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Poincaré inequalities and normal approximation for weighted sums*

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Abstract

Under Poincaré-type conditions, upper bounds are explored for the Kolmogorov distance between the distributions of weighted sums of dependent summands and the normal law. Based on improved concentration inequalities on high-dimensional Euclidean spheres, the results extend and refine previous results to non-symmetric models.

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1 Introduction

Let $X = (X_1, \dots, X_n)$ be an isotropic random vector in $\mathbb{R}^n$ ($n \geq 2$), meaning that $\mathbb{E} X_i X_j = \delta_{ij}$ for all $i, j \leq n$, where δ_{ij} is the Kronecker symbol. Define the weighted sums

$$S_\theta = \theta_1 X_1 + \dots + \theta_n X_n, \qquad \theta = (\theta_1, \dots, \theta_n), \;\; \theta_1^2 + \dots + \theta_n^2 = 1,$$

with coefficients from the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We are looking for natural general conditions on X_k which guarantee that the distribution functions $F_\theta(x) = \mathbb{P}\{S_\theta \leq x\}$ are well approximated for most of $\theta \in \mathbb{S}^{n-1}$ by the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2}\, dy, \quad x \in \mathbb{R}.$$

Of special interest is the question of possible rates in the Kolmogorov distance

$$\rho(F_\theta, \Phi) = \sup_x |F_\theta(x) - \Phi(x)|.$$

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In this problem, going back to the seminal work of Sudakov [35], the well studied classical case of independent components may serve as a basic example for comparison with various models or dependencies. Let us recall that, if X_k are independent and have finite 4-th moments (with mean zero and variance one), there is an upper bound on average

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi) \le \frac{1}{n}\,\bar\beta_4, \quad \bar\beta_4 = \frac{1}{n}\sum_{k=1}^n \mathbb{E}X_k^4, \tag{1.1}$$

where $c > 0$ is an absolute constant, and where we use $\mathbb{E}_\theta$ to denote the integral over the uniform probability measure $\mathfrak{s}_{n-1}$ on the unit sphere. Moreover, for any $r > 0$,

$$\mathfrak{s}_{n-1}\Big\{c\,\rho(F_\theta,\Phi) \ge \frac{1}{n}\,\bar\beta_4\,r\Big\} \ \le\ 2\,e^{-\sqrt{r}}. \tag{1.2}$$

This non-trivial phenomenon was observed by Klartag and Sodin [26]. It shows that when $\bar\beta_4$ is bounded like in the i.i.d. situation, the distances $\rho(F_\theta,\Phi)$ turn out to be typically of order at most $1/n$. This is in contrast to the case of equal coefficients leading to the unimprovable standard $\frac{1}{\sqrt{n}}$-rate (in general, including independent Bernoulli summands X_k). Moreover, in the i.i.d. situation with finite moment $\beta_5 = \mathbb{E}\,|X_1|^5$ and symmetric underlying distributions, the typical rate of normal approximation for F_θ may further be improved to $\beta_5\,n^{-3/2}$ up to a constant (which is best possible as long as $\mathbb{E}X_1^4 \neq 3$, cf. [8]).

As for more general models with not necessarily independent components X_k, the study of this high-dimensional phenomenon has a long history, and we refer an interested reader to the book [15] and a recent paper [13] for an account of various results in this direction. Let us only mention [2], [5], [6], [34], [23], [24], [18], where one can find quantitative variants of Sudakov's theorem on the concentration of F_θ about the typical (average) distribution $F = \mathbb{E}_\theta F$ and/or about the normal law Φ for different metrics and under certain assumptions (of convexity-type, for example). Some papers provide Berry-Esseen-type estimates on the closeness of F_θ to Φ explicitly in terms of θ assuming that the distribution of the random vector X is "sufficiently" symmetric, cf. [29], [30], [19], [25], [21].

Whether or not F itself is close to the standard normal law represents a thin-shell problem on the concentration of the values of the square of the Euclidean norm $|X|$ about its mean $\mathbb{E}\,|X|^2 = n$ (or, in essence, on the concentration of $|X|$ about $\sqrt{n}$). The rate of concentration may be controlled in terms of the functional $\sigma_4^2 = \frac{1}{n}\,\mathrm{Var}(|X|^2)$ which is often of order 1 (including the i.i.d. situation). Once it is the case, one can obtain a standard rate of concentration of F_θ around Φ on average under mild moment assumptions. For example, it is known that, if $\mathbb{E}\,|X|^2 = n$ (without the isotropy hypothesis), then

$$\mathbb{E}_\theta\,\rho(F_\theta,\Phi) \ \le\ c\,\big(M_3^3 + \sigma_4^{3/2}\big)\,\frac{1}{\sqrt{n}},$$

up to an absolute constant $c > 0$, where $M_3^3 = \sup_\theta \mathbb{E}\,|S_\theta|^3$ (cf. [12]).

In order to reach better rates, one has to involve stronger assumptions or functionals such as $\Lambda = \Lambda(X)$ defined as an optimal constant in the inequality

$$\mathrm{Var}\bigg(\sum_{i,j=1}^n a_{ij}X_iX_j\bigg) \le \Lambda \sum_{i,j=1}^n a_{ij}^2 \qquad (a_{ij} \in \mathbb{R}), \tag{1.3}$$

which may be referred to as a second order correlation condition. In terms of Λ, the bound (1.1) has been extended in [13] modulo a logarithmic factor: If additionally to the isotropy assumption the distribution of X is symmetric around the origin, it was shown that

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi) \le \frac{\log n}{n}\,\Lambda. \tag{1.4}$$

The optimal value $\Lambda = \Lambda(X)$ in (1.3) is finite as long as $|X|$ has a finite 4-th moment. It represents the maximal eigenvalue of the covariance matrix associated to the n^2-dimensional random vector $(X_iX_j - \mathbb{E}X_iX_j)_{i,j=1}^n$. This parameter may be effectively estimated in many examples and is related to other standard characteristics. For example, $\Lambda(X) \leq 2\,\max_k\,\mathbb{E}X_k^4$, if X_k are independent. If X is isotropic, and its distribution admits a Poincaré-type inequality

$$\lambda_1\,\mathrm{Var}(u(X))\;\leq\;\mathbb{E}\,|\nabla u(X)|^2 \tag{1.5}$$

with a positive (optimal) constant $\lambda_1 = \lambda_1(X)$ for all smooth functions u on $\mathbb{R}^n$, then we have $\Lambda(X) \leq 4/\lambda_1(X)$.

