On Isoperimetric Functions of Probability Measures Having Log-Concave Densities with Respect to the Standard Normal Law

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Abstract Isoperimetric inequalities are discussed for one-dimensional probability distributions having log-concave densities with respect to the standard Gaussian measure.

Suppose that a probability measure μ on $\mathbf{R}^n$ has a log-concave density f with respect to the standard n-dimensional Gaussian measure γ_n, that is,

$$f(x) = e^{-\frac{1}{2}\,|x|^2 - V(x)}, \qquad x \in \mathbf{R}^n,$$

for some convex function $V : \mathbf{R}^n \to (-\infty, \infty]$. One may also say that μ is log-concave with respect to γ_n. In this case, an important theorem due to D. Bakry and M. Ledoux [B-L] asserts that μ satisfies a Gaussian-type isoperimetric inequality

$$\mu^+(A) \geq \varphi\Big(\Phi^{-1}(\mu(A))\Big), \tag{1}$$

relating the "size" $\mu(A)$ of an arbitrary Borel subset $A \subset \mathbf{R}^n$ to its μ-perimeter

$$\mu^+(A) = \liminf_{\varepsilon \downarrow 0} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}$$

(where A_ε stands for the Euclidean ε-neighborhood of A). Here, Φ^{-1} denotes the inverse to the normal distribution function $\Phi(x) = \gamma_1((-\infty, x])$ with density $\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\,x^2}$ ($x \in \mathbf{R}$). In other words, the isoperimetric function of μ,

$$I_\mu(p) = \inf_{\mu(A)=p} \mu^+(A), \qquad 0 < p < 1$$

(called also an isoperimetric profile) dominates the isoperimetric function $I(p) = \varphi(\Phi^{-1}(p))$ of the measure γ_n, i.e., one has

$$I_\mu(p) \geq I(p) \tag{2}$$

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for all p. The original proof of (1)–(2) given in [B-L] is based on semi-group arguments and a functional form proposed in [B2]. As was shown by L. A. Caffarelli [C], all μ's under consideration represent contractions of γ_n, so the proof of (1)–(2) may be reduced to the purely Gaussian case. An alternative localization approach to the Bakry-Ledoux theorem was later proposed in [B3]; cf. also [B4] for an extension of (1) to a larger class of probability measures. Another approach unifying a number of analytic and isoperimetric inequalities of Gaussian type has been recently developed by P. Ivanisvili and A. Volberg [I-V].

Recently, Raphaël Bouyrie raised the question of whether or not the inequality (2) is strict, even in dimension one, assuming that μ is symmetric and non-Gaussian. Although we do not know the original motivation, this question seems to be rather interesting in itself and not so elementary. Here we give an affirmative answer, involving some arguments from [B3] which were used to prove (1)–(2) in dimension one. Thus, we have:

Theorem 3 *Let μ be a symmetric probability measure on $\mathbf{R}$ which is log-concave with respect to the standard Gaussian measure γ_1. If μ is not Gaussian, then its isoperimetric function satisfies*

$$I_\mu(p) > I(p) \quad \textit{for all} \ \ p \in (0,1).$$

Equivalently, the coincidence $I_\mu(p_0) = I(p_0)$ for some p_0 causes μ to be Gaussian. Of course, this is not true at all without the log-concavity hypothesis (with respect to γ_1). For example, consider the class of symmetric probability measures μ on $\mathbf{R}$ having log-concave densities f with respect to the linear Lebesgue measure (the class of log-concave measures). In this case, the isoperimetric functions have the form

$$J(p) = I_\mu(p) = f(F^{-1}(p)), \tag{3}$$

where F^{-1} is the inverse to the distribution function

$$F(x) = \mu((-\infty,x]) = \int_{-\infty}^{x} f(y)\,dy$$

restricted to the support interval (cf. [B1]). Here, J may be an arbitrary positive concave function on $(0,1)$, symmetric about the point $1/2$. Hence, in this class it may easily happen that $J(p) \geq I(p)$ on $(0,1)$ with equality only at two points p_0 and $1-p_0$ (or even for one point $p_0 = 1/2$, only). Let us also mention that the property $J \geq I$ is another way to say that μ represents a Lipschitz transform of γ_1.

