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Hyperbolic measures on infinite dimensional spaces

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Abstract: Localization and dilation procedures are discussed for infinite dimensional α-concave measures on abstract locally convex spaces (following Borell's hierarchy of hyperbolic measures).

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1. Introduction

The purpose of this note is to review some results about the localization techniques and hyperbolic measures on $\mathbb{R}^n$ and to discuss possible extensions to the setting of abstract (infinite dimensional) locally convex spaces. As a starting point, let us recall the so-called "Localization Lemma" which is due to Lovász and Simonovits.

Theorem 1.1 ([L-S]). *Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous, integrable functions such that*

$$\int_{\mathbb{R}^n} u(x)\,dx > 0, \qquad \int_{\mathbb{R}^n} v(x)\,dx > 0. \tag{1.1}$$

Then, for some points $a, b \in \mathbb{R}^n$ and a positive affine function l on $(0,1)$,

$$\int_0^1 u((1-t)a + tb)\, l(t)^{n-1}\,dt > 0, \qquad \int_0^1 v((1-t)a + tb)\, l(t)^{n-1}\,dt > 0. \tag{1.2}$$

There are some other variants of this theorem, for example, when the first integrals involving the function u are vanishing, cf. [K-L-S]. The approach of Lovász and Simonovits was based on the concept of a needle coming as result of a localization (or bisection) procedure. Later Fradelizi and Guédon [F-G1] proposed an alternative geometric argument with involvement of Krein-Milman's theorem, cf. also [F-G2].

Theorem 1.1 is a powerful tool towards certain integral relations in $\mathbb{R}^n$; it allows reduction to related inequalities in dimension $n = 1$. It is therefore not surprising that this theorem has found numerous applications in different problems of multidimensional Analysis and Geometry, such as isoperimetric problems over convex bodies, log-concave and more general hyperbolic measures, as well as Khinchine and dilation-type inequalities (cf. [K-L-S, G, B1, B2, B3, B6, N-S-V, B-N, F, B-M]). In many such applications, one considers integrals with respect to measures that are different than the Lebesgue measure on $\mathbb{R}^n$, and therefore a more flexible version of Theorem 1.1 involving other measures would be desirable. In addition, having in mind dimension free phenomena and applications to random processes with hyperbolic distributions, it is useful to avoid reference to the dimension and to obtain similar statements about spaces and measures of an infinite dimension.

A positive Radon measure μ on a real, locally convex Hausdorff space. space E is called α-concave ($-\infty \le \alpha \le \infty$) if, for all non-empty Borel sets A and B in E and for all $0 < t < 1$,

$$\mu_*\big((1-t)A + tB\big) \ge \Big[(1-t)\mu(A)^\alpha + t\mu(B)^\alpha\Big]^{1/\alpha}. \tag{1.3}$$

Here, $(1-t)A + tB = \{(1-t)x + ty : x \in A,\ y \in B\}$ stands for the Minkowski weighted sum, and μ_* is the inner measure $\mu_*(U)$ is defined for any $U \subseteq E$, as the $\sup\{\mu(V) : V$ is a measurable subset of $U\}$ (for a possible case when

$(1-t)A+tB$ is not Borel measurable). By the Radon property, (1.3) may equivalently be stated for all non-empty compact subsets of E, and then the inner measure is not needed. Note that we will take positive in the strict sense, meaning that both $\mu(A) \geq 0$ for A measurable, and $\mu(E) > 0$. This will be at no loss of generality, as our results will holds vacuously when $\mu = 0$.

Any measure supported on a one-point set is ∞-concave. In all other cases, necessarily $\alpha \leq 1$. For example, the Lebesgue measure on $\mathbb{R}$ is 1-concave. More generally, the Lebesgue measure on $\mathbb{R}^n$ is $\frac{1}{n}$-concave, which is the content of the Brunn-Minkowski theorem.

Inequality (1.3) strengthens with growing α. In the limit case $\alpha = -\infty$, (1.3) becomes corresponding to

$$\mu_*\big((1-t)A+tB\big) \geq \min\{\mu(A), \mu(B)\}, \tag{1.4}$$

which describes the largest class. Such measures μ are called convex or hyperbolic (not to be confused with a class of special probability distributions used in Statistics, see Example (2.2) in Section (2))". One important case is also $\alpha = 0$, for which (1.3) is understood as

$$\mu_*\big((1-t)A+tB\big) \geq \mu(A)^{1-t}\mu(B)^t.$$

Then the measure μ is called logarithmically concave, or just log-concave.

The class of log-concave measures on $\mathbb{R}^n$ was first considered by Prékopa [Pr] and previously in dimension one by other authors (cf. [I, D-K-H]). The more general classes of α-concave measures (in the setting of an abstract locally convex space) were introduced by Borell [Bor1]. He studied basic properties of α-concave measures, including $0-1$ law, integrability of norms, and convexity properties of measures under convolutions. Borell also gave a characterization of the α-concavity in terms of densities of finite dimensional projections, cf. [Bor1, Bor2], and also [B-L].

Hyperbolic measures are known to satisfy many other important properties that are usually expressed in terms of various relations such as, for example, Khinchin-type inequalities for polynomials of a bounded degree. What is remarkable, most of them involve only the convexity parameter α and do not depend on the dimension of the underlying space E. It is therefore natural to state these relations without unnecessary restrictions on E where possible. For our purposes E will always be assumed to be a locally convex Hausdorff space over $\mathbb{R}$. Additional assumptions will be made where needed.

For example, Theorem 1.1 may be complemented by the following.

Theorem 1.2. *Let μ be a finite α-concave measure on the vector space E, and let $u, v : E \to \mathbb{R}$ be lower semi-continuous μ-integrable functions such that*

$$\int_E u\, d\mu > 0, \qquad \int_E v\, d\mu > 0. \tag{1.5}$$

Then, for some points $a, b \in E$ and some finite α-concave measure ν supported on the segment $\Delta = [a, b]$,

$$\int_\Delta u\, d\nu > 0, \qquad \int_\Delta v\, d\nu > 0. \tag{1.6}$$

Note that lower semicontinuous functions are bounded below on any compact set. Hence, their integrals over compactly supported finite measures such as (1.2) and (1.6) always exist. As an example, the indicator functions of open subsets of E are all lower semicontinuous.

The completeness assumption (meaning that every Cauchy net in E is convergent) is quite natural. It ensures that the closed convex hull of any compact set in E is also compact. In that case any finite Radon measure μ on E has a stronger property

$$\sup\{\mu(K) : K \subset E \text{ convex compact}\} = \mu(E). \tag{1.7}$$

This property, known not to hold in general, is crucial in some applications. (In fact, in the absence of completeness the validity of the above is still open for non-Radon Gaussian measures, see [L-T].)

One can also give a geometric variant of Theorem 1.1 together with a finer formulation of Theorem 1.2 in terms of extreme points of the set $\mathcal{P}_\alpha(u)$ of all α-concave probability measures supported on a convex compact set $K \subset E$ and such that $\int u\, d\mu \geq 0$ (for a continuous function u on K). As we already mentioned, this interesting approach to localization was developed by Fradelizi and Guédon [F-G1]. It was shown there that in case $E = \mathbb{R}^n$ and $\alpha \leq \frac{1}{2}$, any extreme measure is supported on an interval $\Delta \subset K$ with density $l^{(1-\alpha)/\alpha}$ (where l is a non-negative affine function on Δ). Note that we consider Δ to be an (closed) interval if there exist $x, y \in E$ such that $\Delta = \{(1-t)x + ty : t \in [0,1]\}$. When $x = y$, m_Δ corresponds to the Dirac measure at x.

As will be explaned in Section 3, this property extends to general locally convex spaces, and then it easily implies Theorem 1.2.

One interesting application of Theorem 1.2 may be stated in terms of the following operation proposed in [N-S-V]. Given a Borel subset A in a closed convex set $F \subset E$ and a number $\delta \in [0,1]$, define

$$A_\delta = \Big\{x \in A : m_\Delta(A) \geq 1 - \delta \ \text{ for any interval } \Delta \subset F \text{ such that } x \in \Delta\Big\},$$

where m_Δ denotes the normalized one-dimensional Lebesgue measure on Δ (understood to be the Dirac measure in the case that the endpoints coincide).

For example, if $F = E$ and A is the complement to a centrally symmetric, open, convex set $B \subset E$, then $A_\delta = E \setminus (\frac{2}{\delta} - 1)B$ represents the complement to the corresponding dilation of B.

Theorem 1.3. *Let μ be an α-concave probability measure on a complete locally convex space E supported on a closed convex set F ($-\infty < \alpha \leq 1$). For any Borel set A in F and for all $\delta \in [0,1]$ such that $\mu^*(A_\delta) > 0$,*

$$\mu(A) \ \geq \ \big[\,\delta\mu^*(A_\delta)^\alpha + (1-\delta)\big]^{1/\alpha}. \tag{1.8}$$

Here μ^* denotes the outer measure defined by the formula,$\mu^*(A) = \inf\{\mu(U) : A \subset U \text{ measurable}\}$ (which is not needed, when E is a Fréchet space). This relation resembles very much the definition (1.3).

In the important particular case $\alpha = 0$ (i.e., for log-concave measures), (1.8) becomes

$$\mu(A) \geq \mu^*(A_\delta)^\delta.$$

It was discovered in the finite dimensional setting by Nazarov, Sodin and Vol'berg [N-S-V]. The extension of this result to the class of α-concave measures in the form (1.8) is settled in [B-N] and [F], still for finite dimensional spaces. All proofs are essentially based on Theorem 1.1 or its modifications to reduce (1.8) to dimension one (although the one dimensional case appears to be rather delicate). Here we make another step removing the dimensionality of the space assumption, cf. Section 5.

The organization of this note is as follows. In Section 2 we recall basic general facts about α-concave measures, including Borell's characterization of the α-concavity in terms of densities, and describe several examples. Sections 3–4 are devoted to the extension of Fradelizi-Guédon's theorem and Lovász-Simonovits' bisection argument. In particular, the existence of needles which we understand in a somewhat weaker sense is proved for probability measures on Fréchet spaces that satisfy the zero-one law. This can be used as an approach towards Theorems 1.1–1.2, but potentially may have a wider range of applications. Finally, section 5 is devoted to Theorem 1.3, which is then illustrated in the problem of large and small deviations (Section 6). The requisite material on dilation is developed in an appendix.

We do not try to describe in detail results and techniques in dimension one, but mainly focus on their extensions to the setting of infinite dimensional spaces.

2. Support, dimension and characterizations

The support $H_\mu = \mathrm{supp}(\mu)$ of any Radon measure μ on E is defined as the smallest closed subset of E of full measure, so that $\mu(E \setminus H_\mu) = 0$. If μ is hyperbolic, then the set H_μ is necessarily convex, as follows from (1.4). This set has some dimension

$$k = \dim(\mu) = \dim(H_\mu),$$

finite or not, which is called the dimension of the hyperbolic measure μ. If it is finite, absolute continuity of μ will always be understood with respect to the k-dimensional Lebesgue measure on H_μ. When $k = 0$ this is understood to be a point mass δ_x. That is $\delta_x(A) = 1$, when $x \in A$ and is zero otherwise

First, let us recall an important general property of hyperbolic measures proven by Borell.

Theorem 2.1 ([Bor1])**.** *If μ is a hyperbolic probability measure on a locally convex space E, then for any additive subgroup H of E, either $\mu_*(H) = 0$ or $\mu_*(H) = 1$.*

In particular, any μ-measurable affine subspace of E has measure either zero or one.

