Stability of Cramer's Characterization of Normal Laws in Information Distances

Sergey Bobkov, Gennadiy Chistyakov, and Friedrich Götze

Abstract Optimal stability estimates in the class of regularized distributions are derived for the characterization of normal laws in Cramer's theorem with respect to relative entropy and Fisher information distance.

Keywords Characterization of normal laws • Cramer's theorem • Stability problems

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1 Introduction

If the sum of two independent random variables has a nearly normal distribution, then both summands have to be nearly normal. This property is called stability, and it depends on distances used to measure "nearness". Quantitative forms of this important theorem by P. Lévy are intensively studied in the literature, and we refer to [7] for historical discussions and references. Most of the results in this direction describe stability of Cramer's characterization of the normal laws for distances which are closely connected to weak convergence. On the other hand, there is no stability for strong distances including the total variation and the relative entropy, even in the case where the summands are equally distributed. (Thus, the answer to a conjecture from the 1960s by McKean [14] is negative, cf. [4, 5].) Nevertheless, the stability with respect to the relative entropy can be established for *regularized* distributions in the model, where a small independent Gaussian noise is added to the

S. Bobkov
School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455, USA
e-mail: bobkov@math.umn.edu

G. Chistyakov • F. Götze (✉)
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
e-mail: chistyak@math.uni-bielefeld.de; goetze@mathematik.uni-bielefeld.de

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summands. Partial results of this kind have been obtained in [7], and in this note we introduce and develop new technical tools in order to reach optimal lower bounds for closeness to the class of the normal laws in the sense of relative entropy. Similar bounds are also obtained for the Fisher information distance.

First let us recall basic definitions and notations. If a random variable (for short—r.v.) X with finite second moment has a density p, the entropic distance from the distribution F of X to the normal is defined to be

$$D(X) = h(Z) - h(X) = \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{\varphi_{a,b}(x)}\, dx,$$

where

$$\varphi_{a,b}(x) = \frac{1}{b\sqrt{2\pi}}\, e^{-(x-a)^2/2b^2}, \quad x \in \mathbb{R},$$

denotes the density of a Gaussian r.v. $Z \sim N(a, b^2)$ with the same mean $a = \mathbf{E}X = \mathbf{E}Z$ and variance $b^2 = \mathrm{Var}(X) = \mathrm{Var}(Z)$ as for X ($a \in \mathbb{R}$, $b > 0$). Here

$$h(X) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$$

is the Shannon entropy, which is well-defined and is bounded from above by the entropy of Z, so that $D(X) \geq 0$. The quantity $D(X)$ represents the Kullback-Leibler distance from F to the family of all normal laws on the line; it is affine invariant, and so it does not depend on the mean and variance of X.

One of the fundamental properties of the functional h is the entropy power inequality

$$N(X + Y) \geq N(X) + N(Y),$$

which holds for independent random variables X and Y, where $N(X) = e^{2h(X)}$ denotes the entropy power (cf. e.g. [11, 12]). In particular, if $\mathrm{Var}(X + Y) = 1$, it yields an upper bound

$$D(X + Y) \leq \mathrm{Var}(X)D(X) + \mathrm{Var}(Y)D(Y), \tag{1.1}$$

which thus quantifies the closeness to the normal distribution for the sum in terms of closeness to the normal distribution of the summands. The generalized Kac problem addresses (1.1) in the opposite direction: How can one bound the entropic distance $D(X + Y)$ from below in terms of $D(X)$ and $D(Y)$ for sufficiently smooth distributions?

To this aim, for a small parameter $\sigma > 0$, we consider regularized r.v.'s

$$X_\sigma = X + \sigma Z, \qquad Y_\sigma = Y + \sigma Z',$$

where Z, Z' are independent standard normal r.v.'s, independent of X, Y. The distributions of X_σ and Y_σ will be called *regularized* as well. Note that additive white Gaussian noise is a basic statistical model used in information theory to mimic the effect of random processes that occur in nature. In particular, the class of regularized distributions contains a wide class of probability measures on the line which have important applications in statistical theory.

As a main goal, we prove the following reverse of the upper bound (1.1).

Theorem 1.1 *Let X and Y be independent r.v.'s with* $\mathrm{Var}(X+Y) = 1$. *Given* $0 < \sigma \le 1$, *the regularized r.v.'s X_σ and Y_σ satisfy*

$$D(X_\sigma + Y_\sigma) \ge c_1(\sigma)\left(e^{-c_2(\sigma)/D(X_\sigma)} + e^{-c_2(\sigma)/D(Y_\sigma)}\right), \tag{1.2}$$

where $c_1(\sigma) = e^{c\sigma^{-6}\log\sigma}$ and $c_2(\sigma) = c\sigma^{-6}$ with an absolute constant $c > 0$.

Thus, when $D(X_\sigma + Y_\sigma)$ is small, the entropic distances $D(X_\sigma)$ and $D(Y_\sigma)$ have to be small, as well. In particular, if $X + Y$ is normal, then both X and Y are normal, so we recover Cramer's theorem. Moreover, the dependence with respect to the couple $(D(X_\sigma), D(Y_\sigma))$ on the right-hand side of (1.2) can be shown to be essentially optimal, as stated in Theorem 1.3 below.

Theorem 1.1 remains valid even in extremal cases where $D(X) = D(Y) = \infty$ (for example, when both X and Y have discrete distributions). However, the value of $D(X_\sigma)$ for the regularized r.v.'s X_σ cannot be arbitrary. Indeed, X_σ has always a bounded density $p_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathbf{E}\,e^{-(x-X)^2/2\sigma^2} \le \frac{1}{\sigma\sqrt{2\pi}}$, so that $h(X_\sigma) \ge -\log\frac{1}{\sigma\sqrt{2\pi}}$. This implies an upper bound

$$D(X_\sigma) \le \frac{1}{2}\log\frac{e\mathrm{Var}(X_\sigma)}{\sigma^2} \le \frac{1}{2}\log\frac{2e}{\sigma^2},$$

describing a general possible degradation of the relative entropy for decreasing σ. If $D_\sigma \equiv D(X_\sigma + Y_\sigma)$ is known to be sufficiently small, say, when $D_\sigma \le c_1^2(\sigma)$, the inequality (1.2) provides an additional constraint in terms of D_σ:

$$D(X_\sigma) \le \frac{c}{\sigma^6\log(1/D_\sigma)}.$$

Let us also note that one may reformulate (1.2) as an upper bound for the entropy power $N(X_\sigma + Y_\sigma)$ in terms of $N(X_\sigma)$ and $N(Y_\sigma)$. Such relations, especially those of the linear form

$$N(X+Y) \le C\,(N(X) + N(Y)), \tag{1.3}$$

are intensively studied in the literature for various classes of probability distributions under the name "reverse entropy power inequalities", cf. e.g. [1–3, 10]. However,

(1.3) cannot be used as a quantitative version of Cramér's theorem, since it looses information about $D(X+Y)$, when $D(X)$ and $D(Y)$ approach zero.

A result similar to Theorem 1.1 also holds for the Fisher information distance, which may be more naturally written in the standardized form

$$J_{st}(X) = b^2(I(X) - I(Z)) = b^2 \int_{-\infty}^{\infty} \Big(\frac{p'(x)}{p(x)} - \frac{\varphi'_{a,b}(x)}{\varphi_{a,b}(x)}\Big)^2 p(x)\,dx$$

with parameters a and b as before. Here

$$I(X) = \int_{-\infty}^{\infty} \frac{p'(x)^2}{p(x)}\,dx$$

denotes the Fisher information of X, assuming that the density p of X is (locally) absolutely continuous and has a derivative p' in the sense of Radon-Nikodym. Similarly to D, the standardized Fisher information distance is an affine invariant functional, so that $J_{st}(\alpha + \beta X) = J_{st}(X)$ for all $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. In many applications it is used as a strong measure of X being non Gaussian. For example, $J_{st}(X)$ dominates the relative entropy; more precisely, we have

$$\frac{1}{2} J_{st}(X) \geq D(X). \tag{1.4}$$

This relation may be derived from an isoperimetric inequality for entropies due to Stam and is often regarded as an information theoretic variant of the logarithmic Sobolev inequality for the Gaussian measure due to Gross (cf. [6, 9, 16]). Moreover, Stam established in [16] an analog for the entropy power inequality, $\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}$, which implies the following counterpart of the inequality (1.1)

$$J_{st}(X+Y) \leq \mathrm{Var}(X) J_{st}(X) + \mathrm{Var}(Y) J_{st}(Y),$$

for any independent r.v.'s X and Y with $\mathrm{Var}(X+Y) = 1$. We will show that this upper bound can be reversed in a full analogy with (1.2).

Theorem 1.2 *Under the assumptions of Theorem 1.1,*

$$J_{st}(X_\sigma + Y_\sigma) \geq c_3(\sigma) \left(e^{-c_4(\sigma)/J_{st}(X_\sigma)} + e^{-c_4(\sigma)/J_{st}(Y_\sigma)}\right), \tag{1.5}$$

where $c_3(\sigma) = e^{c\sigma^{-6}(\log\sigma)^3}$ *and* $c_4(\sigma) = c\sigma^{-6}$ *with an absolute constant* $c > 0$.

Let us also describe in which sense the lower bounds (1.2) and (1.5) may be viewed as optimal.

Theorem 1.3 *For every* $T \geq 1$, *there exist independent identically distributed r.v.'s* $X = X_T$ *and* $Y = Y_T$ *with mean zero and variance one, such that* $J_{st}(X_\sigma) \to 0$ *as*

$T \to \infty$ for $0 < \sigma \le 1$ and

$$D(X_\sigma - Y_\sigma) \le e^{-c(\sigma)/D(X_\sigma)} + e^{-c(\sigma)/D(Y_\sigma)},$$
$$J_{st}(X_\sigma - Y_\sigma) \le e^{-c(\sigma)/J_{st}(X_\sigma)} + e^{-c(\sigma)/J_{st}(Y_\sigma)}$$

with some $c(\sigma) > 0$ depending on σ only.

In this note we prove Theorem 1.1 and omit the proof of Theorem 1.2. The proofs of these theorems are rather similar and differ in technical details only, which can be found in [8]. The paper is organized as follows. In Sect. 2, we describe preliminary steps by introducing truncated r.v.'s X^* and Y^*. Since their characteristic functions represent entire functions, this reduction of Theorem 1.1 to the case of truncated r.v.'s allows to invoke powerful methods of complex analysis. In Sect. 3, $D(X_\sigma)$ is estimated in terms of the entropic distance to the normal for the regularized r.v.'s X^*_σ. In Sect. 4, the product of the characteristic functions of X^* and Y^* is shown to be close to the normal characteristic function in a disk of large radius depending on $1/D(X_\sigma + Y_\sigma)$. In Sect. 5, we deduce by means of saddle-point methods a special representation for the density of the r.v.'s X^*_σ, which is needed in Sect. 6. Finally in Sect. 7, based on the resulting bounds for the density of X^*_σ, we establish the desired upper bound for $D(X^*_\sigma)$. In Sect. 8 we construct an example showing the sharpness of the estimates of Theorems 1.1 and 1.2.