The aim of these notes is to sharpen (1.4) via a large deviation bound in analogy with (1.2). This turns out to be possible as long as all linear forms S_θ have finite exponential moments. To avoid technical discussions, we restrict ourselves to the case where $\lambda_1 > 0$, which at the same time allows to drop the symmetry assumption.

Theorem 1.1. *Let X be an isotropic random vector in $\mathbb{R}^n$ with mean zero and a positive Poincaré constant λ_1. Then with some absolute constant $c > 0$*

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi)\;\leq\;\frac{\log n}{n}\,\lambda_1^{-1}. \tag{1.6}$$

Moreover, for all $r > 0$,

$$\mathfrak{s}_{n-1}\Big\{c\,\rho(F_\theta,\Phi) \geq \frac{\log n}{n}\,\lambda_1^{-1}\,r\Big\}\;\leq\;2\,e^{-\sqrt{r}}. \tag{1.7}$$

Being restricted to isotropic log-concave distributions, an interesting feature of the bound (1.4) is its connection with certain open problems in Asymptotic Convex Geometry such as the K-L-S and thin-shell conjectures. Namely, modulo n-dependent logarithmic factors, the following three assertions are equivalent up to positive constants c and β (perhaps different in different places) for the entire class of isotropic random vectors X in $\mathbb{R}^n$ having symmetric log-concave distributions (cf. [13]):

(i) $\sup_X \lambda_1^{-1}(X) \leq c\,(\log n)^\beta$;
(ii) $\sup_X \mathrm{Var}(|X|) \leq c\,(\log n)^\beta$;
(iii) $\sup_X \mathbb{E}_\theta\,\rho(F_\theta,\Phi) \leq \frac{c}{n}\,(\log n)^\beta$.

In this connection, let us also mention a recent paper by Jiang, Lee and Vempala [22], which provides a reformulation of (i)-(ii) as a central limit theorem for random variables of the form $\langle X, Y\rangle$, where Y is an independent copy of X.

Note that the implication (i) $\Rightarrow$ (ii) is immediate when applying (1.5) to $u(x) = |x|$, while the reverse statement is a deep theorem due to Eldan [17]. By (1.4), we also have (i) $\Rightarrow$ (iii). As for the implication (iii) $\Rightarrow$ (ii), it holds true in view of a general relation

$$c\,\mathrm{Var}(|X|)\;\leq\;n\,(\log n)^4\;\mathbb{E}_\theta\,\rho(F_\theta,\Phi) + 1$$

(which only requires that all S_θ have a finite and bounded exponential moment).

The symmetry assumption is irrelevant both in (i) and (ii). However, this is not so obvious concerning (iii). Indeed, one may try to use a symmetrization argument by applying (1.4) to the random vector $X' = (X-Y)/\sqrt{2}$. But then we need a quantitative form of a particular variant of Cramer's theorem: If η is an independent copy of a random variable ξ with mean zero and variance one, and if $\xi' = (\xi-\eta)/\sqrt{2}$ is almost standard normal, then so is ξ. The best result in this direction is the following theorem due to Sapogov [33]: Given that $\rho(F_{\xi'},\Phi) \leq \varepsilon \leq 1/e$, we have

$$\rho(F_\xi,\Phi) \leq C\,\big(\log(1/\varepsilon)\big)^{-1/2}$$

up to some absolute constant C, where F_ξ and $F_{\xi'}$ denote the distribution functions of ξ and ξ'. Moreover, the dependence in ε on the right-hand side cannot be improved, as was shown in [16] (cf. also [9] for a related model). Thus, the resulting bound on $\mathbb{E}_\theta\, \rho(F_\theta, \Phi)$ which can be derived this way on the basis of X' cannot yield even a standard rate.

Here, we choose a different route. As we will see, it is possible to remove the symmetry hypothesis, by adding to the right-hand side of (1.4) an additional term responsible for higher order correlations between X_k. More precisely, as a preliminary bound which is based on the Λ-functional only, it will be shown that

$$c\,\mathbb{E}_\theta\, \rho(F_\theta, \Phi) \,\leq\, \frac{\log n}{n}\,\Lambda + \Big(\frac{\log n}{n}\Big)^{1/4} \bigg(\mathbb{E}\,\frac{\langle X, Y\rangle}{\sqrt{|X|^2 + |Y|^2}}\bigg)^{1/2} \tag{1.8}$$

(cf. Proposition 10.1 below). The last expectation is vanishing for symmetric distributions, or, for example, if $|X| = \sqrt{n}$ a.s. together with $\mathbb{E}X = 0$. As another scenario, the second term in (1.8) is of a smaller order in comparison with $\frac{\log n}{n}\,\lambda_1^{-1}$ when (1.5) holds. Nevertheless, in contrast to the bound (1.4), the derivation of (1.8) turns out to be tedious, since it involves a careful analysis of projections of the characteristic functions $f_\theta(t)$ of S_θ as functions of θ onto the subspace of all linear functions in the Hilbert space $L^2(\mathbb{R}^n, \mathfrak{s}_{n-1})$.

The paper is organized as follows. We start with the study of densities of linear functionals on the sphere $\mathbb{S}^{n-1}$ viewed as random variables with respect to the normalized Lebesgue measure $\mathfrak{s}_{n-1}$. Here, the aim will be to refine the asymptotic normality of these distributions in analogy with Edgeworth expansions in the central limit theorem (which we consider up to order 2, Sections 2–3). Then we turn to the problem of deviations of general smooth functions on $\mathbb{S}^{n-1}$ in terms of their Hessians, recalling and extending several results in this direction (Section 4). These results are applied in Sections 5 to characteristic functions $f_\theta(t)$, with a separate treatment of their linear parts in $L^2(\mathfrak{s}_{n-1})$ in the next Section 6. In Section 7, we adapt basic Fourier analytic tools in the form of Berry-Esseen-type bounds to the scheme of weighted sums. Deviations of involved integrals as functions on the sphere are discussed separately in Section 8. Section 9 collects several general facts about Poincaré-type inequalities that will be needed for the proof of Theorem 1.1, while final steps of the proof are deferred to the remaining Sections 10–12.

