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ESTIMATES FOR MOMENTS OF GENERAL MEASURES ON CONVEX BODIES

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Abstract. For $p \geq 1$, $n \in \mathbb{N}$, and an origin-symmetric convex body K in $\mathbb{R}^n$, let

$$d_{\rm ovr}(K, L_p^n) = \inf \left\{ \Big(\frac{|D|}{|K|} \Big)^{1/n} : K \subseteq D,\ D \in L_p^n \right\}$$

be the outer volume ratio distance from K to the class L_p^n of the unit balls of n-dimensional subspaces of L_p. We prove that there exists an absolute constant $c > 0$ such that

$$\frac{c\sqrt{n}}{\sqrt{p \log\log n}} \leq \sup_K d_{\rm ovr}(K, L_p^n) \leq \sqrt{n}.$$

This result follows from a new slicing inequality for arbitrary measures, in the spirit of the slicing problem of Bourgain. Namely, there exists an absolute constant $C > 0$ so that for any $p \geq 1$, any $n \in \mathbb{N}$, any compact set $K \subseteq \mathbb{R}^n$ of positive volume, and any Borel measurable function $f \geq 0$ on K,

$$\int_K f(x)\, dx \ \leq\ C\sqrt{p}\ d_{\rm ovr}(K, L_p^n)\ |K|^{1/n} \sup_H \int_{K\cap H} f(x)\, dx,$$

where the supremum is taken over all affine hyperplanes H in $\mathbb{R}^n$. Combining the above display with a recent counterexample for the slicing problem with arbitrary measures from the work of the second and third authors [J. Funct. Anal. 274 (2018), pp. 2089–2112], we get the lower estimate from the first display.

In turn, the second inequality follows from an estimate for the p-th absolute moments of the function f

$$\min_{\xi \in S^{n-1}} \int_K |(x,\xi)|^p f(x)\ dx \ \leq\ (Cp)^{p/2}\, d_{\rm ovr}^p(K, L_p^n)\ |K|^{p/n} \int_K f(x)\, dx.$$

Finally, we prove a result of the Busemann-Petty type for these moments.

1. Introduction

Suppose that $K \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a centrally-symmetric convex set of volume one (i.e., $K = -K$). Given an even continuous probability density $f : K \to [0, \infty)$, and $p \geq 1$, can we find a direction ξ such that the p-th absolute moment

$$M_{K,f,p}(\xi) = \int_K |(x,\xi)|^p\, f(x)\, dx \tag{1.1}$$

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is smaller than a constant which does not depend on K and f? More precisely and in a more relaxed form, let $\gamma(p,n)$ be the smallest number $\gamma > 0$ satisfying

$$\min_{\xi\in S^{n-1}} M_{K,f,p}(\xi) \le \gamma^p \ |K|^{p/n} \int_K f(x)\, dx \tag{1.2}$$

for all centrally-symmetric convex bodies $K \subseteq \mathbb{R}^n$ and all even continuous functions $f \ge 0$ on K. Here and below, we denote by $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ the Euclidean unit sphere centered at the origin, and $|K|$ stands for volume of appropriate dimension. (Note that the continuity property of f in the definition (1.1) is irrelevant and may easily be replaced by measurability.) As we will see, there is a two-sided bound on $\gamma(p,n)$.

Theorem 1.1. *With some positive absolute constants c and C, for any $p \ge 1$,*

$$\frac{c\sqrt{n}}{\sqrt{\log\log n}} \le \gamma(p,n) \le C\sqrt{pn}.$$

To describe the way the upper bound is obtained, denote by L_p^n the class of the unit balls of n-dimensional subspaces of L_p. Equivalently (see [11, p. 117]), L_p^n is the class of all centrally-symmetric convex bodies D in $\mathbb{R}^n$ such that there exists a finite Borel measure ν_D on S^{n-1} satisfying

$$\|x\|_D^p = \int_{S^{n-1}} |(x,\theta)|^p \ d\nu_D(\theta) \qquad \forall x \in \mathbb{R}^n. \tag{1.3}$$

Here $\|x\|_D = \inf\{a \ge 0 : x \in aD\}$ is the norm generated by D. Note that $L_1^n = \Pi_n^*$ is the class of polar projection bodies which, in particular, contains the cross-polytopes; see [11, Ch. 8] for details.