2 Truncated Random Variables

Turning to Theorem 1.1, let us fix several standard notations. By

$$(F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y), \qquad x \in \mathbb{R},$$

we denote the convolution of given distribution functions F and G. This operation will only be used when $G = \Phi_b$ is the normal distribution function with mean zero and a standard deviation $b > 0$. We omit the index in case $b = 1$, so that $\Phi_b(x) = \Phi(x/b)$ and $\varphi_b(x) = \frac{1}{b}\,\varphi(x/b)$.

The Kolmogorov (uniform) distance between F and G is denoted by

$$\|F - G\| = \sup_{x\in\mathbb{R}} |F(x) - G(x)|,$$

and $\|F - G\|_{\mathrm{TV}}$ denotes the total variation distance. In general, $\|F - G\| \le \frac{1}{2}\,\|F - G\|_{\mathrm{TV}}$, while the well-known Pinsker inequality provides an upper bound for the

total variation in terms of the relative entropy. Namely,

$$\| F - G\|_{\mathrm{TV}}^2 \leq 2 \int_{-\infty}^{\infty} p(x) \, \log \frac{p(x)}{q(x)} \, dx,$$

where F and G are assumed to have densities p and q, respectively.

In the required inequality (1.2) of Theorem 1.1, we may assume that X and Y have mean zero, and that $D(X_\sigma + Y_\sigma)$ is small. Thus, from now on our basic hypothesis may be stated as

$$D(X_\sigma + Y_\sigma) \leq 2\varepsilon \qquad (0 < \varepsilon \leq \varepsilon_0), \tag{2.1}$$

where ε_0 is a sufficiently small absolute constant. By Pinsker's inequality, this yields bounds for the total variation and Kolmogorov distances

$$|| F_\sigma * G_\sigma - \Phi_{\sqrt{1+2\sigma^2}}|| \leq \frac{1}{2} \, || F_\sigma * G_\sigma - \Phi_{\sqrt{1+2\sigma^2}}||_{\mathrm{TV}} \leq \sqrt{\varepsilon} < 1, \tag{2.2}$$

where F_σ and G_σ are the distribution functions of X_σ and Y_σ, respectively. Moreover without loss of generality, one may assume that

$$\sigma^2 \geq \tilde{c}(\log\log(1/\varepsilon)/\log(1/\varepsilon))^{1/3} \tag{2.3}$$

with a sufficiently large absolute constant $\tilde{c} > 0$. Indeed if (2.3) does not hold, the statement of the theorem obviously holds.

We shall need some auxiliary assertions about truncated r.v.'s. Let F and G be the distribution functions of independent, mean zero r.v.'s X and Y with second moments $\mathbf{E}X^2 = v_1^2$, $\mathbf{E}Y^2 = v_2^2$, such that $\mathrm{Var}(X + Y) = 1$. Put

$$N = N(\varepsilon) = \sqrt{1 + 2\sigma^2} \left(1 + \sqrt{2\log(1/\varepsilon)}\right)$$

with a fixed parameter $0 < \sigma \leq 1$.

Introduce truncated r.v.'s at level N. Put $X^* = X$ in case $|X| \leq N$, $X^* = 0$ in case $|X| > N$, and similarly Y^* for Y. Note that

$$\mathbf{E}X^* \equiv a_1 = \int_{-N}^{N} x \, dF(x), \qquad \mathrm{Var}(X^*) \equiv \sigma_1^2 = \int_{-N}^{N} x^2 \, dF(x) - a_1^2,$$

$$\mathbf{E}Y^* \equiv a_2 = \int_{-N}^{N} x \, dG(x), \qquad \mathrm{Var}(\,Y^*) \equiv \sigma_2^2 = \int_{-N}^{N} x^2 \, dG(x) - a_2^2.$$

By definition, $\sigma_1 \leq v_1$ and $\sigma_2 \leq v_2$. In particular,

$$\sigma_1^2 + \sigma_2^2 \leq v_1^2 + v_2^2 = 1.$$

Denote by F^*, G^* the distribution functions of the truncated r.v.'s X^*, Y^*, and respectively by F^*_σ, G^*_σ the distribution functions of the regularized r.v.'s $X^*_\sigma = X^* + \sigma Z$ and $Y^*_\sigma = Y^* + \sigma Z'$, where Z, Z' are independent standard normal r.v.'s that are independent of (X, Y).

Lemma 2.1 *With some absolute constant C we have*

$$0 \le 1 - (\sigma_1^2 + \sigma_2^2) \le CN^2\sqrt{\varepsilon}.$$

Lemma 2.1 can be deduced from the following observations.

Lemma 2.2 *For any $M > 0$,*

$$\begin{aligned} 1 - F(M) + F(-M) &\le 2\left(1 - F_\sigma(M) + F_\sigma(-M)\right) \\ &\le 4\Phi_{\sqrt{1+2\sigma^2}}(-(M-2)) + 4\sqrt{\varepsilon}. \end{aligned}$$

The same inequalities hold true for G.

Lemma 2.3 *With some positive absolute constant C we have*

$$||F^* - F||_{\mathrm{TV}} \le C\sqrt{\varepsilon}, \qquad ||G^* - G||_{\mathrm{TV}} \le C\sqrt{\varepsilon},$$
$$||F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1+2\sigma^2}}||_{\mathrm{TV}} \le C\sqrt{\varepsilon}.$$

The proofs of Lemma 2.1 as well as Lemmas 2.2 and 2.3 are similar to those used for Lemma 3.1 in [7]. For details we refer to [8].

Corollary 2.4 *With some absolute constant C, we have*

$$\int_{|x|>N} x^2\, dF(x) \le CN^2\sqrt{\varepsilon}, \qquad \int_{|x|>2N} x^2\, d(F_\sigma(x) + F^*_\sigma(x)) \le CN^2\sqrt{\varepsilon},$$

and similarly for G replacing F.

Proof By the definition of truncated random variables,

$$v_1^2 = \sigma_1^2 + a_1^2 + \int_{|x|>N} x^2\, dF(x), \qquad v_2^2 = \sigma_2^2 + a_2^2 + \int_{|x|>N} x^2\, dG(x),$$

so that, by Lemma 2.1,

$$\int_{|x|>N} x^2\, d(F(x) + G(x)) \le 1 - (\sigma_1^2 + \sigma_2^2) \le CN^2\sqrt{\varepsilon}.$$

As for the second integral of the corollary, we have

$$\int_{|x|>2N} x^2\, dF_\sigma(x) = \int_{|x|>2N} x^2 \Big[\int_{-\infty}^{\infty} \varphi_\sigma(x-s)\, dF(s) \Big]\, dx$$

$$= \int_{-\infty}^{\infty} dF(s) \int_{|x|>2N} x^2 \varphi_\sigma(x-s)\, dx$$

$$\leq\ 2\int_{-N}^{N} s^2\, dF(s) \int_{|u|>N} \varphi_\sigma(u)\, du + 2\int_{|s|>N} s^2\, dF(s) \int_{-\infty}^{\infty} \varphi_\sigma(u)\, du$$

$$+\, 2\int_{-N}^{N} dF(s) \int_{|u|>N} u^2 \varphi_\sigma(u)\, du + 2\int_{|s|>N} dF(s) \int_{-\infty}^{\infty} u^2 \varphi_\sigma(u)\, du.$$

It remains to apply the previous step and use the bound $\int_N^\infty u^2 \varphi_\sigma(u) du \leq c\sigma N e^{-N^2/(2\sigma^2)}$. The same estimate holds for $\int_{|x|>2N} x^2\, dF_\sigma^*(x)$. □

3 Entropic Distance to Normal Laws for Regularized Random Variables

We keep the same notations as in the previous section and use the relations (2.1) when needed. In this section we obtain some results about the regularized r.v.'s X_σ and X_σ^*, which also hold for Y_σ and Y_σ^*. Denote by p_{X_σ} and $p_{X_\sigma^*}$ the (smooth positive) densities of X_σ and X_σ^*, respectively.

Lemma 3.1 *With some absolute constant C we have, for all $x \in \mathbb{R}$,*

$$|\, p_{X_\sigma}(x) - p_{X_\sigma^*}(x)| \leq C\sigma^{-1}\sqrt{\varepsilon}. \tag{3.1}$$

Proof Write

$$p_{X_\sigma}(x) = \int_{-N}^{N} \varphi_\sigma(x-s)\, dF(s) + \int_{|s|>N} \varphi_\sigma(x-s)\, dF(s),$$

$$p_{X_\sigma^*}(x) = \int_{-N}^{N} \varphi_\sigma(x-s)\, dF(s) + (1 - F(N) + F((-N)-)\, \varphi_\sigma(x).$$

Hence

$$|\, p_{X_\sigma}(x) - p_{X_\sigma^*}(x)| \leq \frac{1}{\sqrt{2\pi}\sigma} \big(1 - F(N) + F(-N)\big).$$

But, by Lemma 2.2, and recalling the definition of $N = N(\varepsilon)$, we have

$$1 - F(N) + F(-N) \le 2(1 - F_\sigma(N) + F_\sigma(-N)) \le C\sqrt{\varepsilon}$$

with some absolute constant C. Therefore, $|\,p_{X_\sigma}(x) - p_{X_\sigma^*}| \le C\sigma^{-1}\sqrt{\varepsilon}$, which is the assertion (3.1). The lemma is proved. □

Lemma 3.2 *With some absolute constant $C > 0$ we have*

$$D(X_\sigma) \le D(X_\sigma^*) + C\sigma^{-3}N^3\sqrt{\varepsilon}. \tag{3.2}$$

Proof In general, if a random variable U has density u with finite variance b^2, then, by the very definition,

$$D(U) = \int_{-\infty}^{\infty} u(x)\log u(x)\,dx + \frac{1}{2}\,\log(2\pi e\,b^2).$$

Hence, $D(X_\sigma) - D(X_\sigma^*)$ is represented as

$$\begin{aligned}
&\int_{-\infty}^{\infty} p_{X_\sigma}(x)\log p_{X_\sigma}(x)\,dx - \int_{-\infty}^{\infty} p_{X_\sigma^*}(x)\log p_{X_\sigma^*}(x)\,dx + \frac{1}{2}\log\frac{v_1^2+\sigma^2}{\sigma_1^2+\sigma^2}\\
&= \int_{-\infty}^{\infty}(p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx + \int_{-\infty}^{\infty} p_{X_\sigma^*}(x)\log\frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)}\,dx\\
&\qquad + \frac{1}{2}\log\frac{v_1^2+\sigma^2}{\sigma_1^2+\sigma^2}.
\end{aligned} \tag{3.3}$$

Since $\mathbf{E}X^2 \le 1$, necessarily $F(-2) + 1 - F(2) \le \frac{1}{2}$, hence

$$\frac{1}{2\sigma\sqrt{2\pi}}\,e^{-(|x|+2)^2/(2\sigma^2)} \le p_{X_\sigma^*}(x) \le \frac{1}{\sigma\sqrt{2\pi}}, \tag{3.4}$$

and therefore

$$|\log p_{X_\sigma^*}(x)| \le C\sigma^{-2}(x^2+4), \quad x \in \mathbb{R}, \tag{3.5}$$

with some absolute constant C. The same estimate holds for $|\log p_{X_\sigma}(x)|$.