As usual, the Euclidean space $\mathbb{R}^n$ is endowed with the canonical norm $|\cdot|$ and the inner product $\langle\cdot,\cdot\rangle$. We denote by c a positive absolute constant which may vary from place to place (if not stated explicitly that c depends on some parameter).

2 Distribution of linear functionals on the sphere

By the rotational invariance of $\mathfrak{s}_{n-1}$, all linear functionals $u(\theta) = \langle\theta, v\rangle$ with $|v| = 1$ have equal distributions. Hence, it is sufficient to focus just on the first coordinate θ_1 of the vector $\theta \in \mathbb{S}^{n-1}$ viewed as a random variable on the probability space $(\mathbb{S}^{n-1}, \mathfrak{s}_{n-1})$. It is well-known that this random variable has density

$$c_n\,\big(1 - x^2\big)_+^{\frac{n-3}{2}}, \quad x \in \mathbb{R}, \qquad c_n = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})},$$

with respect to the Lebesgue measure on the real line, where c_n is a normalizing constant.

We denote by φ_n the density of the normalized first coordinate $\sqrt{n}\,\theta_1$, i.e.,

$$\varphi_n(x) = c_n' \left(1 - \frac{x^2}{n}\right)_+^{\frac{n-3}{2}}, \quad c_n' = \frac{c_n}{\sqrt{n}}.$$

Clearly,

$$\varphi_n(x) \to \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \qquad c_n' \to \frac{1}{\sqrt{2\pi}} = 0.399...$$

as $n \to \infty$, and one can also show that $c_n' < \frac{1}{\sqrt{2\pi}}$ for all n.

Deviations for $\varphi_n(x)$ from $\varphi(x)$ have been considered in [12]. In particular, if $n \geq 3$, then for all $x \in \mathbb{R}$,

$$|\varphi_n(x) - \varphi(x)| \,\leq\, \frac{c}{n}\, e^{-x^2/4}. \tag{2.1}$$

We need to sharpen this bound by obtaining an approximation for $\varphi_n(x)$ with an error of order $1/n^2$ by means of a suitable modification of the standard normal density. Denote by $H_4(x) = x^4 - 6x^2 + 3$ the 4-th Chebyshev-Hermite polynomial.

Proposition 2.1. *For all* $x \in \mathbb{R}$ *and* $n \geq 3$*,*

$$\Big|\varphi_n(x) - \varphi(x)\Big(1 - \frac{H_4(x)}{4n}\Big)\Big| \,\leq\, \frac{c}{n^2}\, e^{-x^2/4}. \tag{2.2}$$

Proof. In the interval $|x| \leq \frac{1}{2}\sqrt{n}$, consider the function $p_n(x) = (1 - \frac{x^2}{n})_+^{\frac{n-3}{2}}$. Using the Taylor expansion for the logarithmic function near zero, one may write

$$\begin{aligned} -\log p_n(x) \;&=\; -\frac{n-3}{2}\,\log\left(1 - \frac{x^2}{n}\right) \\ &=\; \frac{n-3}{2}\,\left(\frac{x^2}{n} + \frac{x^4}{2n^2} + \left(\frac{x^2}{n}\right)^3 \sum_{k=3}^\infty \frac{1}{k}\left(\frac{x^2}{n}\right)^{k-3}\right) \;=\; \frac{x^2}{2} + \delta. \end{aligned}$$

The remainder term has the form

$$\delta = -\frac{3x^2}{2n} + \frac{x^4}{4n} - \frac{1}{n^2}\Big(\frac{3}{4}\,x^4 - \frac{n-3}{3n}\,x^6\varepsilon\Big)$$

with some $0 \leq \varepsilon \leq 1$. By the assumption that $x^2 \leq \frac{1}{4}\,n$, it satisfies

$$\delta \,\geq\, -\frac{3x^2}{2n} + \frac{x^4}{4n} - \frac{3x^4}{4n^2} \,\geq\, -\frac{27x^2}{16n} + \frac{x^4}{4n} \,>\, -\frac{27}{64}.$$

Hence

$$|e^{-\delta} - 1 + \delta| \leq \frac{\delta^2}{2}\, e^{27/64} \leq \delta^2.$$

Moreover, using once more $x^2 \leq \frac{1}{4}\,n$, we get

$$|\delta| \,\leq\, \frac{3x^2}{2n} + \frac{x^4}{4n} + \frac{1}{n^2}\left(\frac{3}{4}\,x^4 + \frac{1}{3}\,x^6\right) \,\leq\, \frac{x^2}{n}\left(\frac{27}{16} + \frac{1}{3}\,x^2\right),$$

which implies

$$\delta^2 \leq \frac{x^4}{n^2}\left(6 + \frac{2}{9}\,x^4\right).$$

Hence, with some $|\varepsilon_1| \leq 1$,

$$e^{x^2/2}\, p_n(x) \,=\, e^{-\delta} \,=\, 1 - \delta + \varepsilon_1 \delta^2 \,=\, 1 + \frac{3x^2}{2n} - \frac{x^4}{4n} + \frac{A}{n^2},$$

where

$$\begin{aligned} |A| \;&\leq\; \left|\frac{3}{4}\,x^4 - \frac{n-3}{3n}\,x^6\varepsilon\right| + x^4\left(6 + \frac{2}{9}\,x^4\right) \\ &\leq\; \frac{3}{4}\,x^4 + \frac{1}{3}\,x^6 + x^4\left(6 + \frac{2}{9}\,x^4\right) \;\leq\; 8x^4 + x^8. \end{aligned}$$

As a result,

$$p_n(x) = e^{-x^2/2} \left[1 + \frac{6x^2 - x^4}{4n} + \frac{\varepsilon}{n^2} \left(8x^4 + x^8 \right) \right], \qquad |\varepsilon| \le 1. \tag{2.3}$$

To derive a similar expansion for $\varphi_n(x)$, denote by Z a standard normal random variable. From (2.3) we obtain that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p_n(x)\, dx &= 1 + \frac{1}{4n} \left(6\, \mathbb{E}Z^2 - \mathbb{E}Z^4 \right) + O\Big(\frac{1}{n^2}\Big) \\ &= 1 + \frac{3}{4n} + O\Big(\frac{1}{n^2}\Big). \end{aligned}$$