For a (bounded) set K in $\mathbb{R}^n$, define the quantity

$$V(K, L_p^n) = \inf\left\{|D|^{1/n} : K \subseteq D,\ D \in L_p^n\right\}.$$

If K is measurable and has positive volume, we have the relation

$$V(K, L_p^n) = d_{\rm ovr}(K, L_p^n)\, |K|^{1/n},$$

with

$$d_{\rm ovr}(K, L_p^n) = \inf\left\{\Big(\frac{|D|}{|K|}\Big)^{1/n} : K \subseteq D,\ D \in L_p^n\right\}. \tag{1.4}$$

For convex K, the latter may be interpreted as the outer volume ratio distance from K to the class of unit balls of n-dimensional subspaces of L_p. The next body-wise estimates refine the upper bound in Theorem 1.1 in terms of the $d_{\rm ovr}$-distance.

Theorem 1.2. *Given a probability measure μ on $\mathbb{R}^n$ with a compact support K, for every $p \ge 1$,*

$$\min_{\xi\in S^{n-1}} \Big(\int |(x,\xi)|^p \, d\mu(x)\Big)^{1/p} \ \le\ C\sqrt{p}\ V(K, L_p^n),$$

where C is an absolute constant. In particular, if f is a non-negative continuous function on a compact set $K \subseteq \mathbb{R}^n$ of positive volume, then

$$\min_{\xi\in S^{n-1}} M_{K,f,p}(\xi) \ \le\ (Cp)^{p/2}\, d_{\rm ovr}^p(K, L_p^n)\ |K|^{p/n} \int_K f(x)\, dx.$$

In the class of centrally-symmetric convex bodies K in $\mathbb{R}^n$, there is a dimensional bound $d_{\rm ovr}(K, L_p^n) \le \sqrt{n}$, which follows from John's theorem and the fact that ellipsoids belong to L_p^n for all $p \ge 1$ (see [6] and [11, Lemma 3.12]). Hence, the second upper bound of Theorem 1.2 is more accurate in comparison with the universal bound of Theorem 1.1.

Moreover, for several classes of centrally-symmetric convex bodies, it is known that the distance $d_{\rm ovr}(K, L_p^n)$ is bounded by absolute constants. These classes include duals of bodies with bounded volume ratio (see [14]) and the unit balls of normed spaces that embed in L_q, $1 \le q < \infty$ (see [15, 18]). In the case $p = 1$, they also include all unconditional convex bodies [14]. The proofs in these papers estimate the distance from the class of intersection bodies, but the actual bodies used there (the Euclidean ball for $p > 1$ and the cross-polytope for $p = 1$) also belong to the classes L_p^n, so the same arguments work for L_p^n.

In order to prove the lower estimate of Theorem 1.1, we first establish the connection between question (1.1) and the slicing problem for arbitrary measures. The slicing problem of Bourgain [2, 3] asks whether $\sup_n L_n < \infty$, where L_n is the minimal positive number L such that, for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,

$$|K| \,\le\, L \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| \, |K|^{1/n}.$$

Here, $\xi^\perp$ is the hyperplane in $\mathbb{R}^n$ passing through the origin and perpendicular to the vector ξ. Bourgain's slicing problem is still unsolved. The best-to-date estimate $L_n \le Cn^{1/4}$ was established by the second-named author [8], removing a logarithmic term from an earlier estimate by Bourgain [4].