Splitting the integration in

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty}(p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx = I_{1,1} + I_{1,2}\\
&= \Big(\int_{|x|\le 2N} + \int_{|x|>2N}\Big)(p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx,
\end{aligned}$$

we now estimate the integrals $I_{1,1}$ and $I_{1,2}$. By Lemma 3.1 and (3.5), we get

$$|I_{1,1}| \le C'\sigma^{-3}N^3\sqrt{\varepsilon}$$

with some absolute constant C'. Applying (3.5) together with Corollary 2.4, we also have

$$\begin{aligned}|I_{1,2}| &\le 4C\sigma^{-2}\left(1 - F_\sigma(2N) + F_\sigma(-2N) + 1 - F^*_\sigma(2N) + F^*_\sigma(-2N)\right)\\ &+ C\sigma^{-2}\Big(\int_{|x|>2N} x^2\,dF_\sigma(x) + \int_{|x|>2N} x^2\,dF^*_\sigma(x)\Big) \;\le\; C'\sigma^{-2}N^2\sqrt{\varepsilon}.\end{aligned}$$

The two bounds yield

$$|I_1| \le C''\sigma^{-3}N^3\sqrt{\varepsilon} \tag{3.6}$$

with some absolute constant C''.

Now consider the integral

$$\begin{aligned}I_2 &= \int_{-\infty}^{\infty} p_{X^*_\sigma}(x)\log\frac{p_{X_\sigma}(x)}{p_{X^*_\sigma}(x)}\,dx = I_{2,1} + I_{2,2}\\ &= \Big(\int_{|x|\le 2N} + \int_{|x|>2N}\Big)\, p_{X^*_\sigma}(x)\log\frac{p_{X_\sigma}(x)}{p_{X^*_\sigma}(x)}\,dx,\end{aligned}$$

which is non-negative, by Jensen's inequality. Using $\log(1+t) \le t$ for $t \ge -1$, and Lemma 3.1, we obtain

$$\begin{aligned}I_{2,1} &= \int_{|x|\le 2N} p_{X^*_\sigma}(x)\log\Big(1 + \frac{p_{X_\sigma}(x) - p_{X^*_\sigma}(x)}{p_{X^*_\sigma}(x)}\Big)\,dx\\ &\le \int_{|x|\le 2N} |\,p_{X_\sigma}(x) - p_{X^*_\sigma}(x)|\,dx \;\le\; 4C\sigma^{-1}N\sqrt{\varepsilon}.\end{aligned}$$

It remains to estimate $I_{2,2}$. We have as before, using (3.5) and Corollary 2.4,

$$|I_{2,2}| \le C\int_{|x|>2N} p_{X^*_\sigma}(x)\frac{x^2+4}{\sigma^2}\,dx \le C'\sigma^{-2}N^2\sqrt{\varepsilon}$$

with some absolute constant C'. These bounds yield

$$I_2 \le C''\sigma^{-2}N^2\sqrt{\varepsilon}. \tag{3.7}$$

In addition, by Lemma 2.1,

$$\log \frac{v_1^2+\sigma^2}{\sigma_1^2+\sigma^2} \le \frac{v_1^2-\sigma_1^2}{\sigma^2} \le C\sigma^{-2}N^2\sqrt{\varepsilon}.$$

It remains to combine this bound with (3.6) and (3.7) and apply them in (3.3). □

4 Characteristic Functions of Truncated Random Variables

Denote by $f_{X^*}(t)$ and $f_{Y^*}(t)$ the characteristic functions of the r.v.'s X^* and Y^*, respectively. As integrals over finite intervals they admit analytic continuations as entire functions to the whole complex plane $\mathbb{C}$. These continuations will be denoted by $f_{X^*}(t)$ and $f_{Y^*}(t)$, $(t \in \mathbb{C})$.

Put $T = \frac{N}{64} = \frac{\sigma'}{64}\left(1+\sqrt{2\log\frac{1}{\varepsilon}}\right)$, where $\sigma' = \sqrt{1+2\sigma^2}$. We may assume that $0 < \varepsilon \le \varepsilon_0$, where ε_0 is a sufficiently small absolute constant.

Lemma 4.1 *For all $t \in \mathbb{C}$, $|t| \le T$,*

$$\frac{1}{2}\,|e^{-t^2/2}| \le |f_{X^*}(t)|\,|f_{Y^*}(t)| \le \frac{3}{2}\,|e^{-t^2/2}|. \tag{4.1}$$

Proof For all complex t,

$$\Big|\int_{-\infty}^{\infty} e^{itx}\,d(F_\sigma^* * G_\sigma^*)(x) - \int_{-\infty}^{\infty} e^{itx}\,d\Phi_{\sigma'}(x)\Big| \le \Big|\int_{-4N}^{4N} e^{itx}\,d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\Big|$$
$$+ \int_{|x|\ge 4N} e^{-x\mathrm{Im}(t)}\,d(F_\sigma^* * G_\sigma^*)(x) + \int_{|x|\ge 4N} e^{-x\mathrm{Im}(t)}\,\varphi_{\sigma'}(x)\,dx. \tag{4.2}$$

Integrating by parts, we have

$$\int_{-4N}^{4N} e^{itx}\,d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x) \;=\; e^{4itN}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(4N)$$
$$- e^{-4itN}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(-4N) - it\int_{-4N}^{4N}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\,e^{itx}\,dx.$$

In view of the choice of T and N, we obtain, using Lemma 2.3, for all $|t| \le T$,

$$\Big|\int_{-4N}^{4N} e^{itx}\,d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\Big| \le 2C\sqrt{\varepsilon}\,e^{4N|\mathrm{Im}(t)|} + 8C|t|\sqrt{\varepsilon}\,e^{4N\,|\mathrm{Im}(t)|}$$
$$\le \frac{1}{6}\,e^{-(1/2+\sigma^2)\,T^2}. \tag{4.3}$$

The second integral on the right-hand side of (4.2) does not exceed, for $|t| \le T$,

$$\begin{aligned}
&\int_{-2N}^{2N} d(F^* * G^*)(s) \int_{|x|\ge 4N} e^{-x\,\mathrm{Im}(t)} \varphi_{\sqrt{2}\sigma}(x-s)\,dx \\
&\le \int_{-2N}^{2N} e^{-s\,\mathrm{Im}(t)}\, d(F^* * G^*)(s) \cdot \int_{|u|\ge 2N} e^{-u\,\mathrm{Im}(t)} \varphi_{\sqrt{2}\sigma}(u)\,du \\
&\le e^{2NT} \cdot \frac{1}{\sqrt{\pi}} \int_{2N/\sigma}^{\infty} e^{\sigma Tu - u^2/4}\,du \le \frac{1}{6}\, e^{-(1/2+\sigma^2)T^2}. \qquad (4.4)
\end{aligned}$$

The third integral on the right-hand side of (4.2) does not exceed, for $|t| \le T$,

$$\sqrt{\frac{2}{\pi}} \int_{4N}^{\infty} e^{Tu - u^2/6}\,du \le \frac{1}{6}\, e^{-(1/2+\sigma^2)\,T^2}. \qquad (4.5)$$

Applying (4.3)–(4.5) in (4.2), we arrive at the upper bound

$$\begin{aligned}
&|e^{-\sigma^2 t^2/2} f_{X^*}(t) e^{-\sigma^2 t^2/2} f_{Y^*}(t) - e^{-(1/2+\sigma^2)t^2}| \\
&\qquad \le \frac{1}{2}\, e^{-(1/2+\sigma^2)T^2} \le \frac{1}{2}\, |e^{-(1/2+\sigma^2)t^2}| \qquad (4.6)
\end{aligned}$$

from which (4.1) follows. □

The bounds in (4.1) show that the characteristic function $f_{X^*}(t)$ does not vanish in the circle $|t| \le T$. Hence, using results from ([13], pp. 260–266), we conclude that $f_{X^*}(t)$ has a representation

$$f_{X^*}(t) = \exp\{g_{X^*}(t)\}, \quad g_{X^*}(0) = 0,$$

where $g_{X^*}(t)$ is analytic on the circle $|t| \le T$ and admits the representation

$$g_{X^*}(t) = ia_1 t - \frac{1}{2}\sigma_1^2 t^2 - \frac{1}{2} t^2 \psi_{X^*}(t), \qquad (4.7)$$

where

$$\psi_{X^*}(t) = \sum_{k=3}^{\infty} i^k c_k \Big(\frac{t}{T}\Big)^{k-2} \qquad (4.8)$$

with real-valued coefficients c_k such that $|c_k| \le C$ for some absolute constant C. In the sequel without loss of generality we assume that $a_1 = 0$. An analogous representation holds for the function $f_{Y^*}(t)$.

5 The Density of the Random Variable X_σ^*

We shall use the following inversion formula

$$p_{X_\sigma^*}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}e^{-\sigma^2t^2/2}f_{X^*}(t)\,dt, \quad x \in \mathbb{R},$$

for the density $p_{X_\sigma^*}(x)$. By Cauchy's theorem, one may change the path of integration in this integral from the real line to any line $z = t+iy$, $t \in \mathbb{R}$, with parameter $y \in \mathbb{R}$. This results in the following representation

$$p_{X_\sigma^*}(x) = e^{yx}e^{\sigma^2y^2/2}f_{X^*}(iy)\cdot I_0(x,y), \quad x \in \mathbb{R}. \tag{5.1}$$

Here

$$I_0(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(t,x,y)\,dt, \tag{5.2}$$

where

$$R(t,x,y) = f_{X^*}(t+iy)e^{-it(x+\sigma^2y)-\sigma^2t^2/2}/f_{X^*}(iy). \tag{5.3}$$

Let us now describe the choice of the parameter $y \in \mathbb{R}$ in (5.1). It is well-known that the function $\log f_{X^*}(iy), y \in \mathbb{R}$, is convex. Therefore, the function $\frac{d}{dy}\log f_{X^*}(iy) + \sigma^2 y$ is strictly monotone and tends to $-\infty$ as $y \to -\infty$ and tends to ∞ as $y \to \infty$. By (4.7) and (4.8), this function is vanishing at zero. Hence, the equation

$$\frac{d}{dy}\log f_{X^*}(iy) + \sigma^2 y = -x \tag{5.4}$$

has a unique continuous solution $y = y(x)$ such that $y(x) < 0$ for $x > 0$ and $y(x) > 0$ for $x < 0$. Here and in the sequel we use the principal branch of $\log z$.