Here we used the property that $p_n(x)$ has a sufficiently fast decay for $|x| \ge \frac{1}{2}\sqrt{n}$, as indicated in (2.1). Since $\varphi_n(x) = c_n'\, p_n(x)$ is a density, we conclude that

$$1 = c_n' \sqrt{2\pi} \left(1 + \frac{3}{4n} + O\Big(\frac{1}{n^2}\Big) \right), \qquad c_n' \sqrt{2\pi} = 1 - \frac{3}{4n} + O\Big(\frac{1}{n^2}\Big).$$

Hence

$$\begin{aligned} \sqrt{2\pi}\, e^{x^2/2}\, \varphi_n(x) &= \left(1 - \frac{3}{4n} + O\Big(\frac{1}{n^2}\Big) \right) \left[1 + \frac{6x^2 - x^4}{4n} + \frac{\varepsilon}{n^2} \Big(8x^4 + x^8 \Big) \right] \\ &= 1 + \frac{6x^2 - x^4}{4n} - \frac{3}{4n} + O\Big(\frac{1 + x^8}{n^2} \Big). \end{aligned}$$

Thus, in the interval $|x| \le \frac{1}{2}\sqrt{n}$,

$$\varphi_n(x) = \varphi(x) \left[1 - \frac{H_4(x)}{4n} + Q_n(x)\, \frac{1 + x^8}{n^2} \right]$$

with a quantity $Q_n(x)$ bounded by a universal constant in absolute value. In view of (2.1), the bound (2.2) follows immediately. □

3 Characteristic function of linear functionals

In the sequel, we denote by $J_n = J_n(t)$ the characteristic function of the first coordinate θ_1 of a random vector $\theta = (\theta_1, \dots, \theta_n)$ which is uniformly distributed on the unit sphere $\mathbb{S}^{n-1}$. In a more explicit form, for any $t \in \mathbb{R}$,

$$J_n(t) \,=\, c_n \int_{-\infty}^{\infty} e^{itx}\, (1 - x^2)_+^{\frac{n-3}{2}}\, dx \,=\, c_n' \int_{-\infty}^{\infty} e^{itx/\sqrt{n}} \Big(1 - \frac{x^2}{n} \Big)_+^{\frac{n-3}{2}}\, dx.$$

This is just a multiple of the Bessel function of the first kind with index $\nu = \frac{n}{2} - 1$ ([3], p. 81).

Thus, the characteristic function of the normalized first coordinate $\theta_1 \sqrt{n}$ is given by

$$\hat{\varphi}_n(t) = J_n\big(t\sqrt{n}\big) = \int_{-\infty}^{\infty} e^{itx} \varphi_n(x)\, dx,$$

which is the Fourier transform of the probability density φ_n. Proposition 2.1 can be used to compare $\hat{\varphi}_n(t)$ with the Fourier transform of the "corrected Gaussian measure", as well as to compare the derivatives of these transforms.

Proposition 3.1. *For all $t \in \mathbb{R}$,*

$$\left| J_n\big(t\sqrt{n}\big) - \Big(1 - \frac{t^4}{4n} \Big)\, e^{-t^2/2} \right| \,\le\, \frac{c}{n^2}.$$

Moreover, for any $k = 1, 2, \dots$*,*

$$\left|\frac{d^k}{dt^k}\, J_n\big(t\sqrt{n}\big) - \frac{d^k}{dt^k}\,\bigg(\Big(1 - \frac{t^4}{4n}\Big)\, e^{-t^2/2}\bigg)\right| \,\le\, \frac{(ck)^{k/2}}{n^2}.$$

Taking $k = 1$, we have

$$\left|\big(J_n\big(t\sqrt{n}\big)\big)' - \Big(\frac{t^5}{4n} - \frac{t^3}{n} - t\Big)\, e^{-t^2/2}\right| \,\le\, \frac{c}{n^2}.$$

One may also add a t-depending factor on the right-hand side. For t of order 1, this can be done just by virtue of Taylor's formula. Indeed, the functions

$$f_n(t) = J_n\big(t\sqrt{n}\big) = \mathbb{E}_\theta\, e^{it\theta_1\sqrt{n}}, \qquad g_n(t) = \Big(1 - \frac{t^4}{4n}\Big)\, e^{-t^2/2}$$

have equal first three derivatives at zero. Since, by Proposition 3.1, $|f_n^{(4)}(t) - g_n^{(4)}(t)| \le \frac{c}{n^2}$, Taylor's formula refines this proposition for the interval $|t| \le 1$.

Corollary 3.2. *For all* $t \in \mathbb{R}$*,*

$$\left|J_n\big(t\sqrt{n}\big) - \Big(1 - \frac{t^4}{4n}\Big)\, e^{-t^2/2}\right| \,\le\, \frac{c}{n^2}\, \min\{1, t^4\},$$

$$\left|\big(J_n\big(t\sqrt{n}\big)\big)' + t\,\Big(1 + \frac{4t^2 - t^4}{4n}\Big)\, e^{-t^2/2}\right| \,\le\, \frac{c}{n^2}\, \min\{1, |t|^3\}.$$

These approximations may be complemented by a Gaussian decay bound

$$\big|J_n\big(t\sqrt{n}\big)\big| \,\le\, 5\, e^{-t^2/2} + 4\, e^{-n/12}, \quad t \in \mathbb{R} \tag{3.1}$$

(cf. [12], Proposition 3.3).