The slicing problem for arbitrary measures was introduced in [12] and considered in [5, 9, 13–15]. In analogy with the original problem, for a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$, let $S_{n,K}$ be the smallest positive number S satisfying

$$\int_K f(x)\,dx \,\le\, S \max_{\xi \in S^{n-1}} \int_{K \cap \xi^\perp} f(x)\,dx \; |K|^{\frac{1}{n}} \tag{1.5}$$

for all even continuous functions $f \ge 0$ in $\mathbb{R}^n$ (where dx on the right-hand side refers to the Lebesgue measure on the corresponding affine subspace of $\mathbb{R}^n$). It was proved in [13] that

$$S_n \,=\, \sup_{K \subseteq \mathbb{R}^n} S_{n,K} \,\le\, 2\sqrt{n}.$$

However, for many classes of bodies, including intersection bodies [12] and unconditional convex bodies [14], the quantity $S_{n,K}$ turns out to be bounded by an absolute constant. In particular, if K is the unit ball of an n-dimensional subspace of L_p, $p > 2$, then $S_{n,K} \le C\sqrt{p}$ with some absolute constant C; see [15]. These results are implied by the following estimate proved in [14]:

Theorem 1.3 ([14]). *For any centrally-symmetric star body $K \subseteq \mathbb{R}^n$ and any even continuous non-negative function f on K,*

$$\int_K f(x)\,dx \,\le\, 2\; d_{\rm ovr}(K, \mathcal{I}_n) \max_{\xi \in S^{n-1}} \int_{K \cap \xi^\perp} f(x)\,dx \; |K|^{1/n},$$

where $d_{\rm ovr}(K, \mathcal{I}_n)$ is the outer volume ratio distance from K to the class $\mathcal{I}_n$ of intersection bodies in $\mathbb{R}^n$.

The class of intersection bodies $\mathcal{I}_n$ was introduced by Lutwak [17]; it can be defined as the closure in the radial metric of radial sums of ellipsoids centered at the origin in $\mathbb{R}^n$.

The slicing problem for arbitrary measures can be modified to include non-central sections. For a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$, let $T_{n,K}$ be the smallest positive number T satisfying

$$\int_K f(x)\,dx \;\le\; T \sup_H \int_{K\cap H} f(x)\,dx\;|K|^{\frac{1}{n}}, \tag{1.6}$$

where the supremum is taken over all affine hyperplanes H in $\mathbb{R}^n$. Let

$$T_n \;=\; \sup_{K\subseteq\mathbb{R}^n} T_{n,K}.$$

We have $T_n \le S_n \le 2\sqrt{n}$. On the other hand, it was shown in [9] that in general the constants T_n and S_n are of the order $\sqrt{n}$, up to a doubly-logarithmic term.

Theorem 1.4 ([9]). *For any $n \ge 3$, there exists a centrally-symmetric convex body $M \subseteq \mathbb{R}^n$ and an even, continuous probability density $f : M \to [0,\infty)$ such that, for any affine hyperplane $H \subseteq \mathbb{R}^n$,*

$$\int_{M\cap H} f(x)\,dx \;\le\; C\frac{\sqrt{\log\log n}}{\sqrt{n}}\,|M|^{-1/n}. \tag{1.7}$$

This implies

$$T_n \ge \frac{c\sqrt{n}}{\sqrt{\log\log n}}.$$

Here C and c are universal constants.

The connection between (1.2) and the slicing inequality for arbitrary measures (1.6) is as follows.

Lemma 1.5. *Given a Borel measurable function $f \ge 0$ on $\mathbb{R}^n$, for any $\xi \in S^{n-1}$ and $p > 0$,*

$$2^p\,(p+1)\left(\sup_{s\in\mathbb{R}}\int_{(x,\xi)=s} f(x)\,dx\right)^p \int |(x,\xi)|^p\, f(x)\,dx \;\ge\; \left(\int f(x)\,dx\right)^{p+1}.$$

If f is defined on a set K in $\mathbb{R}^n$, we then have

$$2^p\,(p+1)\left(\sup_{s\in\mathbb{R}}\int_{K\cap\{(x,\xi)=s\}} f(x)\,dx\right)^p M_{K,f,p}(\xi) \;\ge\; \left(\int_K f(x)\,dx\right)^{p+1}.$$

The lower bound in Theorem 1.1 thus follows, by combining the above inequality with (1.2) and Theorem 1.4.