We shall need one representation of $y(x)$ in the interval $[-(\sigma_1^2+\sigma^2)T_1, (\sigma_1^2+\sigma^2)T_1]$, where $T_1 = c'(\sigma_1^2+\sigma^2)T$ with a sufficiently small absolute constant $c' > 0$. We see that

$$\begin{aligned} q_{X^*}(t) &\equiv \frac{d}{dt}\log f_{X^*}(t) - \sigma^2 t = -(\sigma_1^2+\sigma^2)t - r_1(t) - r_2(t) \\ &= -(\sigma_1^2+\sigma^2)t - t\psi_{X^*}(t) - \frac{1}{2}t^2\psi'_{X^*}(t). \end{aligned} \tag{5.5}$$

The functions $r_1(t)$ and $r_2(t)$ are analytic in the circle $\{|t| \le T/2\}$ and there, by (4.8), they may be bounded as follows

$$|r_1(t)| + |r_2(t)| \le C|t|^2/T \tag{5.6}$$

with some absolute constant C. Using (5.5), (5.6) and Rouché's theorem, we conclude that the function $q_{X^*}(t)$ is univalent in the circle $D = \{|t| \le T_1\}$, and $q_{X^*}(D) \supset \frac{1}{2}(\sigma_1^2 + \sigma^2)D$. By the well-known inverse function theorem (see [15], pp. 159–160), we have

$$q_{X^*}^{(-1)}(w) = b_1 w + ib_2 w^2 - b_3 w^3 + \dots, \qquad w \in \frac{1}{2}(\sigma_1^2 + \sigma^2)D, \tag{5.7}$$

where

$$i^{n-1} b_n = \frac{1}{2\pi i} \int\limits_{|\zeta| = \frac{1}{2}T_1} \frac{\zeta \cdot q'_{X^*}(\zeta)}{q_{X^*}(\zeta)^{n+1}}\, d\zeta, \qquad n = 1, 2, \dots. \tag{5.8}$$

Using this formula and (5.5) and (5.6), we note that

$$b_1 = -\frac{1}{\sigma_1^2 + \sigma^2} \tag{5.9}$$

and that all remaining coefficients $b_2, b_3, \dots$ are real-valued. In addition, by (5.5) and (5.6),

$$-\frac{q_{X^*}(t)}{(\sigma_1^2 + \sigma^2)t} = 1 + q_1(t) \quad \text{and} \quad -\frac{q'_{X^*}(t)}{\sigma_1^2 + \sigma^2} = 1 + q_2(t),$$

where $q_1(t)$ and $q_2(t)$ are analytic functions in D satisfying there $|q_1(t)| + |q_2(t)| \le \frac{1}{2}$. Therefore, for $\zeta \in D$,

$$\frac{q'_{X^*}(\zeta)}{q_{X^*}(\zeta)^{n+1}} = (-1)^n \frac{q_3(\zeta)}{(\sigma_1^2 + \sigma^2)^n \zeta^{n+1}} \equiv (-1)^n \frac{1 + q_2(\zeta)}{(\sigma_1^2 + \sigma^2)^n (1 + q_1(\zeta))^{n+1} \zeta^{n+1}},$$

where $q_3(\zeta)$ is an analytic function in D such that $|q_3(\zeta)| \le 3 \cdot 2^n$. Hence, $q_3(\zeta)$ admits the representation

$$q_3(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \frac{\zeta^k}{T_1^k}$$

with coefficients d_k such that $|d_k| \le 3 \cdot 2^n$. Using this equality, we obtain from (5.8) that

$$b_n = \frac{d_{n-1}}{(\sigma_1^2 + \sigma^2)^n\, T_1^{n-1}} \quad \text{and} \quad |b_n| \le \frac{3 \cdot 2^n}{(\sigma_1^2 + \sigma^2)^n\, T_1^{n-1}}, \quad n = 2, \ldots. \tag{5.10}$$

Now we can conclude from (5.7) and (5.10) that, for $|x| \le T_1/(4|b_1|)$,

$$y(x) = -iq_{X^*}^{(-1)}(ix) = b_1 x - b_2 x^2 + R(x), \quad \text{where} \quad |R(x)| \le 48\, |b_1|^3 |x|^3 / T_1^2. \tag{5.11}$$

In the sequel we denote by θ a real-valued quantity such that $|\theta| \le 1$. Using (5.11), let us prove:

Lemma 5.1 *In the interval $|x| \le c'' T_1/|b_1|$ with a sufficiently small positive absolute constant c'',*

$$y(x)x + \frac{1}{2}\sigma^2 y(x)^2 + \log f_{X^*}(iy(x)) = \frac{1}{2} b_1 x^2 + \frac{c_3 b_1^3}{2T} x^3 + \frac{c\theta b_1^5}{T^2} x^4, \tag{5.12}$$

where c is an absolute constant.

Proof From (5.10) and (5.11), it follows that

$$\frac{1}{2}|b_1 x| \le |\, y(x)| \le \frac{3}{2}|b_1 x|. \tag{5.13}$$

Therefore,

$$\frac{1}{2} y(x)^2 \sum_{k=4}^{\infty} |c_k| \Big(\frac{|\, y(x)|}{T}\Big)^{k-2} \le C\Big(\frac{3}{2}\Big)^4 \frac{b_1^4 x^4}{T^2}.$$

On the other hand, with the help of (5.10) and (5.11) one can easily deduce the relation

$$y(x)x + \frac{1}{2}\,(\sigma^2 + \sigma_1^2)\, y(x)^2 + \frac{1}{2}\, c_3 \frac{y(x)^3}{T} = \frac{1}{2}\, b_1 x^2 + \frac{1}{2}\, c_3 b_1^3 \frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4$$

with some absolute constant c. The assertion of the lemma follows immediately from the two last relations. □

Now, applying Lemma 5.1 to (5.1), we may conclude that in the interval $|x| \le c'' T_1/|b_1|$, the density $p_{X^*}(x)$ admits the representation

$$p_{X_\sigma^*}(x) = \exp\Big\{\frac{1}{2}\, b_1 x^2 + \frac{1}{2}\, c_3 b_1^3 \frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4\Big\} \cdot I_0(x, y(x)) \tag{5.14}$$

with some absolute constant c.

As for the values $|x| > c''T_1/|b_1|$, in (5.1) we choose $y = y(x) = y(c''T_1/|b_1|)$ for $x > 0$ and $y = y(x) = y(-c''T_1/|b_1|)$ for $x < 0$. In this case, by (5.13), we note that $|y| \le 3c''T_1/2$, and we have

$$\Big|\frac{1}{2}\sigma^2 y^2 + \log f_{X^*}(iy)\Big| \le \frac{y^2}{2|b_1|} + \frac{C}{2}\frac{|y|^3}{T}\sum_{k=3}^{\infty}\Big(\frac{|y|}{T}\Big)^{k-3}$$

$$\le \frac{|y|}{2}\left[\frac{3c''T_1}{2|b_1|} + \frac{9}{4}C(c'')^2\frac{T_1^2}{T}\sum_{k=3}^{\infty}\Big(\frac{3c''T_1}{2T}\Big)^{k-3}\right] \le \frac{|y|}{2}\Big(\frac{3}{2}|x| + \frac{1}{4}|x|\Big) \le \frac{7}{8}|yx|.$$

As a result, for $|x| > c''T_1/|b_1|$, we obtain from (5.1) an upper bound $|p_{X^*_\sigma}(x)| \le e^{-\frac{1}{8}|y(x)x|}|I_0(x, y(x))|$, which with the help of left-hand side of (5.13) yields the estimate

$$|p_{X^*_\sigma}(x)| \le e^{-cT|x|/|b_1|}|I_0(x, y(x))|, \quad |x| > c''T_1/|b_1|, \tag{5.15}$$

with some absolute constant $c > 0$.

6 The Estimate of the Integral $I_0(x, y)$

In order to study the behavior of the integral $I_0(x, y)$, we need some auxiliary results. We use the letter c to denote absolute constants which may vary from place to place.

Lemma 6.1 *For $t, y \in [-T/4, T/4]$ and $x \in \mathbb{R}$, we have the relation*

$$\log|R(t, x, y)| = -\gamma(y)t^2/2 + r_1(t, y), \tag{6.1}$$

where

$$\gamma(y) = |b_1|^{-1} + \psi_{X^*}(iy) + 2iy\psi'_{X^*}(iy) \tag{6.2}$$

and

$$|r_1(t, y)| \le ct^2(t^2 + y^2)T^{-2} \quad \textit{with some absolute constant } c. \tag{6.3}$$

Proof From the definition of the function $R(t, x, y)$ it follows that

$$\log|R(t, x, y)| = \frac{1}{2}\Big(\frac{1}{b_1} - \psi_{X^*}(iy) - 2iy\psi'_{X^*}(iy)\Big)t^2$$
$$-\frac{1}{2}(\Re\psi_{X^*}(t + iy) - \psi_{X^*}(iy))(t^2 - y^2) + (\mathrm{Im}\psi_{X^*}(t + iy) + it\psi'_{X^*}(iy))ty. \tag{6.4}$$

Since, for $t, y \in [-T/4, T/4]$ and $k = 4, \ldots,$

$$\begin{aligned}
&\left|\Re(i^k(t+iy)^{k-2} - i^k(iy)^{k-2})\right| \\
&\qquad = \left|\sum_{l=0}^{(k-2)/2} (-1)^{k+1+l}\binom{k-2}{2l} t^{2l} y^{k-2-2l} - (-1)^{k+1} y^{k-2}\right| \\
&\qquad \le t^2 (T/4)^{k-4} \sum_{l=1}^{(k-2)/2} \binom{k-2}{2l} \le 4t^2 (T/2)^{k-4},
\end{aligned}$$

we obtain an upper bound, for the same t and y, namely

$$|\Re\psi_{X^*}(t+iy) - \psi_{X^*}(iy)| \le \sum_{k=4}^{\infty} \frac{|c_k|}{T^{k-2}} |\Re(i^k(t+iy)^{k-2} - i^k(iy)^{k-2})| \le \frac{2^3 C t^2}{T^2}. \tag{6.5}$$

Since, for $t, y \in [-T/4, T/4]$ and $k = 5, \ldots,$

$$\begin{aligned}
&\left|\operatorname{Im}(i^k(t+iy)^{k-2} - i^k(k-2)t(iy)^{k-3})\right| \\
&\qquad = \left|\sum_{l=1}^{(k-3)/2} \binom{k-2}{2l+1} (-1)^{k+l} t^{2l+1} y^{k-3-2l}\right| \\
&\qquad \le |t|^3 (T/4)^{k-5} \sum_{l=1}^{(k-3)/2} \binom{k-2}{2l+1} \le 8|t|^3 (T/2)^{k-5},
\end{aligned}$$

we have

$$\begin{aligned}
&|\operatorname{Im}\psi_{X^*}(t+iy) + it\psi'_{X^*}(iy)| \\
&\le \sum_{k=5}^{\infty} \frac{|c_k|}{T^{k-2}} |\operatorname{Im}(i^k(t+iy)^{k-2} - ti^k(k-2)(iy)^{k-3})| \le \frac{2^4 C |t|^3}{T^3}
\end{aligned} \tag{6.6}$$

for the same t and y. Applying (6.5) and (6.6) in (6.4), we obtain the assertion of the lemma. □