Assuming that V is of class C^2 in the representation (1), we find from (3) that

$$\begin{aligned} V''(x) &= -1 - \left(\frac{f'(x)}{f(x)}\right)' \;=\; -1 - (J'(F(x))' \\ &= -1 - J''(F(x))f(x) \;=\; -1 - J''(p)J(p), \qquad p = F(x). \end{aligned}$$

Hence, in terms of the isoperimetric function, the log-concavity with respect to γ_1 is equivalent to the relation

$$J''(p)J(p) \le -1.$$

For such functions (that are also symmetric about $1/2$), Theorem 3 may be stated as follows: If $J''(p)J(p) = -1$ for some $p \in (0,1)$, then this equality holds true for all p (in which case, necessarily $J = I$). It might be natural to try to prove Theorem 3 using this formulation. However, we prefer to choose a different route, which allows one to avoid the C^2-assumption on the density f, and which also suggests a possible way to quantify the assertion of this theorem. To be more precise, we have:

Theorem 4 *Let μ be a probability measure supported on the interval $(-a,a) \subset \mathbf{R}$ with density $e^{-V(x)}\,\varphi(x)$, where V is an even, convex function, which is differentiable and increasing on $(0,a)$. Then the isoperimetric function of μ satisfies*

$$I_\mu(p) \ \ge\ \frac{1}{2\,\Phi(V'(x))}\, e^{-\frac{1}{2}V'(x)^2 - V'(x)\,y}\,\varphi(y), \tag{4}$$

where $p = \mu((-\infty,x])$, $x \in (-a,0)$, and

$$y = -V'(x) + \Phi^{-1}\left(2p\,\Phi(V'(x))\right).$$

A similar bound also holds for $p > 1/2$, by using $I_\mu(1-p) = I_\mu(p)$.

One can check that equality in (4) is attained for the family of probability measures $\mu = \mu_\lambda$ with densities

$$\varphi_\lambda(x) = \frac{1}{Z}\, e^{-\lambda|x|}\,\varphi(x), \qquad x \in \mathbf{R}, \tag{5}$$

where λ is an arbitrary positive parameter and $Z = Z(\lambda)$ is a normalizing constant.

We now turn to the proofs. As a first step, let us verify Theorem 3 in the particular case of measures μ_λ described in (5).

Lemma 3 *Given $\lambda > 0$, we have $I_{\mu_\lambda}(p) > I(p)$ for all $p \in (0,1)$.*

Proof According to (3), the isoperimetric function of μ_λ is given by

$$I_{\mu_\lambda}(p) = \varphi_\lambda\Big(\Phi_\lambda^{-1}(p)\Big),$$

where Φ_λ denotes the distribution function of μ_λ. Therefore, we need to show that $\Phi_\lambda(y) = \Phi(x) \Rightarrow \varphi_\lambda(y) > \varphi(x)$ for all $x, y \in \mathbf{R}$, where one may additionally assume that $x \le 0$ (using the symmetry).

The increasing map $T(x) = \Phi_\lambda^{-1}(\Phi(x))$ pushes forward γ_1 to μ_λ, so that $\Phi_\lambda(T(x)) = \Phi(x)$. After differentiation we have

$$\varphi_\lambda(T(x))T'(x) = \varphi(x).$$

Hence, it is sufficient to see that $T'(x) < 1$ for all $x < 0$. To this aim, first note that

$$\int_{-\infty}^{x} e^{\lambda y}\, \varphi(y)\, dy = e^{\lambda^2/2}\, \Phi(x-\lambda), \qquad Z = 2\int_{-\infty}^{0} e^{\lambda y}\, \varphi(y)\, dy = 2e^{\lambda^2/2}\, \Phi(-\lambda).$$