In [Bor1, Bor2], Borell also gave a full description of α-concave measures. Similarly to 1.3, a non-negative function f defined on a convex subset H of E is called β-concave, if it satisfies

$$f\big((1-t)x+ty\big) \geq \Big[(1-t)f(x)^\beta + tf(y)^\beta\Big]^{1/\beta} \tag{2.1}$$

for all $t \in (0,1)$ and all points $x, y \in H$ such that $f(x) > 0$ and $f(y) > 0$. The right-hand side is understood in the usual limit sense for the values $\beta = -\infty$, $\beta = 0$ and $\beta = \infty$.

Theorem 2.2 ([Bor1])**.** *If μ is a finite α-concave measure on $\mathbb{R}^n$ of dimension $k = \dim(\mu)$, then $\alpha \leq \frac{1}{k}$. Moreover, μ is absolutely continuous with respect to Lebesgue measure on H_μ and has density f which is positive, finite, and β-concave on the relative interior of H_μ, where*

$$\beta = \frac{\alpha}{1-\alpha k}.$$

Conversely, if a measure μ on $\mathbb{R}^n$ is supported on a convex set H of dimension k and has there a positive, β-concave density f with $\beta \geq -\frac{1}{k}$, then μ is α-concave with α defined implicitly by the same formula above.

Note that β is continuously increasing in the range $[-\frac{1}{k}, \infty]$, when α is varying in $[-\infty, \frac{1}{k}]$.

In the extremal case $\alpha = \frac{1}{k}$, the density $f(x) = \frac{d\mu(x)}{dx}$ is ∞-concave and is therefore constant: Up to a factor, μ must be the k-dimensional Lebesgue measure on H_μ.

More generally, if $\alpha \leq \frac{1}{k}$, $\alpha \neq 0$, the density has the form

$$f(x) = V(x)^{\frac{1}{\alpha} - k}$$

for some function $V : \Omega \to (0, \infty)$ on the relative interior Ω of H_μ, which is concave in case $\alpha > 0$, and is convex in case $\alpha < 0$. In particular, the formula

$$f(x) = V(x)^{-k}$$

describes all k-dimensional hyperbolic measures ($\alpha = -\infty$). If $\alpha = 0$, then necessarily $f(x) = e^{-V(x)}$ for some convex function $V : \Omega \to \mathbb{R}$.

As for general locally convex spaces, another theorem due to Borell reduces the question to Theorem 2.2.

Theorem 2.3 ([Bor1])**.** *A Radon probability measure μ on the locally convex space E is α-concave, if and only if the image of μ under any linear continuous map $T : E \to \mathbb{R}^n$ is an α-concave measure on $\mathbb{R}^n$.*

For special spaces in this characterization one may consider linear continuous maps T from a sufficiently rich family. For example, when $E = C[0,1]$ is the Banach space of all continuous functions on $[0,1]$ with the maximum-norm, the measure μ is α-concave, if and only if the image of μ under any map of the form

$$Tx = (x(t_1), \dots, x(t_n)), \qquad x \in C[0,1], \quad t_1, \dots, t_n \in [0,1],$$

is an α-concave measure on $\mathbb{R}^n$. Similarly, when $E = \mathbb{R}^\infty$, the space of all sequences of real numbers (with the product topology) indexed by $\mathbb{N}_+$, it is sufficient to consider the standard projections

$$T_n x = (x_1, \ldots, x_n), \qquad x = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty. \tag{2.2}$$

The next general observation is that infinite dimensional α-concave measures may not have a positive parameter of convexity. Though a result stated by Borell (see [Bor3]), we include a proof here for emphasis and the convenience of the reader. More precisely we have the following:

Theorem 2.4. *For $\alpha > 0$, any α-concave finite measure μ on a locally convex space E has finite dimension and is compactly supported.*

Our referee should be credited for the improved clarity of this result, as it is their proof of the boundedness of μ's support that we include here.

Proof. As usual, E' denotes the dual spaces of all linear continuous functionals on E and suppose to the contrary that μ is infinite dimensional. We may assume that $H_\mu = \mathrm{supp}(\mu)$ contains the origin. Since H_μ is not contained in any finite dimensional subspace of E, for each n, one can find linearly independent vectors $v_1, \ldots, v_n \in H_\mu$. Each point $x \in E$ has a representation $x = c_1(x)v_1 + \cdots + c_n(x)v_n + y$ with some $c_i \in E'$, where $y = y(x)$ is linearly independent of all v_i (cf. [R], Lemma 4.21). Consider the linear map $T(x) = (c_1(x), \ldots, c_n(x))$, which is continuously acting from E to $\mathbb{R}^n$. Then the image $\nu = \mu T^{-1}$ of μ is a finite α-concave measure on $\mathbb{R}^n$.

Let us see that ν is full dimensional. Otherwise, ν is supported on some hyperplane in $\mathbb{R}^n$ described by the equation $a_1 y_1 + \cdots + a_n y_n = a_0$, where the coefficients $a_i \in \mathbb{R}$ are not all zero. Moreover, since $0 \in H_\mu$, any neighborhood of 0 has a positive μ-measure, so

$$\mu\{x \in E : |T(x)| < \varepsilon\} > 0,$$

for any $\varepsilon > 0$. Hence, necessarily $a_0 = 0$. This implies that μ is supported on the closed linear subspace H of E described by the equation $a_1 c_1(x) + \cdots + a_n c_n(x) = 0$. Here, at least one of the coefficient, say a_i, is non-zero. Since $c_i(v_i) = 1 \neq 0$, we obtain that $v_i \notin H$. But this would mean that $H_\mu \cap H$ is a proper closed subset of the support of μ, while H_μ has a full μ-measure, a contradiction.

Hence, $\dim(\nu) = n$. By Theorem 2.2, this gives $\alpha \leq \frac{1}{n}$, and since n was arbitrary, we conclude that $\alpha \leq 0$ which contradicts to the hypothesis $\alpha > 0$.

Thus, μ must be supported on a finite dimensional affine subspace $H \subset E$. To prove compactness of the support, we may assume that $H = E = \mathbb{R}^n$ and $\dim(\mu) = n$. Then, μ is supported on an open convex set $\Omega \subset \mathbb{R}^n$, where it has density of the form

$$f(x) = V(x)^\gamma, \qquad \gamma = \frac{1}{\alpha} - n \quad \big(0 < \alpha \leq \frac{1}{n}\big),$$

for some concave function $V : \Omega \to (0, \infty)$. The case $\gamma = 0$ is possible, but then $f(x) = c$ for some constant $c > 0$, which implies $\mu(\mathbb{R}^n) = c\,|\Omega|$. Since μ is finite, Ω has to be bounded, and so $H_\mu = \mathrm{clos}(\Omega)$ is compact.

Now fix $a \in \Omega$ and choose $r > 0$ large enough that $a \in \Omega \cap \{|x| < r\} =_{def} D_r$. Let $g : D_r \to [0,1]$ be the concave function with $g(a) = 1$ and level sets $\{g > s\} = sa + (1-s)D_r,\ 0 < s < 1$.

Then

$$\begin{aligned}\int_\Omega f(x)dx &\geq \int_{D_r} (V(a)g(x))^\gamma dx \\ &= m_{\mathbb{R}^n}(D_r)V^\gamma(a)\gamma \int_0^1 s^{\gamma-1}(1-s)^n ds.\end{aligned}$$

As $r \to \infty$ this term tends to infinity, a contradiction. Thus Ω is bounded. $\square$

Example 2.1. The normalized Lebesgue measure on every convex body $K \subset \mathbb{R}^n$ is $\frac{1}{n}$-concave.

Example 2.2. In Statistics, up to scaling parameters, by a (one-dimensional) hyperbolic distribution one often means a probability measure with density of the form

$$f(x) = \frac{1}{Z} \exp\left\{-z\left(\sqrt{(1+p^2)(1+x)} - px\right)\right\}, \qquad x \in \mathbb{R},$$

where $z > 0$, $p \in \mathbb{R}$ are shape parameters, and Z is a normalizing constant. In our terminology, this is a log-concave measure. The name hyperbolic was used as the profile of $log f$ is a hyperbola (cf. e.g. [BN], [P-S] and references therein).

Example 2.3. Any Gaussian Radon measure on a locally convex space E is log-concave. In particular, the Wiener measure on $C[0,1]$ is such.

Example 2.4. The standard Cauchy measure μ_1 on $\mathbb{R}$ with density $f(x) = \frac{1}{\pi(1+x^2)}$ is α-concave with $\alpha = -1$ (which is optimal). More generally, the n-dimensional Cauchy measure μ_n on $\mathbb{R}^n$ with density

$$f_n(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}}$$

is (-1)-concave (c_n is a normalizing constant so that μ_n is probability).

Example 2.5. Although the above density f_n essentially depends on the dimension, the measure μ_n has a dimension-free essense. All marginals of μ_n coincide with μ_1 and moreover, there is a unique Borel probability measure μ on $\mathbb{R}^\infty$ (an infinite dimensional Cauchy measure) which is pushed forward to μ_n by the standard projection T_n from (2.2)). This measure can also be introduced as the distribution of the random sequence

$$X = \Big(\frac{Z_1}{\zeta}, \frac{Z_2}{\zeta}, \dots\Big),$$

where the random variables $\zeta, Z_1, Z_2, \dots$ are independent and all have a standard normal distribution. Thus, μ is (-1)-concave on $\mathbb{R}^\infty$.

Example 2.6. This example is mentioned in [Bor1]. Given $d > 0$ (real), let χ_d be a random variable with χ-distribution, that is χ_d has distribution function

$$f_d(r) = c_d r^{d-1} e^{-r^2/2}$$

with $c_d = 2^{1-\frac{k}{2}}/\Gamma(d/2)$. When d is integer valued, the distribution corresponds to that of the norm of a d-dimensional standard Gaussian. If W is a standard Wiener process, independent of χ_d viewed as a random function in $C[0,1]$, then the random function

$$X(t) = \frac{\sqrt{d}}{\chi_d} W(t), \quad t \in [0,1],$$

is $\alpha = -\frac{1}{d}$-concave. This is called the Student measure (when $d = 1$ this Cauchy similar to the previous example).

Proof. It suffices to prove that for $t_1 < \cdots < t_k$, $(X(t_1), \ldots, X(t_k))$ is $-1/d$-concave, and for this purpose (since α-concavity is affine invariant) it is enough to show that Z/χ_d is $-1/d$-concave, when Z is a k-dimensional standard normal. That is Z has distribution

$$\rho_k(x) = c_k e^{|x|^2/2},$$

where $c_k = (2\pi)^{\frac{k}{2}}$. Thus by Borell's characterization, it is enough to show the density function of Z/χ_d is $-(k+d)^{-1}$-concave. When $g : \mathbb{R}^k \to \mathbb{R}$ is smooth and compactly supported,

$$\begin{aligned}
\mathbb{E}g(Z/\chi_d) &= \int_{\mathbb{R}^k} \int_0^\infty g(z/r)\rho_k(z) f_d(r)\, dr dz \\
&= \int_{\mathbb{R}^k} g(x) \left(\int_0^\infty r^k \rho_k(rx) f_d(r) dr \right) dx.
\end{aligned}$$

Thus it suffices to show that

$$\begin{aligned}
\int_0^\infty r^k \rho_k(rx) f_d(r) dr &= c_k c_d \int_0^\infty r^k e^{-|rx|^2/2} r^{d-1} e^{-r^2/2} dr \\
&= c_k c_d \int_0^\infty r^{k+d} e^{-(1+|x|^2)r^2/2} \ \frac{dr}{r}.
\end{aligned}$$

is a $-(k+d)^{-1}$-concave function. But the substitution $u = r(1+|x|^2)^{1/2}$ yields

$$\begin{aligned}
&(1+|x|^2)^{-(k+d)/2} c_k c_d \int_0^\infty r^{k+d} e^{-r^2/2} \ \frac{dr}{r} \\
&\qquad = C \left((1+|x|^2)^{1/2} \right)^{-(k+d)},
\end{aligned}$$

and since $(1+|x|^2)^{1/2}$ is convex, we have our result. □

3. Extreme α-concave measures

Given a convex compact set K in a locally convex space E, denote by $\mathcal{M}_\alpha(K)$ the collection of all α-concave probability measures with support contained in K, considered as a closed subset of the locally convex space $\mathcal{M}(K)$ of all signed Radon measures on K endowed with the topology of weak convergence (see [Bor1]). For a continuous function u on K, we consider the subcollection

$$\mathcal{P}_\alpha(u) = \left\{\mu \in \mathcal{M}_\alpha(K) : \int u\, d\mu \geq 0\right\}$$

together with its closed convex hull $\widetilde{\mathcal{P}}_\alpha(u)$, taken in $\mathcal{M}(K)$. The latter space is dual to the space $C(K)$ of all continuous functions on K, and $\widetilde{\mathcal{P}}_\alpha(u)$ is compact.