Corollary 1.6. *With some positive absolute constants c and C, for every $p \ge 1$,*

$$\frac{c\sqrt{n}}{\sqrt{\log\log n}} \le T_n \le C\gamma(p,n).$$

Lemma 1.5, in conjunction with Theorem 1.2, leads to a new slicing inequality. In the case of volume, where $f \equiv 1$, this inequality was established earlier by Ball [1] for $p = 1$ and by Milman [18] for arbitrary p.

Theorem 1.7. *Let $f \geq 0$ be a Borel measurable function on a compact set $K \subseteq \mathbb{R}^n$ of positive volume. Then, for any $p > 2$,*

$$\int_K f(x)\,dx \;\leq\; C\sqrt{p}\; d_{\rm ovr}(K, L_p^n)\; |K|^{1/n} \sup_H \int_{K\cap H} f(x)\,dx,$$

where the supremum is taken over all affine hyperplanes H in $\mathbb{R}^n$, and C is an absolute constant.

Theorem 1.7 also holds for $1 \leq p \leq 2$, but in this case it is weaker than Theorem 1.3, because the unit ball of every finite dimensional subspace of L_p, $0 < p \leq 2$, is an intersection body; see [10]. However, for $p > 2$ the unit balls of subspaces of L_p are not necessarily intersection bodies. For example, the unit balls of ℓ_p^n are not intersection bodies if $p > 2$, $n \geq 5$; see [11, Th. 4.13]. So the result of Theorem 1.7 is new for $p > 2$, and generalizes the estimate from [15] in the case where K itself belongs to the class L_p^n.

Theorem 1.7 gives another reason to estimate the outer volume ratio distance $d_{\rm ovr}(K, L_p^n)$ from an arbitrary symmetric convex body to the class of unit balls of subspaces of L_p. As mentioned before,

$$d_{\rm ovr}(K, L_p^n) \leq \sqrt{n},$$

uniformly over all centrally-symmetric convex bodies K in $\mathbb{R}^n$. Surprisingly, the corresponding lower estimates seem to be missing in the literature. Combining Theorems 1.7 and 1.4, we get a lower estimate which shows that $\sqrt{n}$ is optimal up to a doubly-logarithmic term with respect to the dimension n and a term depending on the power p only.

Corollary 1.8. *There exists a centrally-symmetric convex body $M \subseteq \mathbb{R}^n$ such that*

$$d_{\rm ovr}(M, L_p^n) \;\geq\; c\,\frac{\sqrt{n}}{\sqrt{p\log\log n}}$$

for every $p \geq 1$, where $c > 0$ is a universal constant.

We end the Introduction with a comparison result for the quantities $M_{K,f,p}(\xi)$. For $p \geq 1$, introduce the Banach-Mazur distance

$$d_{BM}(M, L_p^n) = \inf\left\{a \geq 1 : \exists\, D \in L_p^n \text{ such that } D \subset M \subset aD\right\}$$

from a star body M in $\mathbb{R}^n$ to the class L_p^n. Recall that L_p^n is invariant with respect to linear transformations. By John's theorem, if M is origin-symmetric and convex, then $d_{BM}(M, L_p^n) \leq \sqrt{n}$. We prove the following:

Theorem 1.9. *Let K and M be origin-symmetric star bodies in $\mathbb{R}^n$, and let $f \geq 0$ be an even continuous function on $\mathbb{R}^n$. Given $p \geq 1$, suppose that for every $\xi \in S^{n-1}$*

$$\int_K |(x,\xi)|^p f(x)\,dx \;\leq\; \int_M |(x,\xi)|^p f(x)\,dx. \tag{1.8}$$

Then

$$\int_K f(x)\,dx \;\leq\; d_{BM}^p(M, L_p^n) \int_M f(x)\,dx.$$

This result is in the spirit of the Busemann-Petty problem for arbitrary measures; see [16, 19]. For example, it was proved in [16] that, with the same notation, if

$$\int_{K\cap\xi^\perp} f(x)\,dx \;\leq\; \int_{M\cap\xi^\perp} f(x)\,dx \qquad \forall \xi \in S^{n-1},$$

then

$$\int_K f(x)\,dx \le d_{BM}(K,\mathcal{I}_n)\int_M f(x)\,dx.$$

We refer the reader to [11, Ch. 5] for more about the Busemann-Petty problem.