Hence, the distribution function of μ_λ is described as

$$\Phi_\lambda(x) = \mu_\lambda((-\infty,x]) = \frac{\Phi(x-\lambda)}{2\Phi(-\lambda)}, \qquad x \leq 0,$$

and, by the symmetry, $\Phi_\lambda(x) = 1 - \Phi_\lambda(-x)$ for $x \geq 0$. It follows that

$$T(x) = \Phi^{-1}(\alpha\Phi(x)) + \lambda \qquad (\alpha = 2\Phi(-\lambda),\ x \leq 0),$$

so, putting $x = \Phi^{-1}(p)$, we get

$$T'(x) = \frac{\alpha\varphi(x)}{I(\alpha\Phi(x))} = \frac{\alpha I(p)}{I(\alpha p)}.$$

But the last ratio is smaller than 1, since $\alpha < 1$ and since $I(p)/p$ is a decreasing function. The latter property is true for any positive, strictly concave function I on $(0, 1)$, which follows from the representation

$$\frac{I(p)}{p} = \frac{I(0+)}{p} + \int_0^1 I'(ps)\, ds. \tag{6}$$

This proves the lemma.

Lemma 4 *Let μ be a symmetric probability measure, which is log-concave with respect to γ_1 with density $f = e^{-V}\varphi$. Suppose that V is monotone in some neighborhood of a point $x \in \mathbf{R}$, and let $p = \mu((-\infty,x])$. Then*

$$I_\mu(p) \geq I_{\mu_\lambda}(p) \quad \textit{for some } \lambda > 0.$$

Proof We prove a stronger statement: Let a positive finite measure μ have density $f(y) = e^{-V(y)}\varphi(y)$ for some convex even function $V : \mathbf{R} \to (-\infty,\infty]$, finite on the interval $(-a,a)$. If a point $x \in (-a,0)$ is such that

$$\mu((-a,x]) \geq p, \qquad \mu((x,0]) \geq \frac{1}{2} - p \qquad \left(0 < p < \frac{1}{2}\right), \tag{7}$$

and if V is monotone in some neighborhood of x, then $f(x) \geq I_{\mu_\lambda}(p)$ for some $\lambda > 0$.

To simplify this assertion, let $l(y) = c - \lambda y$ be an affine function which is tangent to $V(y)$ at x, with necessarily $\lambda > 0$ in view of the monotonicity assumption on V. We extend l from the negative half-axis $(-\infty, 0)$ to $(0,\infty)$ to get an even function, and as a result we obtain a new positive measure μ_0 with density

$$f_0(y) = Ce^{-\lambda|y|}\varphi(y).$$

Since $l(x) = V(x)$ and $l \le V$ everywhere on $(-a, a)$, we have $f \le f_0$, so that $\mu_0((-a, x]) \ge p$ and $\mu_0((x, 0]) \ge \frac{1}{2} - p$. Therefore, in our stronger statement we are reduced to the class of densities of type $f = C\varphi_\lambda$, where C is an arbitrary positive parameter.

For such densities, we have

$$\mu((-\infty, x]) = C\Phi_\lambda(x), \quad \mu((x, 0]) = C\left(\frac{1}{2} - \Phi_\lambda(x)\right), \qquad f(x) = C\varphi_\lambda(x),$$

and involving the assumption (7), we get a constraint on C, namely,

$$C \ge C_0 = \max\left\{\frac{p}{\Phi_\lambda(x)}, \frac{\frac{1}{2} - p}{\frac{1}{2} - \Phi_\lambda(x)}\right\}. \tag{8}$$

Since $C = C_0$ is the worst situation in our conclusion, it remains to show that

$$C_0\, \varphi_\lambda(x) \ge \varphi_\lambda(\Phi_\lambda^{-1}(p)) \equiv I_{\mu_\lambda}(p)$$

with C_0 defined in (8). Putting $q = \Phi_\lambda(x)$, this is the same as

$$\max\left\{\frac{p}{q}, \frac{\frac{1}{2} - p}{\frac{1}{2} - q}\right\} I_{\mu_\lambda}(q) \ge I_{\mu_\lambda}(p).$$