We wish to understand the extreme points of $\widetilde{\mathcal{P}}_\alpha(u)$. Using a general theorem due to D. P. Milman, one can only say that all such points lie in $\mathcal{P}_\alpha(u)$ (cf. [B-S-S], p.124, or [Ph] for a detailed discussion of Krein-Milman's theorem). A full answer to this question is given for $\alpha \leq \frac{1}{2}$ in Fradelizi-Guédon's theorem, which we formulate below in the setting of abstract locally convex spaces. We mention in passing that to our knowledge, a description of the extreme points when $\alpha > \frac{1}{2}$ remains open.

Theorem 3.1. *Given a continuous function u on K and $-\infty \leq \alpha \leq 1$, any extreme point μ in $\widetilde{\mathcal{P}}_\alpha(u)$ has the dimension $\dim(\mu) \leq 1$. Moreover, in case $\alpha \leq \frac{1}{2}$,*

1) *μ is either a mass point at $x \in K$ such that $u(x) \geq 0$; or*
2) *μ is supported on an interval $\Delta = [a, b] \subset K$ with density*

$$\frac{d\mu(x)}{dm_\Delta(x)} = l(x)^{(1-\alpha)/\alpha} \tag{3.1}$$

with respect to the uniform measure m_Δ, where l is a non-negative affine function on Δ such that $\int_a^x u\, d\mu > 0$ and $\int_x^b u\, d\mu > 0$, for all $x \in (a, b)$.

In particular, any α-concave probability measure supported on K, belongs to the closed convex hull of the family of all one-dimensional α-concave probability measures supported on K having density of the form (3.1).

We only consider the first assertion of the theorem. The second part is a purely one dimensional statement, and we refer to [F-G1].

Proof. Suppose that a measure $\mu \in \mathcal{P}_\alpha(u)$ has the dimension $\dim(\mu) \geq 2$. For simplicity, let the origin belong to the relative interior G of the support H_μ of μ. Then one may find linearly independent vectors x and y such that $\pm x$ and $\pm y$ are all in G. On the linear hull $L(x, y)$ of x and y (which is a 2-dimensional linear subspace of E), define linear functionals λ_x and λ_y by putting

$$\lambda_x(x) = \lambda_y(y) = 1,$$
$$\lambda_x(y) = \lambda_y(x) = 0.$$

They are continuous, so by the Hahn-Banach theorem, these functionals may be extended from $L(x, y)$ to the whole space E keeping linearity and continuity.

With these extended functionals, we can associate $\Lambda_\theta = \theta_1 \lambda_x + \theta_2 \lambda_y$, where $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$ (vectors on the unit sphere of $\mathbb{R}^2$). Note that these functionals are uniformly bounded on K, i.e.,

$$\sup_\theta \sup_{z \in K} |\Lambda_\theta(z)| \ \le \ \sup_{z \in K} |\lambda_x(z)| + \sup_{z \in K} |\lambda_y(z)| \ < \ \infty. \tag{3.2}$$

Now, following in essence an argument of [F-G1], define the map $\Phi : \mathbb{S}^1 \to \mathbb{R}$ by

$$\Phi(\theta) = \int_{\{\Lambda_\theta \ge 0\}} u \, d\mu.$$

By the construction, the set $\{\Lambda_\theta = 0\} \cap H_\mu$ represents a proper closed affine subspace of H_μ. So, $\mu\{\Lambda_\theta = 0\} = 0$ according to Theorem 3.1 (the zero-one law for hyperbolic measures). Hence, using (3.2), we may conclude that the map Φ is continuous.

In addition, we have the identity $\Phi(\theta) + \Phi(-\theta) = \int u \, d\mu$. Hence, the intermediate value theorem implies that there exists θ such that with $H_\theta^+ = \{\Lambda_\theta \ge 0\}$ and $H_\theta^- = \{\Lambda_\theta \le 0\}$, we have

$$\int_{H_\theta^+} u \, d\mu = \int_{H_\theta^-} u \, d\mu = \frac{1}{2} \int_E u \, d\mu.$$

Necessarily, $t = \mu(H_\theta^-) > 0$ and $\mu(H_\theta^+) > 0$. Defining α-concave probability measures

$$\mu_0(A) = \frac{\mu(A \cap H_\theta^+)}{\mu(H_\theta^+)}, \qquad \mu_1(A) = \frac{\mu(A \cap H_\theta^-)}{\mu(H_\theta^-)},$$

we arrive at the representation $\mu = (1-t)\mu_0 + t\mu_1$ which means that μ is not extreme. $\square$

One can now return to Theorem 1.2.

Proof of Theorem 1.2. Due to the property (1.7), and by the assumption (1.5),

$$\int_K \min(u, c) \, d\mu > 0, \qquad \int_K \min(v, c) \, d\mu > 0,$$

for some convex compact set $K \subset E$ and a constant $c > 0$. We assume without loss of generality that $\mu(K) = 1$. Moreover, since the function $\min(u, c)$ is lower semicontinuous and bounded, while μ is Radon,

$$\int_K \min(u, c) \, d\mu \ = \ \sup_g \int g \, d\mu,$$

where the sup is taken over all continuous functions on K such that $g \le \min(u, c)$ (cf. e.g. [M], Chapter 2, or [Bog], Chapter 7). A similar identity also holds for

$\min(v, c)$. This allows us to reduce the statement of the theorem to the case where both u and v are continuous on K.

In the latter case, let $u_0 = u - \int_K u\, d\mu$. Consider the functional $T(\eta) = \int_K v\, d\eta$. It is linear and continuous on $\mathcal{M}(K)$, and therefore being restricted to $\mathcal{P}_\alpha(u_0)$ it attains maximum at one of the extreme points of $\tilde{\mathcal{P}}$, say ν. Since $\mu \in \mathcal{P}_\alpha(u_0)$, we conclude that

$$\int_K u_0\, d\nu \geq 0, \quad T(\nu) \geq T(\mu),$$

so, $\int_K u\, d\nu > 0$ and $\int_K v\, d\nu > 0$ which is (1.6). It remains to apply Theorem 3.1. □

A similar argument, based also on the second part of Theorem 3.1, yields Theorem 1.1. Indeed, the n-dimensional integrals (1.1) can be restricted to a sufficently large closed ball $K \subset \mathbb{R}^n$. The normalized Lebesgue measure on K is α-concave with $\alpha = \frac{1}{n}$. Hence, the extreme points in $\tilde{\mathcal{P}}_\alpha(u)$ are at most one dimensional and have densities of the form l^{n-1} (if they are not Dirac measures).

4. Bisection and needles on Fréchet spaces

The notion of a needle was proposed by Lovász and Simonovits for the proof of Theorem 1.1 (Localization Lemma, cf. also [K-L-S]). Previously, it appeared implicitly in [P-W] and may be viewed as development of the Hadwiger-Ohmann bisection approach to the Brunn-Minkowski inequality ([H-O, B-Z], cf. also [G-M] for closely related ideas).

As shown in [L-S], starting from (1.1), one can construct a decreasing sequence of compact convex bodies K_l in $\mathbb{R}^n$ that are shrinking to some segment $\Delta = [a, b]$ and are such that, for each l,

$$\int_{K_l} u(x)\, dx > 0, \qquad \int_{K_l} v(x)\, dx > 0.$$

Moreover, choosing a further subsequence (if necessary) and applying the Brunn-Minkowski inequality in $\mathbb{R}^n$, one gets in the limit

$$\begin{aligned}
\lim_{l\to\infty} \frac{1}{|K_l|} \int_{K_l} u(x)\, dx &= \int_\Delta \psi^{n-1}(x)\, u(x)\, dx, \\
\lim_{l\to\infty} \frac{1}{|K_l|} \int_{K_l} v(x)\, dx &= \int_\Delta \psi^{n-1}(x)\, v(x)\, dx,
\end{aligned}$$

for some non-negative concave function ψ on Δ. Here $|K_l|$ denotes the n-dimensional volume, while the integration on the right-hand side is with respect to the linear Lebesgue measure on the segment. In this way, one may obtain a slightly weaker variant of (1.2) with ψ in place of l, and with non-strict inequalities. An additional argument of a similar flavour was then developed in [L-S] to make ψ affine (while the strict inequalities in (1.2) are easily achieved by applying the conclusion to functions $u - \varepsilon w$ and $v - \varepsilon w$, where $w > 0$ is integrable, continuous, and $\varepsilon > 0$ is small enough). The last step shows that for K_l one may take infinitesimal truncated cylinders with main axis Δ; it is in this sense the limit one dimensional measure $l^{n-1}(x)\,dx$ on Δ may be considered a needle.

The aim of this section is to extend this construction to the setting of separable Fréchet, i.e., complete metrizable locally convex spaces. For example, E may be a Banach space, but there also other important spaces that are not Banach, such as the space $E = \mathbb{R}^\infty$. Note that any finite Borel measure on a Fréchet space is Radon.

While one cannot speak about the Lebesgue measure when E is infinite dimensional, the main hypothesis (1.1) may readily be stated like (1.5) with integration with respect to a given (finite) Borel measure μ on E.

The space of all finite Borel measures on E is endowed with the topology of weak convergence. In particular, $\mu_l \to \mu$ (weakly), if and only if

$$\int u\,d\mu_l \to \int u\,d\mu \qquad (\text{as} \quad l \to \infty)$$

for any bounded continuous functions u on E. As was noticed in [Bor1], the class of all α-concave probability measures on E is closed in the weak topology.

Definition 4.1. Let μ be a finite Borel measure on E. A Borel probability measure ν will be called a needle of μ, if it is supported on a segment $[a,b] \subset E$ and can be obtained as the weak limit of probability measures

$$\mu_l(A) = \frac{1}{\mu(K_l)}\,\mu(A \cap K_l), \qquad (A \text{ is Borel}),$$

where K_l is some decreasing sequence of convex compact sets in E of positive μ-measure such that $\cap_l K_l = [a,b]$.

Here, all μ_l represent normalized restrictions of μ to K_l. In particular, all needles of a given α-concave measure are α-concave, as well. We do not require that K_l be asymptotically close to infinitesimal truncated cylinders.

Definition 4.2. One says that a Borel probability measure μ on E satisfies the zero-one law, if any μ-measurable affine subspace of E has μ-measure either 0 or 1.

For example, this important property holds true for all (Radon) Gaussian measures. More generally, it is satisfied by any hyperbolic probability measure, as follows from Borell's Theorem 2.1.

With these definitions, Theorem 1.2 admits the following refinement.

Theorem 4.1. *Suppose that a Borel probability measure μ on a seperable Fréchet space E satisfies the zero-one law. Let $u, v : E \to \mathbb{R}$ be lower semi-continuous μ-integrable functions such that*

$$\int u\,d\mu > 0, \qquad \int v\,d\mu > 0.$$

Then, these inequalities also hold for some needle ν of μ. Moreover, if μ is supported on a closed convex set F, then ν may be chosen to be supported on F, as well.

