1}{12} = \frac{1}{24e^2}$. Proposition 4.1 is proved. □

Proof of Corollary 1.2. Let X be a random vector in $\mathbf{R}^n$ with distribution μ, and let ν be the normalized restriction of μ to the Euclidean ball centered at the origin and with radius

$$R = \mathbf{E}\,|X| = \int |x|\,\mathrm{d}\mu(x).$$

In terms of the density f of μ, the density of ν is just the function $\frac{1}{p}\,f(x)\,1_{\{|x| \leq R\}}$, where $p = \mathbf{P}\{|X| \leq R\}$. Hence,

$$\|\nu - \mu\|_{\mathrm{TV}} \;=\; \frac{1}{2}\int \left| \frac{1}{p}\,f(x)\,1_{\{|x| \leq R\}} - f(x) \right| \mathrm{d}x \;=\; 1 - p \;\leq\; \alpha,$$

where one may take $\alpha = 1 - \frac{1}{24\,e^2}$, according to Proposition 4.1. Applying the perturbation result of E. Milman (mentioned in Sect. 1), we thus get that

$$h(\mu)^{-1} \leq C\,h(\nu)^{-1}$$

with some absolute constant C. It remains to apply Theorem 1.1 to ν, cf. (1.7), and then we arrive at (1.8). Corollary 1.2 is proved. □

5 Examples

Let us describe how the χ-function may behave in a few basic examples.

1. (Convex body case). As we have already mentioned, when μ is the normalized Lebesgue measure in a convex body $K \subset \mathbf{R}^n$ with density $f = \frac{1}{\mathrm{vol}_n(K)} 1_K$, we have $\chi_{f,\delta} = \chi_K$ for all $0 < \delta < 1$. In particular, if $K = B_R$ is the ball with center at the origin and of radius R,
$$\chi_{f,\delta}(x) = \sqrt{R^2 - |x|^2}.$$
Hence,
$$\int \chi_{f,\delta}\, \mathrm{d}\mu \leq \frac{CR}{\sqrt{n}}$$
with some universal constant C. This leads in Theorem 1.1 to a correct bound on the isoperimetric constant, as emphasized in [15].
2. If μ is the one-sided exponential distribution on the real line with density $f(x) = e^{-x}$ ($x > 0$), then $\chi_{f,\delta}(x) = x$ for $x > 0$, thus independent of δ.
3. If μ is the two-sided exponential distribution on the real line with density $f(x) = \frac{1}{2}\, e^{-|x|}$, then
$$\chi_{f,\delta}(x) = \log(1/\delta) + |x|.$$
4. If $\mu = \gamma_n$ is the standard Gaussian measure on $\mathbf{R}^n$, that is, with density $\varphi_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, we have
$$\chi_{\varphi_n,\delta}(x) = \sqrt{\log(1/\delta)}$$
which is independent of x.
5. Let μ be a probability measure on $\mathbf{R}^n$ having a log-concave density with respect to γ_n. That is, with respect to the Lebesgue measure, μ has density of the form $f = \varphi_n g$, where g is a log-concave function. By Lemma 2.3, $\chi_{f,\delta} \leq \chi_{\varphi_n,\delta}$, so by the previous example,
$$\chi_{f,\delta}(x) \leq \sqrt{\log(1/\delta)}.$$
By Theorem 1.1, this gives $h(\mu)^{-1} \leq C$ with some universal constant C. Such a dimension-free result in a sharper form of the Gaussian isoperimetric inequality was first obtained by Bakry and Ledoux [1]. As was later shown by Caffarelli [9], the measures μ in this example can be treated as contractions of γ_n. This property allows one to extend many Sobolev-type inequalities about γ_n to μ. Localization arguments together with some extensions are discussed in [3] and [6].
6. More generally, let μ be a uniformly log-concave probability measure on $\mathbf{R}^n$ (with respect to the Euclidean norm). This means that μ has a log-concave density $f = e^{-V}$ where the convex function V satisfies
$$\frac{V(x) + V(y)}{2} - V\left(\frac{x+y}{2}\right) \geq \alpha\left(\left|\frac{x-y}{2}\right|\right), \qquad \text{for all} \quad x, y \in \mathbf{R}^n,$$
with some $\alpha = \alpha(t)$, increasing and positive for $t > 0$. The optimal function α is sometimes called the modulus of convexity of V. Note that, the case $\alpha(t) = \frac{t^2}{2}$ returns us to the family described in Example 5. As for the general uniformly log-concave case, various isoperimetric inequalities of the form $\mu^+(A) \geq I(\mu(A))$ in which the behavior

of I near zero is determined by the modulus of convexity have been studied by Milman and Sodin [22].
To see how to apply Theorem 1.1 for obtaining a Cheeger-type isoperimetric inequality like in (1.2), let us rewrite the definition of the uniform log-concavity directly in terms of the density as

$$\sqrt{f(x+h)f(x-h)} \leq e^{-\alpha(|h|)} f(x).$$

Hence, once we know that $\alpha(t) > \alpha(t_0) > 0$, for all $t > t_0$ with some $t_0 > 0$, it follows that

$$\chi_{f,\delta}(x) \leq t_0, \quad \text{for all } x \in \mathbf{R}^n \text{ with } \quad \delta = e^{-\alpha(t_0)}.$$