Throughout this paper, a convex body K in $\mathbb{R}^n$ is a compact, convex set with a non-empty interior. The standard scalar product between $x, y \in \mathbb{R}^n$ is denoted by (x, y) and the Euclidean norm of $x \in \mathbb{R}^n$ by $|x|$. We write log for the natural logarithm.

2. Proofs

In this section we prove Theorem 1.2, Lemma 1.5, and Theorem 1.9. The other results of this paper will follow as explained in the Introduction.

Given a compact set $K \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, put

$$\|x\|_K = \min\{a \ge 0 : \ x \in aK\},$$

if $x \in aK$ for some $a \ge 0$, and $\|x\|_K = \infty$ in the other case. For star bodies, it represents the usual *Minkowski functional* associated with K.

Proof of Theorem 1.2. Let $D \subseteq \mathbb{R}^n$ be the unit ball of an n-dimensional subspace of L_p, so that the relation (1.3) holds for some measure ν_D on the unit sphere S^{n-1}. Then, integrating the inequality

$$\min_{\theta\in S^{n-1}} \int_K |(x,\theta)|^p \, d\mu(x) \le \int_K |(x,\xi)|^p \, d\mu(x) \qquad (\xi \in S^{n-1})$$

over the variable ξ with respect to ν_D, we get the relation

$$\nu_D(S^{n-1}) \min_{\theta\in S^{n-1}} \int_K |(x,\theta)|^p \, d\mu(x) \le \int_K \|x\|_D^p \, d\mu(x).$$

In the case $K \subseteq D$, we have $\|x\|_D \le \|x\|_K \le 1$ on K, so that the last integral does not exceed $\mu(K) = 1$, and thus

$$\nu_D(S^{n-1}) \min_{\theta\in S^{n-1}} \int_K |(x,\theta)|^p \, d\mu(x) \le 1. \tag{2.1}$$

In order to estimate the left-hand side of (2.1) from below, we represent the value $\nu_D(S^{n-1})$ as the integral $\int_{S^{n-1}} |x|^p \, d\nu_D(x)$ and apply the well-known formula

$$|x|^p = \frac{\Gamma(\frac{p+n}{2})}{2\pi^{\frac{n-1}{2}}\,\Gamma(\frac{p+1}{2})} \int_{S^{n-1}} |(x,\theta)|^p \, d\theta, \qquad x \in \mathbb{R}^n$$

(see for example [11, Lemma 3.12]). Using (1.3), this yields the representation

$$\begin{aligned}\nu_D(S^{n-1}) &= \frac{\Gamma(\frac{p+n}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(\frac{p+1}{2})} \int_{S^{n-1}} \int_{S^{n-1}} |(x,\theta)|^p \, d\theta \, d\nu_D(x) \\ &= \frac{\Gamma(\frac{p+n}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(\frac{p+1}{2})} \int_{S^{n-1}} \|\theta\|_D^p \, d\theta.\end{aligned}$$

The last integral may be related to the volume of D, by using the polar formula for the volume of D,

$$n\,|D| = \int_{S^{n-1}} \|\theta\|_D^{-n} \, d\theta = s_{n-1} \int_{S^{n-1}} \|\theta\|_D^{-n} \, d\sigma_{n-1}(\theta),$$

where σ_{n-1} denotes the normalized Lebesgue measure on S^{n-1} and $s_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is its $(n-1)$-dimensional volume. Namely, by Jensen's inequality, we have

$$\int \|\theta\|_D^{-n}\, d\sigma_{n-1}(\theta) \,\geq\, \Big(\int \|\theta\|_D^p\, d\sigma_{n-1}(\theta)\Big)^{-\frac{n}{p}},$$

or equivalently,

$$\int \|\theta\|_D^p\, d\theta \,\geq\, s_{n-1}^{\frac{p+n}{n}}\, (n\,|D|)^{-\frac{p}{n}}.$$