First assume that E is a separable Banach space with norm $\|\cdot\|$, and let E' denote the dual space (of all linear continuous functionals on E) with norm $\|\cdot\|_*$. Suppose that any proper closed affine subspace of E has μ-measure zero. In this case, for the proof of Theorem 4.1 we use the construction similar to the one from the proof of Theorem 3.1.

Given 3 affinely independent points x, y, z in E, define linear functionals λ_x and λ_y on the linear hull $L_z(x, y)$ of $x - z$ and $y - z$ (which is a 2-dimensional linear subspace of E), by putting

$$\lambda_x(x - z) = \lambda_y(y - z) = 1, \tag{4.1}$$
$$\lambda_x(y - z) = \lambda_y(x - z) = 0. \tag{4.2}$$

By the Hahn-Banach theorem, these functionals may be extended by linearity to the whole space E without increasing their norms. This will always be assumed below.

We will also employ the following notation, define the lines

$$\begin{aligned} L_z(x) &= \{z + r(x - z) : r \in \mathbb{R}\}, \\ L_z(y) &= \{z + r(y - z) : r \in \mathbb{R}\}. \end{aligned}$$

Then, for $w \in L_z(x, y)$, $\|w\| \leq 1$,

$$|\lambda_x(w)| \leq \mathrm{dist}^{-1}(x, L_z(y)), \quad |\lambda_y(w)| \leq \mathrm{dist}^{-1}(y, L_z(x)),$$

where we use the notation $\mathrm{dist}(w, A) = \inf\{\|w - a\| : a \in A\}$ (the shortest distance from a point to the set). The extended linear functionals should thus satisfy the above inequalities on the whole space E for all $\|w\| \leq 1$, i.e.,

$$\|\lambda_x\|_* \leq \mathrm{dist}^{-1}(x, L_z(y)), \quad \|\lambda_y\|_* \leq \mathrm{dist}^{-1}(y, L_z(x)). \tag{4.3}$$

Lemma 4.1. *Let $\{(x_n, y_n, z_n)\}_{n\geq 1}$ be affinely independent points in the Banach space E such that $x_n \to x$, $y_n \to y$, $z_n \to z$, where x, y, z are also affinely independent. Then the corresponding linear functionals λ_{x_n} and λ_{y_n} have uniformly bounded norms, i.e.,*

$$\sup_{n\geq 1} \|\lambda_{x_n}\|_* < \infty, \qquad \sup_{n\geq 1} \|\lambda_{y_n}\|_* < \infty.$$

Proof. By shifting, one may assume that $z = 0$, in which case x and y are linearly independent and in particular $\|x\| > 0$ and $\|y\| > 0$. Using (4.3), it is enough to show that

$$\mathrm{dist}(x_n, L_{z_n}(y_n)) \geq c, \quad \text{for all } n \geq n_0,$$

with some n_0 and $c > 0$. Indeed, take an arbitrary point $w = z_n + r(y_n - z_n)$ in $L_{z_n}(y_n)$, $r \in \mathbb{R}$. By the triangle inequality,

$$\|x_n - w\| \geq |r| \, \|y_n - z_n\| - \|x_n - z_n\| \geq 2\|x_n - z_n\|,$$

where the last inequality holds whenever $|r| \geq 3 \, \frac{\|x_n - z_n\|}{\|y_n - z_n\|}$. Hence, by the convergence assumption,

$$\|x_n - w\| \geq \|x\|, \qquad \text{for } \ |r| \geq r_0 = 4 \, \frac{\|x\|}{\|y\|}, \ n \geq n_0.$$

In case $|r| \leq r_0$, again by the triangle inequality,

$$\begin{aligned} \|x_n - w\| &\geq \|x - (z + ry)\| - \|x_n - x\| - |r| \, \|y_n - y\| - r\|z_n\| \\ &\geq \operatorname{dist}(x, L_z(y)) - \|x_n - x\| - r_0 \, \|y_n - y\| - r_0\|z_n\|. \end{aligned}$$

Here, the right-hand side is also separated from zero for sufficiently large n.

By a similar argument, $\operatorname{dist}(y_n, L_{z_n}(x_n)) \geq c$, for all $n \geq n_0$. $\square$

Proof of Theorem 4.1.. We begin with a series of reductions, any Fréchet space with Radon probability measure μ has a subspace E_0 such that $\mu(E_0) = 1$, and in addition there exists a norm $\|\cdot\|$ on E_0 with respect to which E_0 is a separable reflexive Banach space whose closed balls are compact in E (see [Bog], Theorem 7.12.4).

In particular, all Borel subsets of E_0 are Borel in E. By the zero-one law (turning to a smaller subspace if necessary), we may assume that any proper affine subspace of E_0 which is closed for the topology of E_0 has measure zero. That is, for any non-trivial $l \in E_0'$,

$$\mu\{l = c\} = 0, \quad c \in \mathbb{R}. \tag{4.4}$$

Second, it suffices to assume that the support of μ is compact and convex. Indeed, by Ulam's theorem, there is an increasing sequence of compact sets $K_n \subset E_0$ such that $\mu(\cup_n K_n) = 1$. The closed convex hull of any compact set in E_0 is compact (which is true in any Banach and more generally complete locally convex spaces, cf. e.g. [K-A]). Therefore, all K_n may additionally be assumed to be convex.

By the dominated convergence theorem,

$$\lim_{n \to \infty} \int_{K_n} u \, d\mu = \int_E u \, d\mu, \qquad \lim_{n \to \infty} \int_{K_n} v \, d\mu = \int_E v \, d\mu,$$

so that $\int_{K_n} u \, d\mu > 0$ and $\int_{K_n} v \, d\mu > 0$ for large n. Hence, an application of the theorem to μ restricted and normalized to K_n would provide the desired one dimensional measure ν, a needle of μ_n and therefore of μ itself.

Thus, from now on, we may assume that E is a separable Banach space, and μ is a Borel probability measure on E which is supported on a convex compact set $K \subset E$ and is such that (4.4) holds true for all non-trivial $l \in E'$.

We need only to prove the existence of ν such that $\int u\,d\nu \geq 0$ and $\int v\,d\nu \geq 0$. Since in this case we may apply the superficially weaker result to $u-\varepsilon$ and $v-\varepsilon$ for an $\varepsilon > 0$ chosen small enough to preserve the hypothesis.

In addition, it suffices to prove the result when u and v are both continuous. To see this, take u_n and v_n to be sequences continous functions increasing to lower semicontinuous u and v respectively. By the monotone convergence, $\lim_{n\to\infty} \int u_n\,d\mu = \int u\,d\mu > 0$ and $\lim_{n\to\infty} \int v_n\,d\mu = \int v\,d\mu > 0$, so we can take the approximating functions u_n and v_n to be such that $\int u_n\,d\mu > 0$ and $\int v_n\,d\mu > 0$. The theorem produces needles ν_n of μ supported on F and such that

$$\int u_n\,d\nu_n > 0, \qquad \int v_n\,d\nu_n > 0.$$

Since $u \geq u_n$ and $v \geq v_n$, every such measure ν_n will be the required needle.

Let us now turn to the construction procedure.

Given 3 affinely independent points x, y, z in E, consider the linear continuous functionals λ_x and λ_y on E introduced before Lemma 4.1 via the relations (4.1)–(4.2) and the Hahn-Banach theorem. To each point $\theta \in \mathbb{S}^1 = \{(t,s) : t^2+s^2 = 1\}$ we can associate a linear functional $\Lambda_\theta = t\lambda_x + s\lambda_y$ and define the function

$$\Psi : \mathbb{S}^1 \to \mathbb{R}, \qquad \theta = (t,s) \mapsto \int_{\{\Lambda_\theta(\xi - z) \geq 0\}} u(\xi)\,d\mu(\xi).$$

Since $\mu\{\xi : \Lambda_\theta(\xi - z) = 0\} = 0$ (cf. (4.4)), this function is continuous on $\mathbb{S}^1$. In addition, we have the identity

$$\Psi(-\theta) + \Psi(\theta) = \int_E u\,d\mu.$$

Hence, by the intermediate value theorem, there exists $\theta \in \mathbb{S}^1$ such that

$$\int_{\{\Lambda_\theta(\xi - z) \geq 0\}} u(\xi)\,d\mu(\xi) \;=\; \int_{\{\Lambda_\theta(\xi - z) \leq 0\}} u(\xi)\,d\mu(\xi) \;=\; \frac{1}{2} \int_E u\,d\mu.$$

Also,

$$\int_E v\,d\mu \;=\; \int_{\{\Lambda_\theta(\xi - z) \geq 0\}} v(\xi)\,d\mu(\xi) + \int_{\{\Lambda_\theta(\xi - z) \leq 0\}} v(\xi)\,d\mu(\xi) \;>\; 0,$$

so that at least one the last two integrals is positive. Let H^+ denote one of the hyperspaces $\{\Lambda_\theta(\xi - z) \geq 0\}$ or $\{\Lambda_\theta(\xi - z) \leq 0\}$ such that $\int_{H^+} v\,d\mu > 0$. Necessarily, $\mu(H^+) > 0$, and we may consider the normalized restriction μ^+ of μ to H^+ and will have the property that

$$\int_{H^+} u\,d\mu^+ > 0, \quad \int_{H^+} v\,d\mu^+ > 0. \tag{4.5}$$

This procedure can be performed step by step along a sequence $\{(x_n, y_n, z_n)\}_{n\geq 1}$ of affinely independent points, chosen to be dense in $K \times K \times K$.

Let $\nu_1 = \mu^+$ be constructed according to the above procedure for (x_1, y_1, z_1) and with an associated point $\theta_1 = (t_1, s_1) \in \mathbb{S}^1$. Similarly, on the n-th step, given ν_n, let $\nu_{n+1} = \nu_n^+$ be constructed for the triple (x_n, y_n, z_n) and with the associated linear functional

$$\Lambda_{\theta_n} = \Lambda_{(t_n,s_n)} = t_n \lambda_{x_n} + s_n \lambda_{y_n}.$$

Since the space of all Borel probability measures on K is compact and metrizable for the weak topology, the sequence ν_n has a sub-sequential weak limit ν. In particular, from (4.5) we derive the desired property

$$\int_E u\, d\nu \geq 0, \qquad \int_E v\, d\nu \geq 0.$$

It remains to show that $\dim(H_\nu) \leq 1$. Suppose not, in this case there exists affinely independent x, y, z in the relative interior of H_ν that also contains the points $2z - x$ and $2z - y$. Without loss of generality, let $z = 0$, so that $\pm x$ and $\pm y$ belong to the relative interior of H_ν. By the density property, there exists a subsequence, say (x_k, y_k, z_k) such that $(x_k, y_k, z_k) \to (x, y, z)$.

By the construction, the measure ν_k^+ is supported on the half-space H_k^+, which is either $\{\xi : \Lambda_{(t_k,s_k)}(\xi - z_k) \geq 0\}$ or $\{\xi : \Lambda_{(t_k,s_k)}(\xi - z_k) \leq 0\}$. For definiteness, let it be the first half-space. Since all H_k^+ contain x and $-x$, we then have

$$\Lambda_{(t_k,s_k)}(x - z_k) \geq 0, \qquad \Lambda_{(t_k,s_k)}(-x - z_k) \geq 0, \tag{4.6}$$

and similarly for the point y.

Recall that by Lemma 4.1, we can obtain a uniform bound M such that

$$\|\Lambda_{(t_k,s_k)}\|_* \leq \|\lambda_{x_k}\|_* + \|\lambda_{y_k}\|_* \leq M \qquad \text{for all } k.$$

Hence, $\Lambda_{(t_k,s_k)}(z_k) \to 0$ and $\Lambda_{(t_k,s_k)}(x_k - x) \to 0$ as $k \to \infty$. But then by (4.6), necessarily $\Lambda_{(t_k,s_k)}(x_k) \to 0$, as well. By the same argument, $\Lambda_{(t_k,s_k)}(y_k) \to 0$.