Thus, like in Examples 4 and 5, the χ-function is uniformly bounded on the whole space, implying that

$$h(\mu)^{-1} \leq \frac{Ct_0}{1 - e^{-\alpha(t_0)}}$$

with some universal constant C. This example also shows that the uniform log-concavity is the property of a global character, while $\chi_{f,\delta}$ reflects more a local behavior of the density.

7. As an example of a measure which is not uniformly log-concave, let μ be the rotationally invariant probability measure on $\mathbf{R}^n$ with density $f(x) = \frac{1}{Z_n} e^{-\sqrt{n}\,|x|}$ where Z_n is a normalizing constant. Note that, $\int |x|^2 \, d\mu(x)$ is of order n. In order to find $\chi_{f,\delta}(x)$, we need to solve the inequality

$$\frac{1}{2}\sqrt{n}\,|x+h| + \frac{1}{2}\sqrt{n}\,|x-h| \leq \log(1/\delta) + \sqrt{n}\,|x|$$

and determine the maximal possible value for $|h|$. Rewrite it as

$$\frac{1}{2}\sqrt{|x|^2 + 2\langle x,h\rangle + |h|^2} + \frac{1}{2}\sqrt{|x|^2 - 2\langle x,h\rangle + |h|^2} \leq \frac{1}{\sqrt{n}}\log(1/\delta) + |x|.$$

If $|h| = r$ is fixed, the inner product $t = \langle x,h\rangle$ can vary from $-r|x|$ to $r|x|$. Moreover, the left-hand side represents an even convex function in t which thus attains minimum at $t = 0$. In other words, if we want to prescribe to $|h|$ as maximal value, as possible (to meet the inequality), we need to require that $t = 0$. In this case, the inequality is simplified to

$$\sqrt{|x|^2 + |h|^2} \leq \frac{1}{\sqrt{n}}\log(1/\delta) + |x|$$

which is equivalent to

$$|h|^2 \leq \frac{1}{n}\log^2(1/\delta) + \frac{2|x|}{\sqrt{n}}\log(1/\delta).$$

Hence,

$$\chi^2_{f,\delta}(x) = \frac{1}{n}\log^2(1/\delta) + \frac{2|x|}{\sqrt{n}}\log(1/\delta).$$

Integrating this inequality over μ and choosing, for example, $\delta = 1/e$, we get

$$\int \chi^2_{f,\delta}\, d\mu \leq C$$

with some universal constant C. Therefore, by Theorem 1.1, $h(\mu)^{-1} \leq C$ (with some other constant). Note that, $\chi_{f,\delta}$ is bounded by a δ-dependent constant on the ball of radius of order $\sqrt{n}$. Hence, one may also apply Corollary 1.2 without any integration.

Apparently, this example may further be extended to general spherically invariant log-concave probability measures μ on $\mathbf{R}^n$, in which case it is already known that $h(\mu)^{-1}$ is bounded by an absolute constant (under the normalization condition $\int |x|^2 \, \mathrm{d}\mu(x) = n$). Using a different argument, this class was previously considered in [4] and recently in [14], where results of such type are obtained even for more general log-concave densities $f(x) = \rho(\|x\|)$ involving norms on $\mathbf{R}^n$.

8. Finally, as a somewhat "negative" example, let $\mu = \nu^n$ be the product of the two-sided exponential distribution ν from example 2, thus, with density $f(x) = e^{-(x_1+\cdots+x_n)}$, $x = (x_1, \ldots, x_n)$, $x_i > 0$. As easy to see,

$$\chi_{f,\delta}(x) = |x|, \qquad x \in \mathbf{R}^n_+,$$

which by Theorem 1.1, only gives $h(\mu)^{-1} \leq C\sqrt{n}$. Corollary 1.2 allows to improve this estimate to $h(\mu)^{-1} \leq Cn^{1/4}$ which may also be seen from (1.9). However, we cannot obtain the correct bound $h(\mu)^{-1} \leq C$ derived in [7] using the induction argument.

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