Proof of Proposition 3.1. In general, given two integrable functions on the real line, say, p and q, their Fourier transforms

$$\hat p(t) = \int_{-\infty}^{\infty} e^{itx} p(x)\, dx, \qquad \hat q(t) = \int_{-\infty}^{\infty} e^{itx} q(x)\, dx$$

satisfy, for all $t \in \mathbb{R}$,

$$|\hat p(t) - \hat q(t)| \le \int_{-\infty}^{\infty} |p(x) - q(x)|\, dx.$$

Moreover, one may differentiate these transforms k times to get

$$\frac{d^k}{dt^k}\, \hat p(t) \,=\, \int_{-\infty}^{\infty} (ix)^k\, e^{itx}\, p(x)\, dx, \qquad \frac{d^k}{dt^k}\, \hat q(t) \,=\, \int_{-\infty}^{\infty} (ix)^k\, e^{itx}\, q(x)\, dx,$$

as long as the integrands are integrable, which also yields the relation

$$\Big|\frac{d^k}{dt^k}\, \hat p(t) - \frac{d^k}{dt^k}\, \hat q(t)\Big| \,\le\, \int_{-\infty}^{\infty} |x|^k\, |p(x) - q(x)|\, dx.$$

This applies in particular to the functions $p(x) = \varphi_n(x)$ and $q(x) = \varphi(x)\,(1 - \frac{1}{4n}\, H_4(x))$ whose Fourier transform is described as

$$\hat q(t) = e^{-t^2/2}\Big(1 - \frac{t^4}{4n}\Big).$$

Since (by Stirling's formula)

$$\int_{-\infty}^{\infty} |x|^k\, e^{-x^2/4}\, dx \,=\, 2^{k+1}\, \Gamma\Big(\frac{k+1}{2}\Big) \,\le\, (ck)^{k/2},$$

it remains to apply (2.2). $\square$

4 Deviations of smooth functions on the sphere

Smooth functions u on the unit n-sphere with $\mathfrak{s}_{n-1}$-mean zero are known to have fluctuations of order at most $1/\sqrt{n}$ (which is the case for all linear functions). This may be seen from the Poincaré inequality

$$\int |u|^2\, d\mathfrak{s}_{n-1} \leq \frac{1}{n-1} \int |\nabla u|^2\, d\mathfrak{s}_{n-1}. \tag{4.1}$$

Moreover, when u is Lipschitz, that is, $|\nabla u(\theta)| \leq 1$ for all $\theta \in \mathbb{S}^{n-1}$, there is a subgaussian exponential bound on the Laplace transform (cf. [28])

$$\int \exp\Big\{\sqrt{n-1}\, ru\Big\}\, d\mathfrak{s}_{n-1} \,\leq\, e^{r^2/2}, \qquad r \in \mathbb{R}. \tag{4.2}$$

This spherical concentration phenomenon may be strengthened with respect to the dimension n for a wide subclass of smooth functions. We denote by $\nabla^2 u(x)$ the Hessian, that is, the $n \times n$ matrix of second order partial derivative $\partial_{ij} u(x)$, and by $\mathbb{I}_n$ the identity $n \times n$ matrix. Given a symmetric matrix $A = (a_{ij})_{i,j=1}^n$ with real or complex entries, the associated Hilbert-Schmidt and operator norms are defined by

$$\|A\|_{\mathrm{HS}} = \bigg(\sum_{i,j=1}^n |a_{ij}|^2\bigg)^{1/2}, \qquad \|A\| = \max_{|\theta|=1} |\left<A\theta,\theta\right>|.$$

The next proposition summarizes several results from [13] employing a second order concentration on the sphere, a property developed in [10].

Proposition 4.1. *Suppose that a real-valued function u is defined and C^2-smooth in some neighborhood of $\mathbb{S}^{n-1}$. If u is orthogonal to all affine functions in $L^2(\mathfrak{s}_{n-1})$, then*

$$\int |u|^2\, d\mathfrak{s}_{n-1} \leq \frac{5}{(n-1)^2} \int \|\nabla^2 u - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2\, d\mathfrak{s}_{n-1} \tag{4.3}$$

for any $a \in \mathbb{R}$. Moreover, if $\|\nabla^2 u - a\,\mathbb{I}_n\| \leq 1$ on $\mathbb{S}^{n-1}$ and the second integral in (4.3) *is bounded by b, then*

$$\int \exp\Big\{\frac{n-1}{2(1+4b)}\,|u|\Big\}\, d\mathfrak{s}_{n-1} \,\leq\, 2. \tag{4.4}$$

By Markov's inequality, (4.4) yields a corresponding large deviation bound, which may be stated informally as a subexponential stochastic dominance $|u| \leq c_b\,(\frac{1}{\sqrt{n}}Z)^2$ with $Z \sim N(0,1)$. Thus, the deviations of u are of order at most $1/n$.

We will need the following generalization of Proposition 4.1 which is more flexible in applications. Given a function u in the (complex) Hilbert space $L^2 = L^2(\mathbb{R}^n, \mathfrak{s}_{n-1})$, we consider its orthogonal projection

$$l = \mathrm{Proj}_H u$$

onto the linear space H in L^2 generated by the constant and linear functions on $\mathbb{R}^n$. Let us call l an affine part of u.

Proposition 4.2. *Suppose that a complex-valued function u is C^2-smooth in some neighborhood of $\mathbb{S}^{n-1}$ and has $\mathfrak{s}_{n-1}$-mean zero. For any $a \in \mathbb{C}$,*

$$\int |u|^2\, d\mathfrak{s}_{n-1} \,\leq\, \frac{5}{(n-1)^2} \int \|\nabla^2 u - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2\, d\mathfrak{s}_{n-1} + \|l\|_{L^2}^2, \tag{4.5}$$

where l is the affine part of u. Moreover, if $\|\nabla^2 u - a\,\mathbb{I}_n\| \leq 1$ on $\mathbb{S}^{n-1}$, then

$$\|u\|_{\psi_1} \,\leq\, \frac{4}{n-1} + \frac{16}{n-1} \int \|\nabla^2 u - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2\, d\mathfrak{s}_{n-1} + 6\,\|l\|_{L^2}. \tag{4.6}$$

Here we used a standard notation

$$\|u\|_{\psi_1} = \inf \Big\{\lambda > 0 : \mathbb{E}_\theta\, e^{|u|/\lambda} \leq 2\Big\}$$

for the Orlicz norm on the probability space $(\mathbb{S}^{n-1}, \mathfrak{s}_{n-1})$ generated by the Young function $\psi_1(r) = e^{|r|} - 1$ ($r \in \mathbb{R}$).