On the other hand, according to the definition of $\Lambda_{(t_k,s_k)}$ via (4.1)–(4.2), for each k,

$$\Lambda_{(t_k,s_k)}(x_k - z_k) = t_k, \qquad \Lambda_{(t_k,s_k)}(y_k - z_k) = s_k,$$

thus implying that $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = 0$. But this is impossible since $t_k^2 + s_k^2 = 1$. This proves that $\dim(H_\nu) \leq 1$. $\square$

5. The dual form and proof of Theorem 1.3

Following [B-N], let us reformulate Theorem 1.3 in terms of dilated sets. For $\alpha \leq 1$ (Recall, that for α-concave measures this excludes only Dirac measures), $\delta \in (0,1)$, and $p \in [0,1]$ define

$$R_\delta^{(\alpha)}(p) = \begin{cases} 1 - \left[\frac{(1-p)^\alpha - (1-\delta)}{\delta}\right]^{1/\alpha} & \alpha < 0 \text{ or } \alpha > 0, p \leq 1 - (1-\delta)^{1/\alpha} \\ 1 - (1-p)^{1/\delta}, & \alpha = 0. \\ 1 & \alpha > 0, p > 1 - (1-\delta)^{1/\alpha}. \end{cases} \tag{5.1}$$

This definition is motivated by considering $B = F \backslash A$ where F is a closed convex set containing the support of μ and using Lemma A.2, with the inequality

$$\mu(A) \geq \left[\,\delta\mu^*(A_\delta)^\alpha + (1-\delta)\right]^{1/\alpha} \qquad (0 < \delta < 1) \tag{5.2}$$

which is solved as $\mu_*(B^\delta) \geq R_\delta^{(\alpha)}(\mu(B))$.

In the case $\alpha \leq 0$, $R_\delta^{(\alpha)}$ is a strictly concave, increasing function in $p \in [0,1]$. The case $\alpha = 0$ can be derived in the limit. When $0 < \alpha \leq 1$, $R^{(\alpha)}(p)$ is found from the formula above on the interval $0 \leq p \leq 1 - (1-\delta)^{1/\alpha}$ (when the expression makes sense) and we put $R^{(\alpha)}(p) = 1$ on the remaining subinterval of $[0,1]$.

In all cases, $R_\delta^{(\alpha)} : [0,1] \to [0,1]$ represents a concave, continuous, non-decreasing function such that $R_\delta^{(\alpha)}(0) = 0$ and $R_\delta^{(\alpha)}(1) = 1$. Put $R_0^{(\alpha)}(p) = \lim_{\delta\downarrow 0} R_\delta^{(\alpha)}(p) = 1$ for $0 < p \leq 1$ and $R_0^{(\alpha)}(0) = 0$.

Theorem 5.1. *Let μ be an α-concave probability measure on a complete separable locally convex space E supported on a convex closed set F $(-\infty < \alpha \leq 1)$. For any Borel subset B of F and for all $\delta \in [0,1)$,*

$$\mu_*(B^\delta) \geq R_\delta^{(\alpha)}(\mu(B)). \tag{5.3}$$

For example, on the real line $E = \mathbb{R}$ for the Lebesgue measure μ on the unit interval $F = [0,1]$, we have $\alpha = 1$, and (5.3) becomes

$$\mu(B^\delta) \geq \min\left\{\frac{1}{\delta}\,\mu(B), 1\right\}.$$

For the Cauchy measures μ on $\mathbb{R}^n$ and $\mathbb{R}^\infty$ (cf. Examples 2.1), we have $\alpha = -1$, and then (5.1)–(5.3) with $F = E$ yield

$$\mu(B^\delta) \geq \frac{\mu(B)}{1 - (1-\delta)(1-\mu(B))}.$$

Note that when E is a separable Fréchet space and B is Borel, B^δ is universally measurable, so there is no need to use the inner masure.

Let us comment on the extreme values of δ in (5.2) and (5.3). Since the sets B^δ increase for decreasing δ, (5.3) will hold for $\delta = 0$ by continuity, as long as this inequality holds for all $0 < \delta < 1$. In this case, (5.3) with $\delta = 0$ tells as that $\mu(B) > 0 \Rightarrow \mu(B^0) = 1$. Equivalently, after the substitution $A = F \setminus B$ and using Lemma A.2, we get $\mu(A_0) = 0$, that is,

$$\mu\Big\{x \in F : m_\Delta(A) = 1, \text{ for any interval } \Delta \subset F \text{ such that } x \in \Delta\Big\} = 0,$$

as long as $\mu(A) < 1$. This case is however excluded from the formulation of Theorem 1.3 by the assumption $\mu^*(A_\delta) > 0$. Note also that in case $\delta = 1$, (5.2) holds automatically, since then $A_1 = A$.

Thus, both Theorem 1.3 and Theorem 5.1 do not loose generality by assuming that $0 < \delta < 1$ (and we do this below in this section).

Equivalence of Theorem 1.3 and Theorem 5.1. It is straightforward for $\alpha \leq 0$. This case also includes the values $\mu^*(A_\delta) = 0$ in (5.2), since then $\mu_*(B^\delta) = 1$ for $B = F \setminus A$ and thus both (5.2) and (5.3) are immediate.

Consider the case $0 < \alpha \leq 1$. For the implication (5.2) $\Rightarrow$ (5.3), let $p = \mu(B)$, $0 < p < 1$. If $\delta \geq \delta_p = 1 - (1-p)^\alpha$, the formula (5.1) should be applied and then (5.3) becomes

$$\mu_*(B^\delta) \,\geq\, 1 - \left[\frac{(1-\mu(B))^\alpha - (1-\delta)}{\delta}\right]^{1/\alpha}. \tag{5.4}$$

Here the right-hand side tends to 1 as $\delta \downarrow \delta_p$, so necessarily $\mu_*(B^{\delta_p}) = 1$ and hence $\mu_*(B^\delta) = 1$ for all $0 \leq \delta < \delta_p$. Thus, without loss of generality, (5.3) may be stated as (5.4) for the range $\delta \geq \delta_p$. If $\mu_*(B^\delta) = 1$ there is nothing to prove. If $\mu_*(B^\delta) < 1$, then $\mu^*(A_\delta) > 0$ for the set $A = F \setminus B$. In that case, (5.2) is exactly the same as (5.4).

For the implication (5.3) $\Rightarrow$ (5.2), assume that $\mu^*(A_\delta) > 0$. Then $\mu_*(B^\delta) < 1$ for $B = F \setminus A$ which implies that $\mu(B) < p_0 = 1 - (1-\delta)^{1/\alpha}$ according to the definition of $R_\delta^{(\alpha)}(\mu(B))$. Moreover, again the formula (5.1) should be applied to rewrite the hypothesis (5.3) in the form (5.4), which can in turn be rewritten as (5.2).

Proof of Theorem 5.1. First observe that the theorem holds immediately for zero-dimensional measures. Now using Theorem 1.2, let us show how to reduce the desired statement (5.3) to dimension one. Since the sets B^δ may only become larger, when F is getting larger, one may assume that $F = H_\mu$, i.e., the support of μ. Fix $0 < \delta < 1$.

Step 1: First suppose that B is an open set in F such that the boundary ∂B^δ of B^δ in F has μ-measure zero. Fix an arbitrary $p \in (0,1)$. Using the continuity of the functions $R_\delta^{(\alpha)}$, it is sufficient to show that $\mu(B) > p \Rightarrow \mu(D) \geq R_\delta^{(\alpha)}(p)$, where D is the closure of B^δ. If this were not true, we would have

$$\int (1_B - p)\, d\mu > 0, \qquad \int (R_\delta^{(\alpha)}(p) - 1_D)\, d\mu > 0,$$

which is exactly the condition (1.5) for $u = 1_B - p$ and $v = R_\delta^{(\alpha)}(p) - 1_D$ (where 1_A denotes the indicator function of a set A). These functions are lower-semicontinuous, so we may apply Theorem 1.2: There exists an α-concave probability measure ν supported on an interval $\Delta \subset F$, such that (1.6) holds, i.e.,

$$\nu(B) > p, \qquad \nu(D) < R_\delta^{(\alpha)}(\nu(B)).$$

But

$$\nu(D) \geq \nu\big(B^\delta\big) \geq \nu\big((B \cap \Delta)^\delta\big),$$

where $(B\cap\Delta)^\delta$ is the result of the one dimensional dilation operation applied to $B\cap\Delta$ with respect to Δ. Hence, we obtain $\nu((B\cap\Delta)^\delta) < \nu(B)$ which contradicts the relation (5.3) in dimension one.

Step 2: Here we describe one class of open sets to which the previous step may be applied. Let B be a set of the form $T^{-1}(C)\cap F$, where $T : E \to \mathbb{R}^n$ is a continuous linear map and $C \subset \mathbb{R}^n$ is an open polytope ($n \geq 1$ is arbitrary). Then $(T^{-1}(C^\delta))$ represents an intersection of finitely many open half-spaces (Lemma A.3). If $\mu(B) > 0$, then, by the zero-one law, the boundaries of these half-spaces have μ-measure zero and hence $\mu(\partial B^\delta) = 0$, as well.

More generally, let $B = T^{-1}(C) \cap F$, where C is a finite union of open polytopes in $\mathbb{R}^n$. Then C^δ is also a finite union of open polytopes. Using Lemma A.3, we obtain that $\mathrm{clos}(B^\delta) \subset T^{-1}(\mathrm{clos}(C^\delta))$, so $\partial B^\delta \subset T^{-1}(\partial C^\delta)$. Again ∂C^δ is contained in finitely many hyperplanes of $\mathbb{R}^n$ and thus $\mu(\partial B^\delta) = 0$.

Step 3: B is an arbitrary open set in F, assuming that F is a convex, compact set. Denote by $\mathcal{G}$ the collection of all cylindrical sets in F described on the last step. Such sets constitute a base in the original topology on F, since the two coincides by the compactness assumption. Hence $B = \cup G$, where the union is over all $G \in \mathcal{G}$ such that $G \subset B$. Since $\mathcal{G}$ is closed under finite unions, one may apply the Radon property which gives

$$\mu(B) = \sup\{\mu(G) : G \in \mathcal{G},\ G \subset B\}.$$

For any G as above, we have $\mu(G^\delta) \geq R_\delta^{(\alpha)}(\mu(G))$, by the previous steps. Hence, we obtain (5.3) for B, as well.

Step 4: B is an arbitrary Borel set in F. By the strengthened Radon property (1.7), it is sufficient to consider the case of a non-empty compact set B, and we may additionally assume that F is compact.

Any open set in F containing $x \in B$ contains this point together with $B(x)\cap F$, where $B(x) = T_x^{-1}(C(x))$. Here $T_x : E \to \mathbb{R}^n$ is a continuous linear map and $C(x)$ is a Euclidean ball in $\mathbb{R}^n$ (with some n depending on x). Using compactness of B, one can compose its finite covering by the sets of the form

$$G = \big(B(x_1) \cup \cdots \cup B(x_N)\big) \cap F, \qquad x_j \in B,$$

with full intersection being B. Let $\{U_i\}_{i\in I}$ be a decreasing net indexed by a semi-ordered directed set I such that each U_i represents the intersection of finitely many sets G as above. The latter guarantees that $\mu(U_i) \downarrow \mu(B)$ along the net.