Lemma 6.2 *For $|t| \le c''T/\sqrt{|b_1|}$ and $|y| \le c''T/|b_1|$, we have the estimates*

$$\frac{3}{4|b_1|} \le \gamma(y) \le \frac{5}{4|b_1|} \tag{6.7}$$

and

$$|r_1(t, y)| \le t^2/(8|b_1|). \tag{6.8}$$

Proof Recall that the positive absolute constant c'' is chosen to be sufficiently small. Using the following simple bounds

$$|\psi_{X^*}(iy)| \le \sum_{k=3}^{\infty} |c_k| \Big(\frac{|y|}{T}\Big)^{k-2} \le C\frac{|y|}{T} \sum_{k=3}^{\infty} \Big(\frac{c''}{|b_1|}\Big)^{k-3} \le \frac{1}{8|b_1|}, \tag{6.9}$$

$$\begin{aligned} 2|y\psi'_{X^*}(iy)| &\le \frac{2|y|}{T} \sum_{k=3} |c_k|(k-2)\Big(\frac{|y|}{T}\Big)^{k-3} \\ &\le C\frac{2|y|}{T} \sum_{k=3}^{\infty} (k-2)\Big(\frac{c''}{|b_1|}\Big)^{k-3} \le \frac{1}{8|b_1|}, \end{aligned} \tag{6.10}$$

we easily obtain that

$$\begin{aligned} \frac{3}{4|b_1|} &\le \frac{1}{|b_1|} - \psi_{X^*}(iy)| - 2|y\psi'_{X^*}(iy)| \le \gamma(y) \\ &\le \frac{1}{|b_1|} + |\psi_{X^*}(iy)| + 2|y\psi'_{x^*}(iy)| \le \frac{5}{4|b_1|}, \end{aligned}$$

and thus (6.7) is proved. The bound (6.8) follows immediately from (6.3). □

Lemma 6.3 *For $t \in [-T/4, T/4]$ and $x \in [-c''T_1/|b_1|, c''T_1/|b_1|]$, we have*

$$\operatorname{Im} \log R(t, x, y(x)) = \frac{i}{2} t^3 \psi'_{X^*}(iy(x)) + r_2(t, x), \tag{6.11}$$

where

$$|r_2(t, x)| \le c(|t| + |y(x)|)|t|^3 T^{-2} \quad \textit{with some absolute constant } c. \tag{6.12}$$

Proof Write, for $t, y \in [-T/4, T/4]$ and $x \in \mathbb{R}$,

$$\operatorname{Im} \log R(t, y, x) = -tx + \frac{ty}{b_1} - ty\,\Re\psi_{X^*}(t + iy) - \frac{t^2 - y^2}{2} \operatorname{Im} \psi_{X^*}(t + iy). \tag{6.13}$$

Now we choose in this formula $y = y(x)$, where $y(x)$ is the solution of Eq. (5.4) for $x \in [-c''T_1/|b_1|, c''T_1/|b_1|]$. For such x, in view of (5.13), we know that $|y(x)| \le T/4$. Let us rewrite (5.4) [see as well (5.5)] in the form

$$-\frac{1}{b_1} y(x) + y(x)\psi_{X^*}(iy(x)) + \frac{i}{2} y^2 \psi'_{X^*}(iy(x))) = -x.$$

Applying this relation in (6.13), we obtain the formula

$$\begin{aligned}\operatorname{Im}\log R(t,x,y(x)) &= -ty(x)(\Re\psi_{X^*}(t+iy(x)) - \psi_{X^*}(iy(x)))\\ &+ \frac{i}{2}t^3\psi'_{X^*}(iy(x)) - \frac{1}{2}(t^2-y(x)^2)\big(\operatorname{Im}\psi_{X^*}(t+iy(x)) + it\psi'_{X^*}(iy(x))\big).\end{aligned}$$

In view of (6.5) and (6.6), we can conclude that

$$\operatorname{Im}\log R(t,x,y(x)) = \frac{i}{2}t^3\psi'_{X^*}(iy(x)) + r_2(t,x),$$

where

$$|r_2(t,x)| \le 8\,C\,|t|^3|\,y(x)|T^{-2} + 8C|t|^3(t^2+y(x)^2)T^{-3} \le 16\,C(|t|+|\,y(x)|)|t|^3T^{-2}$$

for $|t| \le T/4$ and $|\,y(x)| \le T/4$. Thus, the lemma is proved. □

Our next step is to estimate the integral $I_0(x,y(x))$. To this aim, we need the following lemma.

Lemma 6.4 *With some absolute constants c the following formula holds*

$$I_0(x,y(x)) = \frac{1}{\sqrt{2\pi}\gamma(\,y(x))^{1/2}} + r_0(x), \quad |x| \le c''T_1/|b_1|,$$

where

$$|r_0(x)| \le c(|b_1|^{7/2} + |b_1|^{3/2}y(x)^2)T^{-2}. \tag{6.14}$$

Proof For short we write y in place of $y(x)$. Put $T_2 = c''T/\sqrt{|b_1|}$ and write

$$\int_{-\infty}^{\infty}\Re R(t,x,y)\,dt = I_{01} + I_{02} = \Big(\int_{-T_2}^{T_2} + \int_{|t|\ge T_2}\Big)\Re R(t,x,y)\,dt.$$

First consider the integral I_{01}. We have

$$\begin{aligned}I_{01} = I_{01,1} - I_{01,2} &\equiv \int_{-T_2}^{T_2}|R(t,x,y)|\,dt\\ &- 2\int_{-T_2}^{T_2}|R(t,x,y)|\,\sin^2\Big(\frac{1}{2}\operatorname{Im}\log R(t,x,y)\Big)\,dt.\end{aligned}$$

By (6.1), we see that

$$I_{01,1} = \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\,dt + \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\big(e^{r_1(t,y)}-1\big)\,dt.$$

Using the inequality $|e^z - 1| \le |z|e^{|z|}$, $z \in \mathbb{C}$, and applying Lemma 6.1 together with (6.3), (6.7), (6.8), we have

$$\begin{aligned}
&\Big|\int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\big(e^{r_1(t,y)} - 1\big)\,dt\Big| \le \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}|r_1(t,y)|e^{|r_1(t,y)|}\,dt \\
&\le \int_{-T_2}^{T_2} e^{-\frac{1}{4|b_1|}t^2}|r_1(t,y)|\,dt \le c\int_{-T_2}^{T_2} t^2\,e^{-\frac{1}{4|b_1|}t^2}\frac{t^2+y^2}{T^2}\,dt \\
&\le c|b_1|^{3/2}(|b_1| + y^2)T^{-2}. \qquad (6.15)
\end{aligned}$$

On the other hand

$$\int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\,dt = \frac{\sqrt{2\pi}}{\gamma(y)^{1/2}} - \int_{|t|\ge T_2} e^{-\frac{\gamma(y)}{2}t^2}\,dt, \qquad (6.16)$$

where, by (6.7) and the assumption (2.3),

$$\int_{|t|\ge T_2} e^{-\frac{\gamma(y)}{2}t^2}\,dt \le \frac{c}{\gamma(y)T_2}e^{-\frac{1}{2}(T_2\sqrt{\gamma(y)})^2} \le c|b_1|^{3/2}T^{-1}e^{-c''2\frac{\gamma(y)}{2|b_1|}T^2} \le cT^{-4}. \qquad (6.17)$$

Therefore in view of (6.15)–(6.17), we deduce

$$I_{01,1} = \frac{\sqrt{2\pi}}{\gamma(y)^{1/2}} + c\theta\frac{|b_1|^{3/2}(|b_1| + y^2)}{T^2}. \qquad (6.18)$$

Now let us turn to the integral $I_{01,2}$. By (6.11), we have

$$\begin{aligned}
|I_{01,2}| &\le \frac{1}{2}\int_{-T_2}^{T_2} |R(t,x,y)|\,(\operatorname{Im}\log R(t,x,y))^2\,dt \\
&\le 2\int_{-T_2}^{T_2} |R(t,x,y)|\,\big(t^6|\psi'_{X^*}(iy)|^2 + |r_2(t,x)|^2\big)\,dt.
\end{aligned}$$

By Lemmas 6.1–6.3 and by the estimates (2.3), (6.10), we arrive at the upper bound

$$\begin{aligned}
|I_{01,2}| &\le \frac{c}{T^2}\int_{-\infty}^{\infty} t^6\Big(\frac{t^2+y^2}{T^2} + 1\Big)e^{-\frac{1}{4|b_1|}t^2}\,dt \\
&\le \frac{c}{T^2}|b_1|^{7/2}\Big(\frac{|b_1| + y^2}{T^2} + 1\Big) \le \frac{c|b_1|^{7/2}}{T^2}. \qquad (6.19)
\end{aligned}$$

It remains to estimate the integral I_{02}. By (2.3),

$$|I_{02}| \le 2\int_{T_2}^{\infty} |R(t,x,y)|\,dt \le 2\int_{T_2}^{\infty} e^{-\frac{\sigma^2}{2}t^2}\,dt$$
$$\le 2\int_{c''\sigma T}^{\infty} e^{-\frac{\sigma^2}{2}t^2}\,dt \le c\sigma^{-3}T^{-1}e^{-(c'')^2\sigma^4T^2} \le cT^{-4}. \tag{6.20}$$

The assertion of the lemma follows from (6.18)–(6.20). □

Since for $|x| > c''T_1/|b_1|$ we choose $y(x) = y(\pm c''T_1/|b_1|)$ and since $|y(x)| \le c''T/|b_1|$ for such x, we obtain, using Lemmas 6.1 and 6.2, and the assumption (2.3), that

$$|I_0(x,y(x))| \le \frac{1}{2\pi}\int_{|t|\le T_2} |R(t,x,y(x))|\,dt + \frac{1}{2\pi}\int_{|t|>T_2} |R(t,x,y(x))|\,dt$$
$$\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4|b_1|}}\,dt + \frac{1}{2\pi}\int_{|t|>T_2} e^{-\frac{\sigma^2t^2}{2}}\,dt$$
$$\le c\Big(|b_1|^{\frac{1}{2}} + T_2^{-1}\sigma^{-2}e^{-\frac{\sigma^2T_2^2}{2}}\Big) \le c|b_1|^{\frac{1}{2}} \tag{6.21}$$

with some absolute constant c. The bound (6.21) holds for $|x| \le c''T_1/|b_1|$ as well. Thus (6.21) is valid for all real x.

Lemma 6.4 and the upper bound (6.21) allow us to control the behavior of the integral $I_0(x,y(x))$.

7 End of the Proof of Theorem 1.1

Starting from the hypothesis (2.1), we need to derive a good upper bound for $D(X_\sigma)$, which is equivalent to bounding the relative entropy $D(X_\sigma^*)$, according to Lemma 3.2. This will be done with the help of the relations (5.14), (5.15), Lemma 6.4, and (6.21) for the density $p_{X_\sigma^*}(x)$ of the r.v. X_σ^*. First, let us prove the following lemma.