Thus,

$$\begin{aligned} \nu_D(S^{n-1}) \quad &\geq \quad \frac{\Gamma(\frac{p+n}{2})\, s_{n-1}^{\frac{p+n}{n}}}{2\pi^{\frac{n-1}{2}}\,\Gamma(\frac{p+1}{2})\, n^{\frac{p}{n}}\, |D|^{\frac{p}{n}}} \\ &= \quad \sqrt{\pi}\, \frac{\Gamma(\frac{p+n}{2})}{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{n}{2})} \Big(\frac{s_{n-1}}{n\,|D|}\Big)^{\frac{p}{n}} \,\geq\, \frac{c^p}{\Gamma(\frac{p+1}{2})\, |D|^{\frac{p}{n}}}, \end{aligned}$$

where $c > 0$ is an absolute constant. Here we used the well-known asymptotic relation $\sqrt{n}\, s_{n-1}^{\frac{1}{n}} \to c_0$ as $n \to \infty$, for some absolute $c_0 > 0$, as well as the estimate $\Gamma(\frac{p+n}{2})/\Gamma(\frac{n}{2}) \geq (cn)^{p/2}$.

Applying this lower estimate on the left-hand side of (2.1), we get

$$\min_{\theta \in S^{n-1}} \int_K |(x,\theta)|^p \, d\mu(x) \,\leq\, C^p\, \Gamma\Big(\frac{p+1}{2}\Big)\, |D|^{\frac{p}{n}}.$$

It remains to take the minimum over all admissible D and note that $\Gamma\left(\frac{p+1}{2}\right)^{1/p} \leq c\sqrt{p}$ for $p \geq 1$. $\square$

To prove Lemma 1.5, we need the following simple assertion.

Lemma 2.1. *Given a measurable function* $g : \mathbb{R} \to [0,1]$, *the function*

$$q \mapsto \Big(\frac{q+1}{2} \int_{-\infty}^{\infty} |t|^q\, g(t)\, dt\Big)^{\frac{1}{q+1}}$$

is non-decreasing on $(-1,\infty)$.

Proof. The standard argument is similar to the one used in the proof of Lemma 2.4 in [7]. Given $-1 < q < p$, let $A > 0$ be defined by

$$\int_{-\infty}^{\infty} |t|^q\, g(t)\, dt = \int_{-A}^{A} |t|^q\, dt = \frac{2}{q+1}\, A^{q+1}.$$

Using

$$|t|^p \leq A^{p-q}\, |t|^q \quad (|t| \leq A) \quad \text{and} \quad |t|^p \geq A^{p-q}\, |t|^q \quad (|t| \geq A),$$

together with the assumption $0 \leq g \leq 1$, we then have

$$\begin{aligned} &\int_{|t|\leq A} (1-g(t))\, |t|^p\, dt - \int_{|t|>A} g(t)\, |t|^p\, dt \\ &\qquad \leq\, A^{p-q} \Big(\textstyle\int_{|t|\leq A} (1-g(t))\, |t|^q\, dt - \int_{|t|>A} g(t)\, |t|^q\, dt \Big) \,=\, 0. \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} g(t)\, |t|^p\, dt \,\geq\, \int_{-A}^{A} |t|^p\, dt \,=\, \frac{2}{p+1}\, A^{p+1},$$

that is,

$$\left(\frac{p+1}{2}\int_{-\infty}^{\infty} g(t)\,|t|^p\,dt\right)^{\frac{1}{p+1}} \;\geq\; A \;=\; \left(\frac{q+1}{2}\int_{-\infty}^{\infty} g(t)\,|t|^q\,dt\right)^{\frac{1}{q+1}}.$$