Proof of Proposition 4.2. The Poincaré-type inequalities (4.1) and (4.3) continue to hold in the class of all complex-valued functions u with $\mathfrak{s}_{n-1}$-mean zero, while (4.2) and (4.4) require slight modifications. Indeed, (4.4) may be applied separately to the real part $u_0 = \mathrm{Re}(u)$ and to the imaginary part $u_1 = \mathrm{Re}(u)$ of u, which results in

$$\int \exp\Big\{\frac{n-1}{2(1+4b_k)}\,|u_k|\Big\}\,d\mathfrak{s}_{n-1} \leq 2, \quad b_k = \int \|\nabla^2 u_k - a_k\,\mathbb{I}_n\|_{\mathrm{HS}}^2\,d\mathfrak{s}_{n-1}, \tag{4.7}$$

for $k = 0$ and $k = 1$, assuming that the following conditions are fulfilled:

$a)$ u_0 and u_1 (that is, u) are C^2-smooth and orthogonal to all affine functions in $L^2(\mathfrak{s}_{n-1})$;

$b)$ $\|\nabla^2 u_k - a_k\,\mathbb{I}_n\| \leq 1$ on $\mathbb{S}^{n-1}$ with $a_0 = \mathrm{Re}(a)$ and $a_1 = \mathrm{Im}(a)$.

The latter requirement is met as long as

$$\|\nabla^2 u - a\,\mathbb{I}_n\| \,\equiv\, \max_{|\theta|=1} \big|\left\langle(\nabla^2 u - a\,\mathbb{I}_n)\theta, \theta\right\rangle\big| \,\leq\, 1. \tag{4.8}$$

As for the exponential bounds in (4.7), they may equivalently be written in terms of the Orlicz ψ_1-norm as

$$\|u_k\|_{\psi_1} \,\leq\, \frac{2}{n-1} + \frac{8b_k}{n-1}, \quad k = 0, 1.$$

Applying the triangle inequality $\|u\|_{\psi_1} \leq \|u_0\|_{\psi_1} + \|u_1\|_{\psi_1}$ in the Orlicz space and noting that $b_0 + b_1$ is just the integral on the right-hand side in (4.5)–(4.6), we conclude that

$$\|u\|_{\psi_1} \,\leq\, \frac{4}{n-1} + \frac{16}{n-1}\int \|\nabla^2 u - aI_n\|_{\mathrm{HS}}^2\,d\mathfrak{s}_{n-1}. \tag{4.9}$$

This is a "complex" variant of the inequality (4.4), which holds for all $a \in \mathbb{C}$ under the assumption that u is C^2-smooth in some neighbourhood of $\mathbb{S}^{n-1}$, is orthogonal to all affine functions in $L^2(\mathfrak{s}_{n-1})$, and satisfies (4.8).

One may now start with an arbitrary C^2-smooth function u with mean zero, but apply these hypotheses and the conclusions to the projection Tu of u onto the orthogonal complement of the space H of all linear functions in $L^2(\mathfrak{s}_{n-1})$. This space has dimension n, and one may choose for the orthonormal basis in H the canonical functions

$$l_k(\theta) = \sqrt{n}\,\theta_k, \quad k = 1, \dots, n, \quad \theta = (\theta_1, \dots, \theta_n) \in \mathbb{S}^{n-1}.$$

Therefore, the "linear" part $l = Tu - u$ of u is described as the orthogonal projection in $L^2(\mathfrak{s}_{n-1})$ onto H, namely

$$\begin{aligned} l(\theta) \;&=\; \sum_{k=1}^n \langle u, l_k\rangle_{L^2}\, l_k(\theta) \;=\; \sum_{k=1}^n \Big(\int u(x) l_k(x)\,d\mathfrak{s}_{n-1}(x)\Big)\, l_k(\theta) \\ &=\; n\int \Big(u(x)\sum_{k=1}^n x_k\theta_k\Big)\,d\mathfrak{s}_{n-1}(x). \end{aligned}$$

In other words,

$$l(\theta) = \langle v, \theta\rangle \quad \text{with } v = n\int x u(x)\,d\mathfrak{s}_{n-1}(x),$$

which implies, in particular, that

$$\|l\|_{L^2}^2 = \frac{1}{n}\,|v|^2 = n \iint \langle x,y\rangle\, u(x)\bar{u}(y)\, d\mathfrak{s}_{n-1}(x)d\mathfrak{s}_{n-1}(y). \tag{4.10}$$

The functions Tu and u have identical Euclidean second derivatives. Hence, (4.5) follows from (4.3) when the latter is applied to Tu, since Tu and l are orthogonal in L^2. Applying (4.9) with Tu in place of u, we similarly have

$$\|Tu\|_{\psi_1} \,\le\, \frac{4}{n-1} + \frac{16}{n-1}\int \|\nabla^2 u - a\,\mathbb{I}_n\|_{\rm HS}^2\, d\mathfrak{s}_{n-1}, \tag{4.11}$$

provided that $\|\nabla^2 Tu - a\,\mathbb{I}_n\| = \|\nabla^2 u - a\,\mathbb{I}_n\| \le 1$ on $\mathbb{S}^{n-1}$ as in (4.8).

To derive (4.6), it remains to use the fact that the linear functions on the sphere behave like Gaussian random variables. This can be seen from (4.2), which may be applied with $r=1$ to the real and imaginary parts of $l/\|l\|_{\rm Lip}$. Then it gives

$$\int \exp\Big\{\sqrt{n-1}\,|l|/4\,\|l\|_{\rm Lip}\Big\}\, d\mathfrak{s}_{n-1} \le 2,$$

so that

$$\|l\|_{\psi_1} \le \frac{4}{\sqrt{n-1}}\,\|l\|_{\rm Lip} = \frac{4\sqrt{n}}{\sqrt{n-1}}\,\|l\|_{L^2} \le 6\,\|l\|_{L^2}.$$

The latter should be combined with (4.11), and we arrive at (4.6) due to the triangle inequality $\|u\|_{\psi_1} \le \|Tu\|_{\psi_1} + \|l\|_{\psi_1}$. □

5 Concentration of characteristic functions

Given a random vector $X = (X_1,\dots,X_n)$ in $\mathbb{R}^n$, we consider the smooth functions

$$u_t(\theta) = f_\theta(t) = \mathbb{E}\, e^{it\langle X,\theta\rangle}, \quad \theta\in\mathbb{R}^n, \tag{5.1}$$

where $t\in\mathbb{R}$ serves as a parameter. For any fixed $\theta\in\mathbb{R}^n$, $t\to f_\theta(t)$ represents the characteristic function of the weighted sum $S_\theta = \langle X,\theta\rangle$ with distribution function F_θ, while the $\mathfrak{s}_{n-1}$-mean

$$f(t) = \mathbb{E}_\theta f_\theta(t) = \mathbb{E}_\theta\, \mathbb{E}\, e^{it\langle X,\theta\rangle}$$

is the characteristic function of the average distribution function $F(x) = \mathbb{E}_\theta\,\mathbb{P}\{S_\theta \le x\}$, $x\in\mathbb{R}$. Recall that we use $\mathbb{E}_\theta$ to denote integrals with respect to the uniform measure $\mathfrak{s}_{n-1}$.