Lemma 7.1 *For* $|x| \le c''T_1/|b_1|$,

$$\log\frac{p_{X_\sigma^*}(x)}{\varphi_{\sqrt{1/|b_1|}}(x)} = \frac{c_3}{2T}\Big((b_1x)^3 + 3b_1y(x)\Big) + \tilde{r}(x),$$

where with some absolute constant c

$$|\tilde{r}(x)| \le \frac{c}{T^2}\big(b_1^2y(x)^2 + |b_1|^3 + |b_1|^5x^4\big). \tag{7.1}$$

Proof By (5.14) and Lemma 6.4, we have, for $|x| \le c'' T_1/|b_1|$,

$$\log \frac{p_{X_\sigma^*}(x)}{\varphi_{1/\sqrt{|b_1|}}(x)} = \frac{1}{2}\, c_3 b_1^3 \frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4 - \frac{1}{2}\log(|b_1|\gamma(y(x)) + \log\Big(1 + \sqrt{\frac{\gamma(y(x))}{2\pi}}\, r_0(x)\Big). \tag{7.2}$$

Recalling (6.2) and (4.8), we see that

$$\begin{aligned} |b_1|\gamma(y(x)) &= 1 + |b_1|(\psi_{X^*}(iy(x)) + 2iy(x)\psi'_{X^*}(iy(x))) \\ &= 1 + 3c_3|b_1|\, y(x)T^{-1} + \rho_1(x), \end{aligned} \tag{7.3}$$

where

$$\rho_1(x) \equiv |b_1| \sum_{k=4}^{\infty} i^k (2k-3)\, c_k \Big(\frac{iy(x)}{T}\Big)^{k-2}.$$

It is easy to see that

$$|\rho_1(x)| \le 8C|b_1|\Big(\frac{y(x)}{T}\Big)^2 \le \frac{1}{4}. \tag{7.4}$$

Since $\frac{|3c_3\, b_1 y(x)|}{T} \le \frac{1}{4}$, and using $|\log(1+u) - u| \le u^2$ ($|u| \le 1/2$), we get from (7.3) that

$$\log(|b_1|\,\gamma(y(x))) = \frac{3c_3|b_1|\, y(x)}{T} + c\theta\Big(\frac{b_1 y(x)}{T}\Big)^2 \tag{7.5}$$

with some absolute constant c. Now we conclude from (6.7) and (6.14) that

$$\sqrt{\frac{\gamma(y(x))}{2\pi}}|r_0(x)| \le c\,|b_1|\frac{b_1^2 + y(x)^2}{T^2} \le \frac{1}{4}$$

and arrive as before at the upper bound

$$\Big|\log\Big(1 + \sqrt{\frac{\gamma(y(x))}{2\pi}}\, r_0(x)\Big)\Big| \le c\,|b_1|\frac{b_1^2 + y(x)^2}{T^2}. \tag{7.6}$$

Applying (7.5) and (7.6) to (7.2), we obtain the assertion of the lemma. □

To estimate the quantity $D(X_\sigma^*)$, we represent it as

$$J_1 + J_2 = \Big(\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} + \int_{|x|>c''T_1/|b_1|}\Big) p_{X_\sigma^*}(x) \log \frac{p_{X_\sigma^*}(x)}{\varphi_{\sqrt{1/|b_1|}}(x)}\, dx. \tag{7.7}$$

First let us estimate J_1, using the letters c, C' to denote absolute positive constants which may vary from place to place. By Lemma 7.1,

$$\begin{aligned} J_1 &= \frac{c_3}{T} J_{1,1} + J_{1,2} \\ &= \frac{c_3}{2T} \int_{-\frac{c'' T_1}{|b_1|}}^{\frac{c'' T_1}{|b_1|}} p_{X_\sigma^*}(x) \Big((b_1 x)^3 + 3 b_1 y(x) \Big) dx + \int_{-\frac{c'' T_1}{|b_1|}}^{\frac{c'' T_1}{|b_1|}} p_{X_\sigma^*}(x) \tilde{r}(x)\, dx. \end{aligned} \tag{7.8}$$

Using (5.14) and Lemma 6.4, we note that

$$\begin{aligned} \int_{-c'' T_1/|b_1|}^{c'' T_1/|b_1|} x^3 p_{X_\sigma^*}(x)\, dx &= \int_{-c'' T_1/|b_1|}^{c'' T_1/|b_1|} x^3 (p_{X_\sigma^*}(x) - \varphi_{\sqrt{1/|b_1|}}(x))\, dx \\ &= J_{1,1,1} + J_{1,1,2} + J_{1,1,3} \\ &= \int_{-c'' T_1/|b_1|}^{c'' T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x) \Big(\frac{1}{\sqrt{\gamma(y(x))|b_1|}} - 1 \Big) e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2}\, dx \\ &\quad + \int_{-c'' T_1/|b_1|}^{c'' T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x) \Big(e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} - 1 \Big)\, dx \\ &\quad + \frac{1}{\sqrt{2\pi |b_1|}} \int_{-c' T_1/|b_1|}^{c'' T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x) e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} r_0(x)\, dx. \end{aligned}$$

It is easy to see that

$$\frac{|c_3|\, |b_1|^3 |x|^3}{2T} + \frac{c |b_1|^5 x^4}{T^2} \le \frac{|b_1| x^2}{4} \quad \text{for} \quad |x| \le \frac{c'' T_1}{|b_1|}. \tag{7.9}$$

Using (7.3) and (7.4) and the bound $|(1+u)^{-1/2} - 1| \le |u|$, $|u| \le \frac{1}{2}$, we get

$$|\, (\gamma(x)|b_1|)^{-1/2} - 1| \le c\, \frac{|b_1|\, |\, y(x)|}{T}.$$

The last estimates and (5.13) lead to

$$\begin{aligned} |\, J_{1,1,1}| &\le \frac{c|b_1|}{T} \int_{-\frac{c'' T_1}{|b_1|}}^{\frac{c'' T_1}{|b_1|}} |x|^3 |\, y(x)| \sqrt{|b_1|}\, e^{-|b_1| x^2/4}\, dx \\ &\le \frac{c |b_1|^{5/2}}{T} \int_{-\infty}^{\infty} x^4 e^{-|b_1| x^2/4}\, dx \le \frac{c}{T}. \end{aligned} \tag{7.10}$$

Applying $|e^u - 1| \le |u|e^{|u|}$, we have, for $|x| \le c'T_1/|b_1|$,

$$\left|e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} - 1\right| \le c|b_1|^3|x|^3\Big(\frac{1}{2T} + \frac{b_1^2|x|}{T^2}\Big)\, e^{|b_1|x^2/4}.$$

Therefore, we deduce the estimate

$$|\,J_{1,1,2}| \le c|b_1|^{7/2}\int_{-\infty}^{\infty} x^6\Big(\frac{1}{T} + \frac{b_1^2|x|}{T^2}\Big)\, e^{-|b_1|x^2/4}\,dx \le c\Big(\frac{1}{T} + \frac{|b_1|^{3/2}}{T^2}\Big). \tag{7.11}$$

By (5.13) and (6.14), we immediately get

$$|\,J_{1,1,3}| \le \frac{c}{T^2}\int_{-\frac{c''T_1}{|b_1|}}^{\frac{c''T_1}{|b_1|}} |x|^3\left(|b_1|^{7/2} + |b_1|^{3/2}\,y(x)^2\right) e^{-|b_1|\,x^2/4}\,dx \le \frac{c|b_1|^{3/2}}{T^2}. \tag{7.12}$$

Hence, by (7.10)–(7.12) and (2.3),

$$\Big|\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 p_{X_\sigma^*}(x)\,dx\Big| \le c\left(\frac{1}{T} + \frac{|b_1|^{3/2}}{T^2}\right) \le \frac{c}{T}. \tag{7.13}$$

In the same way,

$$\Big|\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x p_{X_\sigma^*}(x)\,dx\Big| \le c\left(\frac{|b_1|}{T} + \frac{|b_1|^{5/2}}{T^2}\right) \le \frac{c|b_1|}{T}. \tag{7.14}$$

Recalling (5.11), we see that $y(x) = b_1 x + c\theta b_1^2 x^2/T_1$. As a result, using (7.13) and (7.14) and the property $\mathrm{Var}(X) \le 1$, we come to the upper bound

$$|\,J_{1,1}| \le c\,|b_1|^3 T^{-1}. \tag{7.15}$$

In order to estimate $J_{1,2}$, we employ the inequality (7.1). Recalling (5.14), (6.21) and (7.9), we then have

$$|\,J_{1,2}| \le \frac{c}{T^2}\int_{-c'T_1/|b_1|}^{c'T_1/|b_1|} \left(b_1^2 y(x)^2 + |b_1|^3 + |b_1|^5 x^4\right)\sqrt{|b_1|}\,e^{-|b_1|x^2 4}\,dx \le \frac{c|b_1|^3}{T^2}. \tag{7.16}$$

Combining (7.15) and (7.16), we arrive at

$$|\,J_1| \le c|b_1|^3 T^{-2}. \tag{7.17}$$

Let us estimate J_2. From (5.15), (6.21), we have, for all $|x| > c''T_1/|b_1|$,

$$p_{X_\sigma^*}(x) \le C'\sqrt{|b_1|}\,e^{-cT|x|/|b_1|} \le C'\sqrt{|b_1|}\,e^{-cc'c''T^2/|b_1|^3} < 1. \tag{7.18}$$

Here we also used (2.3) and the assumption that $0 < \varepsilon \le \varepsilon_0$, where ε_0 is a sufficiently small absolute constant. Using (7.18) and (2.3), we easily obtain

$$\begin{aligned} J_2 &\le -\int_{|x|>c''T_1/|b_1|} p_{X_\sigma^*}(x) \log \varphi_{\sqrt{1/|b_1|}}(x)\, dx \\ &= \frac{1}{2} \log \frac{2\pi}{|b_1|} \int_{|x|>c''T_1/|b_1|} p_{X_\sigma^*}(x)\, dx + \frac{|b_1|}{2} \int_{|x|>c''T_1/|b_1|} x^2 p_{X_\sigma^*}(x)\, dx \\ &\le C' \sqrt{|b_1|} \int_{|x|>c''T_1/|b_1|} \frac{1}{2} (\log(4\pi) + |b_1| x^2) e^{-cT|x|/|b_1|}\, dx \\ &\le C' \left(|b_1|^{3/2} T^{-1} + |b_1|^{-3/2} T\right) e^{-cc'c''T^2/|b_1|^3} \le C' T^{-2}. \end{aligned} \tag{7.19}$$

Thus, we derive from (7.17) and (7.19) the inequality $D(X_\sigma^*) \le c|b_1|^3 T^{-2}$. Recalling (3.2) and Lemma 2.1, we finally conclude that

$$D(X_\sigma) \le c \frac{|b_1|^3}{T^2} + c\Big(\frac{N}{\sigma}\Big)^3 \sqrt{\varepsilon} \le \frac{c}{(v_1^2 + \sigma^2)^3 T^2} + c\Big(\frac{N}{\sigma}\Big)^3 \sqrt{\varepsilon} \le \frac{c}{(v_1^2 + \sigma^2)^3 T^2}. \tag{7.20}$$

An analogous inequality also holds for the r.v. Y_σ, and thus Theorem 1.1 follows from these estimates.