Now, let $\delta < \delta' < 1$. Given $x \in F$, the property $x \notin B^\delta$ means that $m_{[x,y]}(B) = \inf_{i\in I} m_{[x,y]}(U_i) \leq \delta$ for any $y \in F$. In that case, there is $i \in I$ such that $m_{[x,y]}(U_i) < \delta'$, and hence the increasing sets

$$V_i(x) = \big\{y \in F : m_{[x,y]}(U_i) < \delta'\big\}, \quad i \in I,$$

cover F. By the construction, for each i, the function $\varphi(x,y) = m_{[x,y]}(U_i)$ is of the type

$$\varphi(y) \ = \ \mathrm{mes}\Big\{t \in (0,1) : \forall\, k \leq l \ \ \exists j \leq N_k \quad (1-t)T_{x_{kj}}(x) + tT_{x_{kj}}(y) \in C(x_{kj})\Big\}$$

for some continuous linear maps $T_{x_{kj}} : E \to \mathbb{R}^{n(x_{kj})}$ and some Euclidean balls $C(x_{kj})$ in $\mathbb{R}^{n(x_{kj})}$. As the boundaries of Euclidean balls do not contain non-degenerate intervals, any such function φ must be continuous on F. Therefore, all the sets $V_i(x)$ are open in F, so that by compactness, $V_i(x) = F$ for some $i = i(x)$. Thus, given $x \in F \setminus B^\delta$, we have $m_{[x,y]}(U_{i(x)}) < \delta'$ for any $y \in F$, and hence $F \setminus B^\delta$ is contained in

$$\bigcup_i \big\{x \in F : m_{[x,y]}(U_i) < \delta' \ \text{ for all } y \in F\big\}.$$

It follows that B^δ contains the intersection of the open sets

$$U_i^{\delta'} \ = \big\{x \in F : m_{[x,y]}(U_i) > \delta' \ \text{ for some } y \in F\big\}$$

and thus, by the Radon property,

$$\mu_*(B^\delta) \ \geq \ \mu\big(\cap_i U_i^{\delta'}\big) = \lim_i \mu\big(U_i^{\delta'}\big).$$

On the other hand, by Step 3, $\mu(U_i^{\delta'}) \geq R_{\delta'}^{(\alpha)}(\mu(U_i))$, and taking the limit along the net we get $\mu_*(B^\delta) \geq R_{\delta'}^{(\alpha)}(\mu(B))$. It remains to let $\delta' \downarrow \delta$ and use the contunuity of $R_\delta^{(\alpha)}$ with respect to δ. $\square$

6. Large and small deviations

As is known, the dilation-type inequality (1.8) of Theorem 1.3 may equivalently be stated on functions (which is often more convenient in applications). Namely, with every Borel measurable function u on E with values in the extended line $[-\infty, \infty]$, one associates its "modulus of regularity"

$$\delta_u(\varepsilon) \ = \ \sup\, \mathrm{mes}\big\{t \in (0,1) : |\, u((1-t)x + ty)\,| \leq \varepsilon\, |u(x)|\big\}, \qquad 0 \leq \varepsilon \leq 1,$$

where the supremum is running over all points $x, y \in E$ such that $u(x)$ is finite.

The behavior of δ_u near zero is used to control the probabilities of large and small deviations of u under hyperbolic measures by involving the parameter α, only (cf. [B4, B-N, F]). In particular, there is the following recursive functional inequality, which is stated below, in the setting of an abstract complete locally convex space E.

We assume that μ is an α-concave probability measure on E with $-\infty < \alpha \leq 1$ and that u is a Borel measurable, μ-a.e. finite function on E.

Theorem 6.1. *Given $0 < \lambda < \operatorname{ess\,sup} |u|$ such that $\mu\{|u| \geq \lambda\}$, for all $\varepsilon \in (0,1)$,*

$$\mu\{|u| > \lambda\varepsilon\} \ \geq \ \Big[\delta\, \mu\{|u| \geq \lambda\}^\alpha + (1-\delta)\Big]^{1/\alpha}, \tag{6.1}$$

where $\delta = \delta_u(\varepsilon)$.

In case $\alpha = 0$, this relation turns into

$$\mu\{|u| > \lambda\varepsilon\} \ \geq \ \big(\mu\{|u| \geq \lambda\}\big)^{\delta}. \tag{6.2}$$

Note that for $\alpha \leq 0$, the assumption $\lambda < \operatorname{ess\,sup}|u|$ may be removed.

If μ is supported on a convex closed set F in E, the inequalities (6.1)–(6.2) continue to hold when u is defined on F (rather than on the whole space). In that case, in the definition of δ_u the supremum should be taken over all points $x, y \in F$.

Proof of Theorem 6.1. Let us recall a simple argument based on Theorem 1.3. The latter is applied with $F = E$ to the set

$$A = \{x \in E : \lambda\varepsilon < |u(x)| < \infty\}.$$

By the definiton,

$$\begin{aligned} A_\delta &= \{x \in E : m_{[x,y]}(A) \geq 1 - \delta \ \ \forall y \in E\} \\ &= \big\{x \in E : \mathrm{mes}\{t \in (0,1) : \lambda\varepsilon < |u((1-t)x + ty)| < \infty\} \geq 1 - \delta \ \ \forall y \in E\big\}. \end{aligned}$$

Suppose that $\lambda \leq |u(x)| < \infty$. Then, for any $y \in E$, we have $|u((1-t)x+ty)| \leq \lambda\varepsilon \Rightarrow |u((1-t)x+ty)| \leq \varepsilon|u(x)|$, so that

$$\mathrm{mes}\big\{t \in (0,1) : |u((1-t)x + ty)| \leq \lambda\varepsilon\big\} \ \leq$$

$$\mathrm{mes}\big\{t \in (0,1) : |u((1-t)x + ty)| \leq \varepsilon|u(x)|\big\} \ \leq \ \delta_u(\varepsilon).$$

Hence,

$$\mathrm{mes}\big\{t \in (0,1) : \lambda\varepsilon < |u((1-t)x + ty)| < \infty\big\} \ \geq \ 1 - \delta_u(\varepsilon)$$

which implies that $x \in A_\delta$ with $\delta = \delta_u(\varepsilon)$. This gives the inclusion

$$\{x \in E : \lambda \leq |u(x)| < \infty\} \subset A_\delta$$

and also that $\mu_*(A_\delta) > 0$ (due to the assumption on λ). It remains to apply (1.8)). □

In the next two corrolaries we follow [B-N], cf. also [F]. Denote by m, a median of $|u|$ under μ, i.e., a real number such that

$$\mu\{|u| > m\} \leq \frac{1}{2}, \quad \mu\{|u| < m\} \leq \frac{1}{2}.$$

Corollary 6.1. *Assuming that $m > 0$, for all $r > 1$,*

$$\mu\{|u| \geq mr\} \ \leq \ \bigg[1 + \frac{2^{-\alpha} - 1}{\delta_u(\frac{1}{r})}\bigg]^{1/\alpha}. \tag{6.3}$$

When $\alpha = 0$, the right-hand side is understood as the limit at zero, that is,

$$\mu\{|u| \geq mr\} \,\leq\, 2^{-1/\delta_u(\frac{1}{r})}. \tag{6.4}$$

If $\alpha < 0$, the inequality (6.3) may be simplified as

$$\mu\{|u| \geq mr\} \,\leq\, C_\alpha\, \delta_u(1/r)^{-1/\alpha} \tag{6.5}$$

with constant $C_\alpha = (2^{-\alpha} - 1)^{1/\alpha}$. Note $C_\alpha \to \frac{1}{2}$, as $\alpha \to -\infty$. As is easy to see, we also have a uniform bound, such as, for example, $C_\alpha \leq 1$ in the region $\alpha \leq -1$.

Proof. To derive (6.3) in case $\alpha \neq 0$, apply (6.1) with $\lambda = mr$ and $\varepsilon = 1/r$. Then $\mu\{|u| > \lambda\varepsilon\} \leq \frac{1}{2}$, and letting $p = \mu\{|u| \geq \lambda\}$, we get $\frac{1}{2} \geq (\delta p^\alpha + (1-\delta))^{1/\alpha}$. It remains to solve this inequality in terms of p. Note that when $\alpha > 0$, necessarily $\frac{1}{2} \geq (1-\delta)^{1/\alpha}$ or $\frac{2^{-\alpha}-1}{\delta} \geq -1$, so the right-hand side of (6.3) makes sense. By a similar argument, (6.4) follows from (6.2) in the log-concave case. □

Now, let us turn to the problem of small deviations.

Corollary 6.2. *If $m > 0$, for all $0 < \varepsilon < 1$,*

$$\mu\{|u| \leq m\varepsilon\} \,\leq\, C_\alpha\, \delta_u(\varepsilon) \tag{6.6}$$

with constant $C_\alpha = \frac{2^{-\alpha}-1}{-\alpha}$.

Proof. One may assume that $\alpha \neq 0$ and $m = 1$. From (6.1) with $\lambda = 1$, we obtain that $\mu\{|u| \leq \varepsilon\} \leq \varphi(x)$, where $\varphi(x) = 1 - (1+x)^{1/\alpha}$ and $x = (2^{-\alpha} - 1)\, \delta_u(\varepsilon)$. Since this function is concave in $x > -1$, we have $\varphi(x) \leq \varphi(0) + \varphi'(0)x = \frac{2^{-\alpha}-1}{-\alpha}\, \delta_u(\varepsilon)$. When $\alpha = 0$, (6.6) holds with $C_0 = \lim_{\alpha\to 0} C_\alpha = \log 2$. □

Finally, let us illustrate Corollaries 6.1–6.2 on the example of the semi-norms.

Lemma 6.1. *If u is a Borel measurable semi-norm on E (not identically zero), then*

$$\delta_u(\varepsilon) = \frac{2\varepsilon}{1+\varepsilon}, \qquad 0 < \varepsilon \leq 1.$$

Proof. One may assume that both $u(x)$ and $u(y)$ are finite in the definition of δ_u. Moreover, it is a matter of normalization alone, to assume that $c = u(y) \leq u(x) = 1$. Then, by the triangle inequality,

$$u((1-t)x + ty) \,\geq\, |(1-t)u(x) - tu(y)| \,=\, |(1+c)t - 1|,$$

so

$$\begin{aligned}\mathrm{mes}\big\{t \in (0,1) : u((1-t)x + ty) \leq \varepsilon\, u(x)\big\} &\leq \mathrm{mes}\big\{t \in (0,1) : |(1+c)t - 1| \leq \varepsilon\big\} \\ &= \min\{t_1, 1\} - t_0,\end{aligned}$$

where $t_1 = \frac{1+\varepsilon}{1+c}$, $t_0 = \frac{1-\varepsilon}{1+c}$. In case $c \geq \varepsilon$, we have $t_1 - t_0 = \frac{2\varepsilon}{1+c} \leq \frac{2\varepsilon}{1+\varepsilon}$. In case $c \leq \varepsilon$, similarly $1 - t_0 = \frac{c+\varepsilon}{1+c} \leq \frac{2\varepsilon}{1+\varepsilon}$. Thus, $\delta_u(\varepsilon) \leq \frac{2\varepsilon}{1+\varepsilon}$ in both cases. Here, the equality is attained by taking $y = -x$ with $0 < u(x) < \infty$. □

Any non-trivial Borel measurable semi-norm u on E is generated by a centrally symmetric, Borel measurable, convex set B in E, so that

$$B = \{x \in E : u(x) \leq 1\}.$$

Let us first assume $\mu(B) > 0$, we are then in position to apply Corollary 6.1. More conveniently, starting from (6.1) with $\lambda = r$ and $\varepsilon = \frac{1}{r}$ $(r > 1)$, Lemma 6.1 gives

$$\begin{aligned} 1 - \mu(B) &= \mu\{u(x) > 1\} \\ &\geq \big[\delta\, \mu\{u(x) \geq r\}^\alpha + (1-\delta)\big]^{1/\alpha} \\ &\geq \big[\delta\,(1 - \mu(rB))^\alpha + (1-\delta)\big]^{1/\alpha}, \qquad \delta = \frac{2}{r+1}. \end{aligned}$$

At this step, the assumption $\mu(B) > 0$ may be removed. Recalling also Corollary 6.2, we arrive at:

Corollary 6.3. *Given a symmetric, Borel measurable, convex set $B \subset E$, for all $r > 1$ with $\mu(rB) < 1$,*

$$1 - \mu(B) \geq \left[\frac{2}{r+1}\,\big(1 - \mu(rB)\big)^\alpha + \frac{r-1}{r+1}\right]^{1/\alpha}. \tag{6.7}$$

In the limit case $\alpha = 0$, the above is the same as

$$1 - \mu(rB) \leq \big(1 - \mu(B)\big)^{(r+1)/2}.$$

This inequality is due to Lovász and Simonovits [L-S] in case of Euclidean balls B in $\mathbb{R}^n$. Guédon [G] extended it to general symmetric convex sets in $\mathbb{R}^n$ and also found a precise relation in the case $\alpha > 0$. Namely, (6.7) is solved in terms of $1 - \mu(rB)$ as

$$1 - \mu(rB) \leq \max{}^{1/\alpha}\left\{\frac{r+1}{2}\,\big(1 - \mu(B)\big)^\alpha - \frac{r-1}{2},\, 0\right\}.$$

As for the range $\alpha < 0$, (6.7) may be then rewritten as

$$1 - \mu(rB) \leq \left[\frac{r+1}{2}\,\big(1 - \mu(B)\big)^\alpha - \frac{r-1}{2}\right]^{1/\alpha}.$$

These large deviations bounds provide a sharp form of Borell's Lemma 3.1 in [Bor1].