$\square$

Proof of Lemma 1.5. One may assume that f is integrable. For $t \in \mathbb{R}$, introduce the hyperplanes $H_t = \{(x,\xi) = t\}$. Since f is Borel measurable on $\mathbb{R}^n$, the function

$$g(t) = \frac{\int_{H_t} f(x)\,dx}{\sup_s \int_{H_s} f(x)\,dx}$$

is Borel measurable on the line and satisfies $\|g\|_\infty = 1$. By Fubini's theorem,

$$\int_{-\infty}^{\infty} |t|^p\, g(t)\,dt \;=\; \frac{\int |(x,\xi)|^p\, f(x)\,dx}{\sup_s \int_{H_s} f(x)\,dx},$$

$$\int_{-\infty}^{\infty} g(t)\,dt \;=\; \frac{\int f(x)\,dx}{\sup_s \int_{H_s} f(x)\,dx}.$$

Applying Lemma 2.1 to the function g with $q = 0$ and p, we get

$$\frac{1}{2}\int_{-\infty}^{\infty} g(t)\,dt \;\leq\; \left(\frac{p+1}{2}\int_{-\infty}^{\infty} |t|^p\, g(t)\,dt\right)^{\frac{1}{p+1}},$$

which in our case becomes

$$\left(\int f(x)\,dx\right)^{p+1} \;\leq\; (p+1)\left(2\sup_s \int_{H_s} f(x)\,dx\right)^p \int |(x,\xi)|^p\, f(x)\,dx.$$

$\square$

Proof of Theorem 1.9. Let $D \in L_p^n$ be such that the distance $d_{BM}(M, L_p^n)$ is almost realized, i.e., for small $\delta > 0$, suppose that $D \subseteq M \subseteq (1+\delta)\, d_{BM}(M, L_p^n)\, D$.

Integrating both sides of (1.8) over $\xi \in S^{n-1}$ with respect to the measure ν_D from (1.3), we get

$$\int_K \|x\|_D^p\, f(x)\,dx \;\leq\; \int_M \|x\|_D^p\, f(x)\,dx.$$

Equivalently, using the integrals in spherical coordinates, we have

$$0 \;\leq\; \int_{S^{n-1}} \|\theta\|_D^p \left(\int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} r^{n+p-1} f(r\theta)\,dr\right) d\theta \;=\; \int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p}\, I(\theta)\,d\theta,$$

where

$$I(\theta) \;=\; \|\theta\|_M^p \int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} r^{n+p-1} f(r\theta)\,dr.$$

For $\theta \in S^{n-1}$ such that $\|\theta\|_K \geq \|\theta\|_M$, the latter quantity is non-negative, and one may proceed by writing

$$\begin{aligned} I(\theta) \;&=\; \int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} \left(\|\theta\|_M^p - r^{-p}\right) r^{n+p-1} f(r\theta)\,dr + \int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\,dr \\ &\leq\; \int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\,dr. \end{aligned}$$

But, in the case $\|\theta\|_K \le \|\theta\|_M$, we have

$$-I(\theta) = \|\theta\|_M^p \int_{\|\theta\|_M^{-1}}^{\|\theta\|_K^{-1}} r^p\, r^{n-1} f(r\theta)\, dr \ge \int_{\|\theta\|_M^{-1}}^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr,$$

which is the same upper bound on $I(\theta)$ as before. Thus,

$$0 \le \int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p} \left(\int_{\|\theta\|_K^{-1}}^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta,$$

that is,

$$\int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \le \int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p} \left(\int_0^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta.$$

Now, by the choice of D,

$$\|\theta\|_M \le \|\theta\|_D \le (1+\delta)\, d_{BM}(M, L_p^n) \|\theta\|_M$$

for every $\theta \in S^{n-1}$. Hence

$$\begin{aligned}
\int_K f(x)\, dx &= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \\
&\le \int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \\
&\le \int_{S^{n-1}} \frac{\|\theta\|_D^p}{\|\theta\|_M^p} \left(\int_0^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \\
&\le (1+\delta)\, d_{BM}^p(M, L_p^n) \int_{S^{n-1}} \left(\int_0^{\|\theta\|_M^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta \\
&= (1+\delta)\, d_{BM}^p(M, L_p^n) \int_M f(x)\, dx.
\end{aligned}$$

Sending δ to zero, we get the result. □

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