Remark 7.2 Under the assumptions of Theorem 1.1, a stronger inequality than (1.2) follows from (7.20). Namely, $D(X_\sigma + Y_\sigma)$ may be bounded from below by

$$e^{c\sigma^{-6} \log \sigma} \Big[\exp\Big\{ -\frac{c}{(\mathrm{Var}(X_\sigma))^3\, D(X_\sigma)} \Big\} + \exp\Big\{ -\frac{c}{(\mathrm{Var}(Y_\sigma))^3\, D(Y_\sigma)} \Big\} \Big].$$

8 Proof of Theorem 1.3

In order to construct r.v.'s X and Y with the desired properties, we need some auxiliary results. We use the letters $c, c', \tilde{c}$ (with indices or without) to denote absolute positive constants which may vary from place to place, and θ may be any number such that $|\theta| \le 1$. First we analyze the function v_σ with Fourier transform

$$f_\sigma(t) = \exp\{-(1+\sigma^2)\, t^2/2 + it^3/T\}, \qquad t \in \mathbb{R}.$$

Lemma 8.1 *If the parameter $T > 1$ is sufficiently large and $0 \le \sigma \le 2$, the function f_σ admits the representation*

$$f_\sigma(t) = \int_{-\infty}^{\infty} e^{itx} v_\sigma(x)\, dx \tag{8.1}$$

with a real-valued infinitely differentiable function $v_\sigma(x)$ which together with its all derivatives is integrable and satisfies

$$v_\sigma(x) > 0, \qquad \textit{for} \quad x \le (1+\sigma^2)^2\, T/16; \tag{8.2}$$

$$|v_\sigma(x)| \le e^{-(1+\sigma^2)Tx/32}, \quad \textit{for} \quad x \ge (1+\sigma^2)^2\, T/16. \tag{8.3}$$

In addition, for $|x| \le (1+\sigma^2)^2\, T/16$,

$$c_1\, e^{-2(5-\sqrt{7})\,|xy(x)|/4} \le |v_\sigma(x)| \le c_2\, e^{-4|xy(x)|/9}, \tag{8.4}$$

where

$$y(x) = \frac{1}{6}\, T\,\big(-(1+\sigma^2) + \sqrt{(1+\sigma^2)^2 - 12x/T}\,\big). \tag{8.5}$$

The right inequality in (8.4) continues to hold for all $x \le (1+\sigma^2)^2 T/16$.

Proof Since $f_\sigma(t)$ decays very fast at infinity, the function v_σ is given according to the inversion formula by

$$v_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} f_\sigma(t)\, dt, \quad x \in \mathbb{R}. \tag{8.6}$$

Clearly, it is infinitely many times differentiable, and all its derivatives are integrable. It remains to prove (8.2)–(8.4). By the Cauchy theorem, one may also write

$$v_\sigma(x) = e^{yx} f_\sigma(iy)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} R_\sigma(t,y)\, dt, \quad \text{where } R_\sigma(t,y) = \frac{f_\sigma(t+iy)}{f_\sigma(iy)}, \tag{8.7}$$

for every fixed real y. Here we choose $y = y(x)$ according to the equality in (8.5) for $x \le (1+\sigma^2)^2\, T/16$. In this case, it is easy to see that

$$\begin{aligned} e^{-ixt} R_\sigma(t, y(x)) &= \exp\Big\{-\frac{(1+\sigma^2)t^2}{2}\Big(1 + \frac{6y(x)}{(1+\sigma^2)T}\Big) + i\frac{t^3}{T}\Big\} \\ &\equiv \exp\Big\{-\frac{\alpha(x)}{2}\, t^2 + i\frac{t^3}{T}\Big\}. \end{aligned}$$

Note that $\alpha(x) \ge (1+\sigma^2)/2$ for x as above.

For a better understanding of the behaviour of the integral in the right-hand side of (8.7), put

$$\tilde{I} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} R_\sigma(t,y)\, dt$$

and rewrite it in the form

$$\tilde{I} = \tilde{I}_1 + \tilde{I}_2 = \frac{1}{2\pi}\Big(\int_{|t|\le T^{1/3}} + \int_{|t|>T^{1/3}}\Big) e^{-ixt} R_\sigma(t, y(x))\, dt. \tag{8.8}$$

Using $|\cos u - 1 + u^2/2| \le u^4/4!$ ($u \in \mathbb{R}$), we easily obtain the representation

$$\begin{aligned}\tilde{I}_1 &= \frac{1}{2\pi}\int_{|t|\le T^{1/3}} \Big(1 - \frac{t^6}{2T^2}\Big) e^{-\alpha(x)t^2/2}\, dt + \frac{\theta}{4!\,T^4}\,\frac{1}{2\pi}\int_{|t|\le T^{1/3}} t^{12} e^{-\alpha(x)t^2/2}\, dt \\ &= \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\,T^2} + \frac{c\theta}{\alpha(x)^6 T^4}\Big) \\ &\quad - \frac{1}{2\pi}\int_{|t|>T^{1/3}} \Big(1 - \frac{t^6}{2T^2}\Big) e^{-\alpha(x)t^2/2}\, dt. \end{aligned} \tag{8.9}$$

The absolute value of last integral does not exceed $c(T^{1/3}\alpha(x))^{-1} e^{-\alpha(x)T^{2/3}/2}$. The integral $\tilde{I}_2$ admits the same estimate. Therefore, we obtain from (8.8) the relation

$$\tilde{I} = \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\,T^2} + \frac{c\theta}{\alpha(x)^6\,T^4}\Big). \tag{8.10}$$

Applying (8.10) in (8.7), we deduce for the half-axis $x \le (1+\sigma^2)^2\,T/16$, the formula

$$v_\sigma(x) = \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\,T^2} + \frac{c\theta}{\alpha(x)^6\,T^4}\Big) e^{y(x)x} f_\sigma(iy(x)). \tag{8.11}$$

We conclude immediately from (8.11) that (8.2) holds. To prove (8.3), we use (8.7) with $y = y_0 = -(1+\sigma^2)T/16$ and, noting that

$$x + \frac{1+\sigma^2}{2} y_0 \ge \frac{x}{2} \quad \text{for} \quad x \ge \frac{(1+\sigma^2)^2}{16}\,T,$$

we easily deduce the desired estimate

$$|v_\sigma(x)| \le e^{-(1+\sigma^2)Tx/32}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-5(1+\sigma^2)^2 t^2/16}\, dt \le e^{-(1+\sigma^2)Tx/32}.$$

Finally, to prove (8.4), we apply the formula (8.11). Using the explicit form of $y(x)$, write

$$\begin{aligned} e^{y(x)x} f_\sigma(iy(x)) &= \exp\Big\{y(x)x + \frac{1+\sigma^2}{2}\,y^2(x) + \frac{y(x)^3}{T}\Big\} \\ &= \exp\Big\{\frac{y(x)}{3}\Big(2x + \frac{1+\sigma^2}{2}\,y(x)\Big)\Big\} \end{aligned} \tag{8.12}$$

for $x \le \frac{(1+\sigma^2)^2}{16}\,T$. Note that the function $y(x)/x$ is monotonically decreasing from zero to $-\frac{4}{3}\,(1+\sigma^2)^{-1}$ and is equal to $= -\frac{8}{3}\,\big(-1+\sqrt{\frac{7}{4}}\big)\,(1+\sigma^2)^{-1}$ at the point $x = -\frac{(1+\sigma^2)^2}{16}\,T$. Using these properties in (8.12), we conclude that in the interval $|x| \le \frac{(1+\sigma^2)^2}{16}\,T$,

$$e^{-2(5-\sqrt{7})|\,y(x)x|/9} \le e^{y(x)x} f_\sigma(iy(x)) \le e^{-4|\,y(x)x|/9}, \tag{8.13}$$

where the right-hand side continues to hold for all $x \le \frac{(1+\sigma^2)^2}{16}\,T$. The inequalities in (8.4) follow immediately from (8.11) and (8.13). □

Now, introduce independent identically distributed r.v.'s U and V with density

$$p(x) = d_0 v_0(x)\, I_{(-\infty,T/16]}(x), \quad \frac{1}{d_0} = \int_{-\infty}^{T/16} v_0(u)\,du, \tag{8.14}$$

where I_A denotes the indicator function of a set A. The density p depends on T, but for simplicity we omit this parameter. Note that, by Lemma 8.1, $|1-d_0| \le e^{-cT^2}$.

Consider the regularized r.v. U_σ with density $p_\sigma = p * \varphi_\sigma$, which we represent in the form

$$p_\sigma(x) = d_0 v_\sigma(x) - w_\sigma(x), \quad \text{where} \quad w_\sigma(x) = d_0\,((v_0 I_{(T/16,\infty)}) * \varphi_\sigma)(x).$$

The next lemma is elementary, and we omit its proof.

Lemma 8.2 *We have*

$$\begin{aligned} &|w_\sigma(x)| \le \varphi_\sigma(|x| + T/16)\, e^{-cT^2}, && x \le 0,\\ &|w_\sigma(x)| \le e^{-cT^2}, && 0 < x \le T/16,\\ &|w_\sigma(x)| \le e^{-cTx}, && x > T/16. \end{aligned}$$

Lemma 8.3 *For all sufficiently large $T > 1$ and $0 < \sigma \le 2$,*

$$D(U_\sigma) = \frac{3}{(1+\sigma^2)^3\,T^2} + \frac{c\theta}{T^3}.$$

Proof Put $\mathbf{E}\,U_\sigma = a_\sigma$ and $\mathrm{Var}(U_\sigma) = b_\sigma^2$. By Lemma 8.2, $|a_\sigma| + |b_\sigma^2 - 1 - \sigma^2| \le e^{-cT^2}$. Write

$$\begin{aligned} D(U_\sigma) = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 &= d_0 \int_{|x|\le c'T} v_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\,dx \\ &- \int_{|x|\le c'T} w_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\,dx + \int_{|x|>c'T} p_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\,dx, \end{aligned} \tag{8.15}$$

where $c' > 0$ is a sufficiently small absolute constant. First we find two-sided bounds on $\tilde{J}_1$, which are based on some additional information about v_σ.