Let us also mention an immediate consequence from Corollary 6.2 and Lemma 6.1 concerning measures of small balls.

Corollary 6.4. *Given a symmetric, Borel measurable, convex set $B \subsetneq E$ such that $\mu(B) \leq \frac{1}{2}$, we have*

$$\mu(\varepsilon B) \leq C_\alpha\, \varepsilon \qquad (0 \leq \varepsilon \leq 1)$$

with constant $C_\alpha = \frac{2(2^{-\alpha} - 1)}{-\alpha}$.

Appendix A: Dilation and its properties

We will discuss the elementary properties of dilation as an operation on Borel sets, $A \to A_\delta$, where $\delta \in [0,1]$ is viewed as parameter. To help acquaint the reader unfamiliar with dilation, we include proofs of the operation's elementary properties on polytopes, and allows the proof of Theorem 5.1 to be self-contained.

Let F be a closed convex subset of a locally convex space E with respect to which this operation is defined:

$$A_\delta = \Big\{x \in A : m_\Delta(A) \geq 1-\delta, \text{ for any interval } \Delta \subset F \text{ such that } x \in \Delta\Big\}.$$

As before, m_Δ denotes a uniform distribution on Δ (again understood as the Dirac measure when the endpoints coincide). In this definition, by the intervals Δ we mean closed intervals $[a,b]$ connecting arbitrary points a, b in F. Moreover, the requirement $x \in \Delta$ may equivalently be replaced by the condition that x is one of the endpoints of Δ.

Note that $A_1 = A$. If $0 \leq \delta < 1$, as an equivalent definition one could put

$$A_\delta = \Big\{x \in F : m_\Delta(A) \geq 1-\delta, \text{ for any interval } \Delta \subset F \text{ such that } x \in \Delta\Big\}.$$

Indeed, in this case, if $x \in F \setminus A$, then $m_{[x,x]}(A) = 0 < 1-\delta$ meaning that $x \notin A_\delta$ according to the second definition. Thus, for $\delta \in [0,1)$, both definitions lead to the same set and we have the property $A_\delta \subset A$.

Lemma A.1. *a) If $A \subset F$ is closed, then every set A_δ is closed as well.*

b) If E is a separable Fréchet space and A is Borel measurable in F, then every set A_δ is universally measurable.

Let us recall that a set in a Hausdorff topological space E is called universally measurable, if it belongs to the Lebesgue completion of the Borel σ-algebra with respect to an arbitrary Borel probability measure on E. In that case one may freely speak about the measures of these sets.

Proof. For a Borel set A in F, consider the function

$$\psi(x,y) = \int 1_A \, dm_{[x,y]} = \int_0^1 1_A((1-t)x + ty)\, dt, \qquad x, y \in F.$$

First assume that A is closed. Then, given a net $x_i \to x$, $y_i \to y$ in F indexed by a semi-ordered set I, we have

$$\limsup_{i \in I} 1_A((1-t)x_i + ty_i) \leq 1_A((1-t)x + ty).$$

After integration this implies

$$\limsup_{i \in I} \psi(x_i, y_i) \leq \psi(x,y).$$

Indeed, the space $L^1[0,1]$ is separable, so the above relation is only to be checked for increasing sequences $i = i_n$ in I. But in that case one may apply the Lebesgue dominated convergence theorem. This means that ψ is upper semicontinuous on $F \times F$, and thus A_δ represents the intersection over all $y \in F$ of the closed sets $\{x \in A : \psi(x,y) \geq 1 - \delta\}$.

In part b), assume that E is a Fréchet space. If A is Borel, then the function ψ is Borel measurable on $F \times F$, so the complement of A_δ in A,

$$A \setminus A_\delta = \{x \in A : \psi(x,y) < 1 - \delta, \text{ for some } y \in A\},$$

represents the x-projection of a Borel set in $E \times E$. But every Borel set in a Polish space is Souslin, and therefore both $A \setminus A_\delta$ and A_δ are universally measurable (cf. [Bog], Corollary 6.6.7 and Theorem 7.4.1). $\square$

There is an opposite operation representing a certain dilation or enlargement of sets. Given a Borel measurable set $B \subset F$ and $\delta \in [0,1)$, define

$$B^\delta \ = \bigcup_{m_\Delta(B) > \delta} \Delta \ = \ \big\{x \in F : m_{[x,y]}(B) > \delta \ \text{ for some } y \in F\big\}. \tag{A.8}$$

Here the union is running over all intervals $\Delta \subset F$ such that $m_\Delta(B) > \delta$.

Note that B^δ contains B (since all singletons in B participate in the above union via $m[x,x]$ the Dirac measure at the point x). Also, although we will be most interested in the case that F is closed and convex, there will be occasion to dilate with respect to more general sets, notice that operation above is well defined as soon as B is Borel.

Lemma A.2. *For any $\delta \in [0,1)$ and any Borel set $B \subset F$, the complement $A = F \setminus B$ satisfies the dual relations*

$$F \setminus A_\delta = (F \setminus A)^\delta \qquad \text{and} \qquad F \setminus B^\delta = (F \setminus B)_\delta.$$

In particular, B^δ is open in F, once B is open in F.

Proof. Given $x \in F$, the property $x \notin A_\delta$ means that, for some interval $\Delta \subset F$ containing x, we have $m_\Delta(A) < 1 - \delta$, that is, $m_\Delta(B) > \delta$ meaning that $\Delta \subset B^\delta$. Therefore, $x \notin A_\delta \Leftrightarrow x \in B^\delta$. For the last assertion, it remains to recall Lemma A.1. $\square$

A.1. Dilation of Polytopes

Though some of the results here may apply in more general settings, we will consider only the case that that $F = E = \mathbb{R}^n$.

Theorem A.1. *When C is convex,*

$$C^\delta \subseteq (1 - \frac{1}{\delta})C + \frac{1}{\delta}C \tag{A.9}$$

$$= \{x : \exists c_1, c_2 \in C, x = c_1 + \frac{1}{\delta}(c_2 - c_1)\}. \tag{A.10}$$

When C is also open, we have equality of the three sets.

Proof. Given $x \in C^\delta$, by definition there exists an $[a,b]$ with $m_{[a,b]}(C) > \delta$, by comparing to $[a,x]$ and $[x,b]$ we may assume without loss of generality that $b = x$. Defining $T = \sup\{\lambda : (1-\lambda)a\lambda x \in C\}$ and $t = \inf\{\lambda : (1-\lambda)a + \lambda x \in C\}$, it follows that $m_{[a,x]}(C) = T - t > \delta$. It is a straight forward computation that

$$m_{[(1-t)a+tx,x]}(C) = (T-t)/(1-t) > \delta,$$

so we may assume without loss of generality that $t = 0$.

With this the case, for small $\varepsilon > 0$, $(1-\varepsilon)a + \varepsilon x \in C$, and $(1-(T-\varepsilon))a + (T-\varepsilon)x \in C$. Taking ε small enough will provide the existence of $c_i \in C$ such that $c_2 = c_1 + \sigma(x - c_1)$ for some $\sigma > \delta$. Setting $c = c_1 + \delta(x - c_1)$ we have an element of C, which after unpacking the definitions $x = c_1 + \frac{1}{\delta}(c - c_1)$.

Now for the reverse inclusion when C is open. For $x = (1 - \frac{1}{\delta})c_1 + \frac{1}{\delta}c_2$, $c_i \in C$, rearranging and using convexity, $c_2 = (1-\delta)c_1 + \delta x$ are such that the $\mathbb{1}_C((1-t)c_1 + tx) = 1$ for $t \in [0,\delta]$. Now by the continuity of vector space operations and the fact that C is open, it must actually hold for some $\varepsilon > 0$,that $\mathbb{1}_C((1-t)c_1 + tx) = 1$ for $t \in [0, \delta + \varepsilon]$. Thus $x \in C^\delta$. □

Notice that C convex (resp. open) implies that $(1-\frac{1}{\delta})C + \frac{1}{\delta}C$ is convex (resp. open) and as well. Combining this observation with the identity above we have the following;

Corollary A.1. *For C convex and open, C^δ is convex and open as well.*

Define $P \subset E$ to be a *polytope* when P is the convex hull of finitely points. Equivalently in a finite dimensional space, when P is a bounded set realized as the intersection of finitely many closed half spaces.

Theorem A.2. *P a polytope implies that P is compact and convex.*

Proof. Let $\{p_k\}_1^n$ be an enumeration of the extreme points of P. Define $\Delta^n = \{x \in [0,1]^n : x_i \geq 0, \sum_j x_j = 1\}$ and $T : \Delta^n \to E$, $x \mapsto \sum_i x_i p_i$. Then $P = T(\Delta^n)$ □

A.2. Stability of Euclidean Polytopes Under Dilation

We will call A an open polytope when it is the relative interior of a polytope (with respect to the affine hull of the polytope), or equivalently when it is a bounded set, realized as the intersection of finitely many open halfspaces.

Theorem A.3. *If A is an open polytope, A^δ is as well.*

We will first prove an analogous result for (closed) polytopes, that will be helpful in our proof.

Theorem A.4. *If P is a polytope then the closure of P^δ is as well.*

Proof. First we will prove

$$\overline{P^\delta} = P^{\overline{\delta}} := \{x : \exists p \in P \text{ s.t. } m_{[p,x]}(P) \geq \delta\}$$

We start with $\supseteq$, take $x \in \overline{P^{\delta}}$, if $x \in P$ then $x \in \overline{P^\delta}$ immediately, so assume $x \in P^c$ and a be $p \in P$ such that $m_{[p,x]}(P) \geq \delta$. Since $x \notin P$ a closed set there is a neighborhood of x disjoint from P. Thus $p - (\frac{n-1}{n})(p-x)$ is a sequence in P^δ and clearly it converges to x.

In the opposite direction, for $x \in \overline{P^\delta}$ take $x_n \in P^\delta$ converging to x, and $p_n \in P$ such that $m_{[p_n,x_n]}(P) > \delta$. Restricting to a subsequence, we may assume that p_n is convergent to some $p \in P$. By convexity $p_n + t(x_n - p_n) \in P$ for $t \in [0,\delta]$. Since P is closed, taking the limit for $t \in [0,\delta]$ implies $p + t(x-p) \in P$, and hence $m_{[p,x]}(P) \geq \delta$. Thus the reverse inclusion holds, and we have our claim.