In order to control deviations of u_t from $f(t)$ on $\mathbb{S}^{n-1}$ at the standard rate, the spherical concentration inequalities (4.1)–(4.2) are sufficient. Indeed, the function u_t has a gradient described in the vector form as

$$\langle \nabla u_t(\theta), w\rangle \,=\, it\,\mathbb{E}\,\langle X,w\rangle\, e^{it\langle X,\theta\rangle}, \qquad w\in\mathbb{C}^n.$$

Hence, under the isotropy assumption, writing $w = w_0 + iw_1$ ($w_0,w_1\in\mathbb{R}^n$), we have

$$\begin{aligned} |\langle \nabla u_t(\theta), w\rangle|^2 \;&\le\; \mathbb{E}\,|\langle X,w\rangle|^2 \\ &=\; \mathbb{E}\,\langle X,w_0\rangle^2 + \mathbb{E}\,\langle X,w_1\rangle^2 \,=\, |w_0|^2 + |w_1|^2 \,=\, |w|^2, \end{aligned}$$

that is, $|\langle \nabla u_t(\theta), w\rangle| \le |t|\,|w|$ for all $w\in\mathbb{C}^n$. This gives a uniform bound $|\nabla u_t(\theta)| \le |t|$, so that, by the spherical Poincaré inequality (4.1),

$$\mathbb{E}_\theta\, |f_\theta(t) - f(t)|^2 \,\le\, \frac{t^2}{n-1}. \tag{5.2}$$

A similar inequality is also true for the Orlicz ψ_2-norm of $f_\theta(t) - f(t)$ generated by the Young function $\psi_2(r) = e^{r^2} - 1$.

As it turns out, this rate of concentration may be improved under a second order correlation condition (1.3) at least for values of t which are not too large, by involving the characteristic $\Lambda = \Lambda(X)$. In the isotropic case, this condition is described as the relation

$$\mathbb{E}\,\Big|\sum_{j,k=1}^n z_{jk}\,(X_j X_k - \delta_{jk})\Big|^2 \,\le\, \Lambda \sum_{j,k=1}^n |z_{jk}|^2, \quad z_{jk} \in \mathbb{C}. \tag{5.3}$$

Here, Λ is necessarily bounded away from zero. Indeed, (5.3) includes $\mathbb{E}\,X_j^2 X_k^2 - \delta_{jk} \le \Lambda$ as partial cases. Summing this over all $j,k = 1,\dots,n$ leads to $\mathbb{E}\,|X|^4 - n \le n^2\Lambda$. But $\mathbb{E}\,|X|^4 \ge (\mathbb{E}\,|X|^2)^2 = n^2$ implying that

$$\Lambda \ge \frac{n-1}{n} \ge \frac{1}{2}.$$

As was proved in [13] on the basis of Proposition 4.1, if the distribution of X is isotropic and symmetric about the origin, the characteristic functions $f_\theta(t)$ satisfy in the interval $|t| \le An^{1/5}$

$$c\,\mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \,\le\, \frac{\Lambda t^4}{n^2}, \tag{5.4}$$

where the constant $c > 0$ depends on the parameter $A \ge 1$ only. Moreover,

$$\mathbb{E}_\theta\,\exp\Big\{\frac{cn}{\Lambda t^2}\,|f_\theta(t) - f(t)|\Big\} \le 2. \tag{5.5}$$

Note that, in the symmetric case, the functions $\theta \to f_\theta(t)$ are even, so, all u_t have zero linear parts when projecting them onto the subspace H of all linear functions in $L^2(\mathbb{R}^n, \mathfrak{s}_{n-1})$.

To drop the symmetry assumption, consider an orthogonal decomposition

$$u_t(\theta) = f(t) + l_t(\theta) + v_t(\theta), \tag{5.6}$$

where

$$l_t(\theta) = c_1(t)\,\theta_1 + \dots + c_n(t)\,\theta_n, \qquad \theta = (\theta_1,\dots,\theta_n) \in \mathbb{R}^n,$$

is the orthogonal projection of $u_t - f(t)$ onto H (the linear part) and where $v_t(\theta) = u_t(\theta) - f(t) - l_t(\theta)$ is the non-linear part of u_t. By the orthogonality,

$$\mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \,=\, \mathbb{E}_\theta\,|l_t(\theta)|^2 + \mathbb{E}_\theta\,|v_t(\theta)|^2. \tag{5.7}$$

With these notations, the bounds (5.4)–(5.5) should be properly modified.

Proposition 5.1. *Given an isotropic random vector X in $\mathbb{R}^n$, in the interval $|t| \le An^{1/5}$,*

$$c\,\mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \,\le\, \|l_t\|_{L^2}^2 + \frac{\Lambda t^4}{n^2} \tag{5.8}$$

with some constant $c > 0$ depending on the parameter $A > 0$. Here, l_t is the linear part of $f_\theta(t)$ in $L^2(\mathbb{R}^n, \mathfrak{s}_{n-1})$ from the orthogonal decomposition (5.6)*. Moreover, if $|t| \le An^{1/6}$, then*

$$c\,\|f_\theta(t) - f(t)\|_{\psi_1} \,\le\, \|l_t\|_{L^2} + \frac{\Lambda t^2}{n}. \tag{5.9}$$

If the distribution of X is symmetric about the origin, then $l_t(\theta) = 0$, and we return in (5.8)–(5.9) to (5.4)–(5.5). The linear part l_t is also vanishing, when X has mean zero and a constant Euclidean norm, i.e. when $|X| = \sqrt{n}$ a.s. (this will be clarified in the next section).