If $p \ge q$, it holds true, since $\frac{p}{q} I_{\mu_\lambda}(q) \ge I_{\mu_\lambda}(p)$, which in turn follows from the fact that the function I_{μ_λ} is strictly concave (so that $I_{\mu_\lambda}(p)/p$ is strictly decreasing). In case $p \le q$, we use

$$\frac{\frac{1}{2} - p}{\frac{1}{2} - q} I_{\mu_\lambda}(q) \ge I_{\mu_\lambda}(p),$$

or equivalently, after the change $p' = \frac{1}{2} - p,\ q' = \frac{1}{2} - q$,

$$\frac{p'}{q'} I_{\mu_\lambda}\left(\frac{1}{2} - q'\right) \ge I_{\mu_\lambda}\left(\frac{1}{2} - p'\right).$$

Here again $p' \ge q'$ and we deal with the concave function $\tilde{I}(p') = I_{\mu_\lambda}(\frac{1}{2} - p')$ on the interval $(0, 1/2)$. Hence, $\tilde{I}(p')/p'$ is strictly decreasing, which is seen from the general identity (6).

Lemma 4 is proved.

Proof of Theorem 3 If a probability measure μ on the line is log-concave with respect to γ_1, it has a density

$$f(x) = e^{-V(x)}\varphi(x),$$

for some convex even function V on the interval $(-a, a)$, finite or not, and one may put $V = \infty$ outside that interval. Since V attains its minimum at zero, necessarily $V(0) < 0$, as long as μ is non-Gaussian. In particular, in this case

$$I_\mu(1/2) = f(0) > \varphi(0) = I(1/2).$$

Moreover, let $[-x_0, x_0]$ be the longest interval, where V is constant, so that $V(x_0-) = V(0)$. Then similarly

$$I_\mu(p) = I_\mu(1/2) > I(1/2)$$

for all $p \in [p_0, 1-p_0]$, where $p_0 = \mu((-a, x_0])$.

In case $0 < p < p_0$, $p = \mu((-a, x])$, the point x necessarily belongs to the interval $(-a, x_0)$, where V is strictly decreasing. Therefore, one may apply Lemma 4 and combine it with Lemma 3, which then leads to the required assertion $I_\mu(p) \geq I_{\mu_\lambda}(p) > I(p)$.

Theorem 3 is thus proved.

Proof of Theorem 4 If an even, convex function V in the representation $f = e^{-V}\varphi$ for the density of μ is differentiable and is increasing on $(0, a)$, the assumption of Lemma 4 is fulfilled for all points $x \neq 0$ from the supporting interval of the measure μ. In this case, since the tangent affine function in the proof of Lemma 4 is given by $l(y) = V(x) + V'(x)(y-x)$, necessarily $\lambda = \lambda(x) = -V'(x)$ $(-a < x < 0)$. Hence, we obtain that

$$I_\mu(p) \geq I_{\mu_{\lambda(x)}}(p), \qquad p = \mu((-a, x]). \tag{9}$$

The expression $I_{\mu_{\lambda(x)}}(p)$ may be written in a more explicit form. Recall that, for $0 < p < 1/2$,

$$y \equiv \Phi_\lambda^{-1}(p) = \Phi^{-1}(\alpha p) + \lambda, \qquad Z = 2e^{\lambda^2/2}\,\Phi(-\lambda),$$

where $\alpha = 2\Phi(-\lambda)$, so that

$$I_{\mu_\lambda}(p) = \frac{1}{Z}\,e^{-\lambda\,|y|}\,\varphi(y) = \frac{1}{2\Phi(-\lambda)}\,e^{-\lambda^2/2+\lambda y}\,\varphi(y).$$

Hence, (9) turns into (4), thus proving Theorem 4.

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