Using a Taylor expansion for the function $\sqrt{1-u}$ about zero in the interval $-\frac{3}{4} \le u \le \frac{3}{4}$, we easily obtain that, for $|x| \le (1+\sigma^2)^2 T/16$,

$$\begin{aligned}\frac{6y(x)}{(1+\sigma^2)T} &= -1 + \sqrt{1 - \frac{12x}{(1+\sigma^2)^2}\,T} \\ &= -\frac{6x}{(1+\sigma^2)^2\,T} - \frac{18x^2}{(1+\sigma^2)^4\,T^2} - \frac{108x^3}{(1+\sigma^2)^6\,T^3} + \frac{c\theta x^4}{T^4},\end{aligned}$$

which leads to the relation

$$\begin{aligned}y(x)x + \frac{1+\sigma^2}{2}\,y(x)^2 + \frac{y(x)^3}{T} &= -\frac{x^2}{2(1+\sigma^2)} - \frac{x^3}{(1+\sigma^2)^3\,T} \\ &\quad - \frac{9x^4}{2(1+\sigma^2)^5\,T^2} + \frac{c\theta x^5}{T^3}.\end{aligned} \tag{8.16}$$

In addition, it is easy to verify that

$$\alpha(x) = (1+\sigma^2)\Big(1 - \frac{6x}{(1+\sigma^2)^2\,T} - \frac{18x^2}{(1+\sigma^2)^4\,T^2} + \frac{c\theta x^3}{T^3}\Big). \tag{8.17}$$

Finally, using (8.16) and (8.17), we conclude from (8.11) that v_σ is representable as

$$\begin{aligned}v_\sigma(x) &= g(x)\varphi_{\sqrt{1+\sigma^2}}(x)e^{h(x)} \\ &= \Big(1 + \frac{3x}{(1+\sigma^2)^2\,T} + \frac{15}{2}\,\frac{3x^2-(1+\sigma^2)}{(1+\sigma^2)^4\,T^2} + \frac{c\theta|x|(1+x^2)}{T^3}\Big) \\ &\quad \cdot \varphi_{\sqrt{1+\sigma^2}}(x)\exp\Big\{-\frac{x^3}{(1+\sigma^2)^3\,T} - \frac{9x^4}{2(1+\sigma^2)^5\,T^2} + \frac{c\theta x^5}{T^3}\Big\}\end{aligned} \tag{8.18}$$

for $|x| \le (1+\sigma^2)^2 T/16$.

Now, from (8.18) and Lemma 8.2, we obtain a simple bound

$$|w_\sigma(x)/v_\sigma(x)| \le 1/2 \quad \text{for} \quad |x| \le c'T. \tag{8.19}$$

Therefore we have the relation, using again Lemmas 8.1 and 8.2,

$$\begin{aligned}\tilde{J}_1 &= d_0 \int_{|x|\le c'T} v_\sigma(x)\log\frac{v_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\,dx + 2\theta\int_{|x|\le c'T}|w_\sigma(x)|\,dx + \theta e^{-cT^2} \\ &= \int_{|x|\le c'T} v_\sigma(x)\log\frac{v_\sigma(x)}{\varphi_{\sqrt{1+\sigma^2}}(x)}\,dx + \theta e^{-cT^2}.\end{aligned} \tag{8.20}$$

Let us denote the integral on the right-hand side of (8.20) by $\tilde{J}_{1,1}$. With the help of (8.18) it is not difficult to derive the representation

$$\tilde{J}_{1,1} = \int_{|x|\le c'T} \varphi_{\sqrt{1+\sigma^2}}(x) e^{h(x)} \Big(-\frac{x^3}{(1+\sigma^2)^3\,T} - \frac{15x^4}{2(1+\sigma^2)^5\,T^2} + \frac{3x}{(1+\sigma^2)^2\,T} + \frac{54x^2 - 15(1+\sigma^2)}{2(1+\sigma^2)^4\,T^2} + \frac{c\theta|x|(1+x^4)}{T^3} \Big)\, dx. \tag{8.21}$$

Since $|e^{h(x)} - 1 - h(x)| \le \frac{1}{2} h(x)^2 e^{|h(x)|}$, and $\varphi^2_{\sqrt{1+\sigma^2}}(x) e^{2h(x)} \le \varphi_{\sqrt{1+\sigma^2}}(x)$ for $|x| \le c'T$, we easily deduce from (8.21) that

$$\tilde{J}_{1,1} = \int_{|x|\le c'T} \varphi_{\sqrt{1+\sigma^2}}(x) \Big(\frac{3(1+\sigma^2)x - x^3}{(1+\sigma^2)^3\,T} + \frac{54x^2 - 15(1+\sigma^2)}{2(1+\sigma^2)^4\,T^2} - \frac{21(1+\sigma^2)x^4 - 2x^6}{2(1+\sigma^2)^6\,T^2} \Big)\, dx + \frac{c\theta}{T^3} = \frac{3}{(1+\sigma^2)^3\,T^2} + \frac{c\theta}{T^3}. \tag{8.22}$$

It remains to estimate the integrals $\tilde{J}_2$ and $\tilde{J}_3$. By (8.19) and Lemma 8.2,

$$|\tilde{J}_2| \le \int_{|x|\le c'T} |w_\sigma(x)| (-\log \varphi_{a_\sigma, b_\sigma}(x) + \log \frac{3}{2} + |\log v_\sigma(x)|)\, dx \le \tilde{c} T^3 e^{-cT^2} \le e^{-cT^2}, \tag{8.23}$$

while by Lemmas 8.1 and 8.2,

$$|\tilde{J}_3| \le \int_{|x|>c'T} (|v_\sigma(x)| + |w_\sigma(x)|)(\sqrt{2\pi} b_\sigma + \frac{x^2}{2b_\sigma^2} + |\log(|v_\sigma(x) + |w_\sigma(x)|)\, dx \le \tilde{c} \int_{|x|>c'T} (1+x^2) e^{-cT|x|} dx + \int_{|x|>c'T} (|v_\sigma(x)| + |w_\sigma(x)|)^{1/2}\, dx \le e^{-cT^2}. \tag{8.24}$$

The assertion of the lemma follows from (8.22)–(8.24). □

To complete the proof of Theorem 1.3, we need yet another lemma.

Lemma 8.4 *For all sufficiently large $T > 1$ and $0 < \sigma \le 2$, we have*

$$D(U_\sigma - V_\sigma) \le e^{-cT^2}.$$

Proof Putting $\bar{p}_\sigma(x) = p_\sigma(-x)$, we have

$$D(U_\sigma - V_\sigma) = \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}\, dx$$
$$+ \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}{\varphi_{\sqrt{\mathrm{Var}(X_\sigma - Y_\sigma)}}(x)}\, dx. \tag{8.25}$$

Note that $\bar{p}_\sigma(x) = d_0 \bar{v}_\sigma(x) - \bar{w}_\sigma(x)$ with $\bar{v}_\sigma(x) = v_\sigma(-x)$, $\bar{w}_\sigma(x) = w_\sigma(-x)$, and

$$p_\sigma * \bar{p}_\sigma = d_0^2 (v_\sigma * \bar{v}_\sigma)(x) - d_0 (v_\sigma * \bar{w}_\sigma)(x) - d_0 (\bar{v}_\sigma * w_\sigma)(x) + (w_\sigma * \bar{w}_\sigma)(x). \tag{8.26}$$

By the very definition of v_σ, $v_\sigma * \bar{v}_\sigma = \varphi_{\sqrt{2(1+\sigma^2)}}$. Since $|\mathrm{Var}(U_\sigma - V_\sigma) - 2(1 + \sigma^2)| \le e^{-cT^2}$, using Lemma 8.1, we note that the second integral on the right-hand side of (8.25) does not exceed e^{-cT^2}. Using Lemmas 8.1 and 8.2, we get

$$|(v_\sigma * \bar{w}_\sigma)(x)| + |(\bar{v}_\sigma * w_\sigma)(x)| + |w_\sigma * \bar{w}_\sigma(x)| \le e^{-cT^2}, \quad |x| \le \tilde{c}T, \tag{8.27}$$

$$|(v_\sigma * \bar{w}_\sigma(x)| + |(\bar{v}_\sigma * w_\sigma)(x)| + |(w_\sigma * \bar{w}_\sigma)(x)| \le e^{-cT|x|}, \quad |x| > \tilde{c}T. \tag{8.28}$$

It follows from these estimates that

$$\frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)} = 1 + c\theta e^{-cT^2} \tag{8.29}$$

for $|x| \le c'T$. Hence, with the help of Lemmas 8.1 and 8.2, we may conclude that

$$\left| \int_{|x| \le c'T} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}\, dx \right| \le e^{-cT^2}. \tag{8.30}$$

A similar integral over the set $|x| > c'T$ can be estimated with the help of (8.27) and (8.28), and here we arrive at the same bound as well. Therefore, the assertion of the lemma follows from (8.25). □

Introduce the r.v.'s $X = (U - a_0)/b_0$ and $Y = (V - a_0)/b_0$. Since $D(X_\sigma) = D(U_{b_0\sigma})$ and $D(X_\sigma - Y_\sigma) = D(U_{b_0\sigma} - V_{b_0\sigma})$, the statement of Theorem 1.3 for the entropic distance D immediately follows from Lemmas 8.3 and 8.4. As for the distance J_{st}, we need to prove corresponding analogs of Lemmas 8.3 and 8.4 for $J_{st}(U_\sigma)$ and $J_{st}(U_\sigma - V_\sigma)$, respectively. By the Stam inequality (1.4) and Lemma 8.3, we see that

$$J_{st}(U_\sigma) \ge c(\sigma)\, T^{-2} \quad \text{for sufficiently large } T > 1, \tag{8.31}$$

where $c(\sigma)$ denote positive constants depending on σ only. We estimate the quantity $J_{st}(U_\sigma - V_\sigma)$, by using the formula

$$\frac{J_{st}(U_\sigma - V_\sigma)}{2\mathrm{Var}(U_\sigma)} = -\int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)''(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{\mathrm{Var}(U_\sigma - V_\sigma)}}(x)}\, dx. \tag{8.32}$$

It is not difficult to conclude from (8.26), using our previous arguments, that

$$(p_\sigma * \bar{p}_\sigma)''(x) = d_0^2 \varphi''_{\sqrt{2(1+\sigma^2)}}(x) + R_\sigma(x), \tag{8.33}$$

where $|R_\sigma(x)| \le c(\sigma)e^{-cT^2}$ for $|x| \le \tilde{c}T$ and $|R_\sigma(x)| \le c(\sigma)e^{-cT|x|}$ for $|x| > \tilde{c}T$. Applying (8.33) in the formula (8.32) and repeating the argument that we used in the proof of Lemma 8.4, we obtain the desired result, namely

$$J_{st}(U_\sigma - V_\sigma) \le c(\sigma)\mathrm{Var}(X_\sigma)e^{-cT^2} \quad \text{for sufficiently large } T > 1. \tag{8.34}$$

By Theorem 1.2, $J_{st}(U_\sigma) \le -c(\sigma)/(\log J_{st}(U_\sigma - V_\sigma))$, so $J_{st}(U_\sigma) \to 0$ as $T \to \infty$. Since $J_{st}(X_\sigma) = J_{st}(U_{b_0\sigma})$ and $J_{st}(X_\sigma - Y_\sigma) = J_{st}(U_{b_0\sigma} - V_{b_0\sigma})$, the statement of Theorem 1.3 for J_{st} follows from (8.31) and (8.34).

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