Now, observing that $\overline{P^\delta} = \overline{P^{\delta}}$ is also bounded and convex, to show that $\overline{P^\delta}$ is in fact a polytope, what remains to prove is that it possess only a finite number of extreme points. Notice that compactness of P for any $x \in \overline{P^{\delta}}$ there exists $p, q \in P$ such that the normalized measure of the interval $m_{[p,x]}(P) \geq \delta$ is maximal, and $q = p + m_{[p,x]}(P)(x-p)$. We claim that $x \in \mathcal{E}(\overline{P^{\delta}})$ implies that the p and q described above are elements of $\mathcal{E}(P)$. Notice, $x \in \mathcal{E}(\overline{P^{\delta}})$ implies $m_{[p,x]} = \delta$, and that $x = p + \frac{1}{\delta}(q-p)$. Now take $p_i \in P$ such that $p_1/2 + p_2/2 = p$ then there exists $p_i \in P$ such that $\frac{1}{2}p_1 + \frac{1}{2}p_2 = p$. Defining

$$x_i = p_i + \frac{1}{\delta}(q - p_i),$$

the x_i are elements of $\overline{P^{\delta}}$ such that $x_1/2 + x_2/2 = x$, hence $x_i = x$, from which it follows that $p_i = p$ and p is an extreme point of P. Now consider the possibility of $q_i \in P$ such that $\frac{q_1+q_2}{2} = q$. and define

$$x_i = p + \frac{1}{\delta}(q_i - p)$$

Similarly to the previous, $x_1/2 + x_2/2 = x$ so that by x extreme, $x_i = x$ from which it follows $q_i = q$ and that q is extreme in $\mathcal{E}(\overline{P^\delta})$. Thus, if $\{\rho_i\}$ is an enumeration of the extreme points of P then $\mathcal{E}(\overline{P^\delta}) \subseteq \{\rho_i + \frac{1}{\delta}(\rho_j - \rho_i)\}$, in particular $\overline{P^\delta}$ is a polytope. $\square$

We are now in position to verify theorem A.3.

Proof. If A is an open polytope, then $A = P^\circ$ for some polytope P. By theorem A.4, we will prove our result if we can show $A^\delta = \overline{P^\delta}^\circ$. Since A^δ is an open subset of P^δ, $\subseteq$ is immediate. Now given $x \in P^\delta$, there exists $p, q \in P$ such that $x = p + (1 - \frac{1}{\delta})(q-p)$. Since P is convex, $A = P^\circ$ is dense in P, so take p_n and q_n sequences in A converging to p and q respectively. Then $x_n = p_n + (1 - \frac{1}{\delta})(q_n - p_n) \in A^\delta$ and $x_n \to x$.

This actually completes the proof since this shows$(\overline{A^\delta})^\circ \supseteq (\overline{P^\delta})^\circ$, but by A^δ convex and open, $\overline{A^\delta}^\circ = (A^\delta)^\circ = A^\delta$. $\square$

Theorem A.5. *When $E = F$ and A is complement of a centrally symmetric open, convex set $B \subseteq E$, then $A_\delta = E \setminus (\frac{2}{\delta} - 1)B$ represents the complement to the corresponding dilation of B. That is*

$$(B^\delta)^c = E \setminus (\frac{2}{\delta} - 1)B$$

Proof. For $x \in (B^\delta)^c$, every Δ containing x, $m_\Delta(B) < \delta$ □

We can now prove the following lemma used in the proof of Theorem 5.1,

Lemma A.3. *Let F be a convex closed set in E, and let T be a linear continuous map from E to another locally convex space E_1. For any Borel set $C \subset T(F)$,*

$$\big(T^{-1}(C) \cap F\big)^\delta = T^{-1}\big(C^\delta\big) \cap F,$$

where the operation $C \to C^\delta$ is understood with respect to the image $T(F)$.

Proof. For all $a, b \in F$, the map T pushes forward the unform measure $m_{[a,b]}$ to $m_{[Ta,Tb]}$. Therefore, the pre-image $B = T^{-1}(C)$ has measure $m_{[a,b]}(B) = m_{[Ta,Tb]}(C)$, so

$$\big(B \cap F\big)^\delta = \bigcup_{m_{[Ta,Tb]}(C) > \delta} [a,b] = \bigcup_{m_{[x,y]}(C) > \delta} T^{-1}([x,y]) \cap F = T^{-1}\big(C^\delta\big) \cap F. \quad \square$$

When $E_1 = \mathbb{R}^n$ and C is a polytope, the dilated set C^δ is a polytope, as well. Hence, by Lemma A.3, $(T^{-1}(C))^\delta$ represents the intersection of finitely many half-spaces.

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References

[B1] Bobkov, S. G. Remarks on the growth of L^p-norms of polynomials. Geometric aspects of functional analysis, 27–35, Lecture Notes in Math., 1745, Springer, Berlin, 2000. MR1796711

[B2] Bobkov, S. G. Some generalizations of Prokhorov's results on Khinchin-type inequalities for polynomials. (Russian) Teor. Veroyatnost. i Primenen. 45 (2000), no. 4, 745–748. Translation in: Theory Probab. Appl. 45 (2002), no. 4, 644–647. MR1968725

[B3] Bobkov, S. G. Localization proof of the isoperimetric Bakry-Ledoux inequality and some applications. Teor. Veroyatnost. i Primenen. 47

(2002), no. 2, 340–346. Translation in: Theory Probab. Appl. 47 (2003), no. 2, 308–314. MR2001838

[B4] Bobkov, S. G. Large deviations via transference plans. Advances in mathematics research, Vol. 2, 151–175, Adv. Math. Res., 2, Nova Sci. Publ., Hauppauge, NY, 2003. MR2035184

[B5] Bobkov, S. G. Large deviations and isoperimetry over convex probability measures. Electron. J. Probab. 12 (2007), 1072–1100. MR2336600

[B6] Bobkov, S. G. On isoperimetric constants for log-concave probability distributions. Geometric aspects of functional analysis, 81–88, Lecture Notes in Math., 1910, Springer, Berlin, 2007. MR2347041

[B-M] Bobkov, S. G., Madiman, M. Concentration of the information in data with log-concave distributions. Ann. Probab. 39 (2011), no. 4, 1528–1543. MR2857249

[B-N] Bobkov, S. G., Nazarov, F. L. Sharp dilation-type inequalities with fixed parameter of convexity. J. Math. Sci. (N.Y.) 152 (2008), no. 6, 826–839. Translation from: Zap. Nauchn. Sem. POMI 351 (2007), Veroyatnost i Statistika, 12, 54–78. MR2742901

[BN] Barndorff-Nielsen, O. Hyperbolic distributions and distributions on hyperbolae. Scand. J. Statist. 5 (1978), no. 3, 151–157 MR0509451

[Bog] Bogachev, V. I. Measure Theory. Vol. I, II. Springer-Verlag, Berlin, 2007. Vol. I: xviii+500 pp., Vol. II: xiv+575 pp. MR2267655

[B-S-S] Bogachev, V. I., Smolyanov, O. G., Sobolev, V. I. Topological vector spaces and their applications (Russian). Moscow, Izhevsk, 2012, 584 pp.

[Bor1] Borell, C. Convex measures on locally convex spaces. Ark. Math. 12 (1974), 239–252. MR0388475

[Bor2] Borell, C. Convex set functions in d-space. Period. Math. Hungar. 6 (1975), no. 2, 111–136. MR0404559

[Bor3] Borell, Christer. Convexity of measures in certain convex cones in vector space $sigma$-algebras. Mathematica Scandinavica 53 (1983), 125–144. MR0733944

[B-L] Brascamp, H. J., Lieb, E. H. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Funct. Anal. 22 (1976), no. 4, 366–389. MR0450480

[B-Z] Burago Yu. D., Zalgaller, V. A. Geometric inequalities. Springer-Verlag, Berlin, 1988. Translated from the Russian by A. B. Sosinskii, Springer Series in Soviet Mathematics, xiv+331 pp. MR0936419

[D-K-H] Davidovich, Ju. S., Korenbljum, B. I., Hacet, B. I. A certain property of logarithmically concave functions. (Russian) Dokl. Akad. Nauk SSSR 185 (1969), 1215–1218. MR0241584

[F] Fradelizi, M. Concentration inequalities for s-concave measures of dilations of Borel sets and applications. Electron. J. Probab. 14 (2009), no. 71, 2068–2090. MR2550293

[F-G1] Fradelizi, M., Guédon, O. The extreme points of subsets of s -concave probabilities and a geometric localization theorem. Discrete Comput. Geom. 31 (2004), no. 2, 327–335. MR2060645

[F-G2] Fradelizi, M., Guédon, O. A generalized localization theorem and geometric inequalities for convex bodies. Adv. Math. 204 (2006), no. 2, 509–529. MR2249622

[G-M] Gromov, M., Milman, V. D. Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. Composition Math. 62 (1987), 263–282. MR0901393

[G] Guédon, O. Kahane-Khinchine type inequalities for negative exponent. Mathematika 46 (1999), no. 1, 165–173. MR1750653

[H-O] Hadwiger, H., Ohmann, D. Brunn-Minkowskischer Satz und Isoperimetrie. Math. Z., 66 (1956), 1–8. MR0082697

[I] Ibragimov, I. A. On the composition of unimodal distributions. (Russian) Teor. Veroyatnost. i Primenen. 1 (1956), 283–288. MR0087249

[K-L-S] Kannan, R., Lovász, L. Simonovits, M. Isoperimetric problems for convex bodies and a localization lemma. Discrete Comput. Geom. 13 (1995), no. 3–4, 541–559. MR1318794

[K-A] Kantorovich, L. V., Akilov, G. P. Functional Analysis. Translated from the Russian by Howard L. Silcock. Second edition. Pergamon Press, Oxford-Elmsford, N.Y., 1982. xiv+589 pp. MR0788496

[K-S] Kotz, Samuel, and Saralees Nadarajah. Multivariate t-distributions and their applications. Cambridge University Press, 2004. MR2038227

[L-T] Ledoux, M.,Talagrand, M. Probability in Banach Spaces: isoperimetry and processes. Vol. 23. Springer, 1991. MR2814399

[L-S] Lovász, L. Simonovits, M. Random walks in a convex body and an improved volume algorithm. Random Structures Algor. 4 (1993), no. 4, 359–412. MR1238906

[M] Meyer, P.-A. Probability and potentials. Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1966 xiii+266 pp. MR0205288

[N-S-V] Nazarov, F., Sodin, M., Vol'berg, A. The geometric Kannan-Lovász-Simonovits lemma, dimension-free estimates for the distribution of the values of polynomials, and the distribution of the zeros of random analytic functions. (Russian) Algebra i Analiz 14 (2002), no. 2, 214–234. Translation in: St. Petersburg Math. J. 14 (2003), no. 2, 351–366. MR1925887

[P-S] Puig, P., Stephens, M, A. Goodness-of-fit tests for the hyperbolic distribution. Canad. J. Statist. 29 (2001), no. 2, 309–320. MR1840711

[P-W] Payne, L. E., Weinberger, H. F. An optimal Poincaré inequality for convex domains. Arch. Rational Mech. Anal. 5 (1960), 286–292. MR0117419

[Ph] Phelps, R. R. Lectures on Choquet's theorem. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966 v+130 pp. MR1835574

[Pr] Prékopa, A. Logarithmic concave measures with application to stochastic programming. Acta Sci. Math. (Szeged) 32 (1971), 301–316. MR0315079

[R] Rudin, W. Functional Analysis. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, xiii+397 pp. MR1157815