Proof. To employ Propositions 4.1–4.2, we need to choose a suitable value $a \in \mathbb{C}$ and estimate the operator norm $\|\nabla^2 u_t - a\,\mathbb{I}_n\|$ and the Hilbert-Schmidt norm $\|\nabla^2 u_t - a\,\mathbb{I}_n\|_{\rm HS}$. First note that, by differentiation of (5.1), for any fixed $t \in \mathbb{R}$,

$$\big[\nabla^2 u_t(\theta)\big]_{jk} = \frac{\partial^2}{\partial\theta_j \partial\theta_k}\, f_\theta(t) = -t^2\, \mathbb{E}\, X_j X_k\, e^{it\langle X,\theta\rangle}.$$

Hence, a good choice is $a = -t^2 f(t)$ in order to balance the diagonal elements in the matrix of second derivatives of u_t. For any vector $w \in \mathbb{C}^n$, using the canonical inner product in the complex n-space, we have

$$\big\langle \nabla^2 u_t(\theta)\, w, w\big\rangle = -t^2\, \mathbb{E}\,|\,\langle X,w\rangle\,|^2\, e^{it\langle X,\theta\rangle}.$$

Hence, by the isotropy assumption,

$$\big|\,\big\langle (\nabla^2 u_t(\theta) - a\,\mathbb{I}_n)\, w, w\big\rangle\,\big| \le t^2\, \mathbb{E}\,|\,\langle X,w\rangle\,|^2 + |a|\,|w|^2 \le 2t^2, \qquad |w| = 1.$$

In terms of the norm defined as in (4.8), this bound insures that

$$\|\nabla^2 u_t(\theta) - a\,\mathbb{I}_n\| \le 2t^2. \tag{5.10}$$

In addition, putting $a(\theta) = -t^2 f_\theta(t)$, we have

$$\begin{aligned}
\big\|\nabla^2 u_t(\theta) - a(\theta)\,\mathbb{I}_n\big\|_{\rm HS}^2 &= \sum_{j,k=1}^n \big|\nabla^2 u_t(\theta)_{jk} - a(\theta)\,\delta_{jk}\big|^2 \\
&= \sup \bigg|\sum_{j,k=1}^n z_{jk}\,\big(\nabla^2 u_t(\theta)_{jk} - a(\theta)\,\delta_{jk}\big)\bigg|^2 \\
&\le t^4\, \sup\, \mathbb{E}\,\bigg|\sum_{j,k=1}^n z_{jk}\,(X_j X_k - \delta_{jk})\bigg|^2,
\end{aligned}$$

where the supremum is running over all complex numbers z_{jk} such that $\sum_{j,k=1}^n |z_{jk}|^2 = 1$. But, under this constraint, due to the second order correlation condition, the last expectation is bounded by Λ. Since u_t and v_t have equal Hessians, we conclude that

$$\big\|\nabla^2 v_t(\theta) - a(\theta)\,\mathbb{I}_n\big\|_{\rm HS}^2 \,\le\, \Lambda t^4 \tag{5.11}$$

for all θ. On the other hand, by (5.2),

$$\mathbb{E}_\theta\,\big\|(a(\theta) - a)\,\mathbb{I}_n\big\|_{\rm HS}^2 \,=\, nt^4\, \mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \,\le\, 2t^6. \tag{5.12}$$

The two last bounds give

$$\mathbb{E}_\theta\,\big\|\nabla^2 v_t(\theta) - a\,\mathbb{I}_n\big\|_{\rm HS}^2 \,\le\, 2\Lambda t^4 + 4t^6,$$

which, by Proposition 4.1, yields

$$\mathbb{E}_\theta\,|v_t(\theta)|^2 \,\le\, \frac{5}{(n-1)^2}\,(2\Lambda t^4 + 4t^6).$$

One can sharpen this bound for the range $|t| \le An^{1/5}$. Applying it in (5.7), we get

$$\mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \,\le\, \mathbb{E}_\theta\,|l_t(\theta)|^2 + \frac{5}{(n-1)^2}\,(2\Lambda t^4 + 4t^6),$$

which, according to the identity in (5.12), gives

$$\mathbb{E}_\theta \|(a(\theta) - a)\,\mathbb{I}_n\|_{\mathrm{HS}}^2 \,\le\, nt^4\,\mathbb{E}_\theta\,|l_t(\theta)|^2 + \frac{5n}{(n-1)^2}\,(2\Lambda t^8 + 4t^{10}).$$

Combining this with (5.11), we get

$$\mathbb{E}_\theta \|\nabla^2 v_t(\theta) - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2 \,\le\, 2nt^4\,\mathbb{E}_\theta\,|l_t(\theta)|^2 + 2\Lambda t^4 + \frac{10\,n}{(n-1)^2}\,(2\Lambda t^8 + 4t^{10}).$$

Hence, by Proposition 4.1 once more,

$$\mathbb{E}_\theta\,|v_t(\theta)|^2 \,\le\, \frac{10\,nt^4}{(n-1)^2}\,\mathbb{E}_\theta\,|l_t(\theta)|^2 + \frac{10}{(n-1)^2}\,\Lambda t^4 + \frac{50\,n}{(n-1)^4}\,(\Lambda t^8 + 2t^{10}),$$

so that, by (5.7),

$$\begin{aligned}\mathbb{E}_\theta\,|f_\theta(t) - f(t)|^2 \quad &\le \quad \Big(1 + \frac{10\,nt^4}{(n-1)^2}\Big)\,\mathbb{E}_\theta\,|l_t(\theta)|^2 \\ &\quad + \frac{10}{(n-1)^2}\,\Lambda t^4 + \frac{50\,n}{(n-1)^4}\,(\Lambda t^8 + 2t^{10}).\end{aligned}$$

According to the identity in (5.12), this gives

$$\begin{aligned}\mathbb{E}_\theta \|(a(\theta) - a)\,\mathbb{I}_n\|_{\mathrm{HS}}^2 \quad &\le \quad nt^4\,\Big(1 + \frac{10\,nt^4}{(n-1)^2}\Big)\,\mathbb{E}_\theta\,|l_t(\theta)|^2 \\ &\quad + \frac{10\,n}{(n-1)^2}\,\Lambda t^8 + \frac{50\,n^2}{(n-1)^4}\,(\Lambda t^{12} + 2t^{14}).\end{aligned}$$

One can combine this with (5.11) to obtain that

$$\begin{aligned}\mathbb{E}_\theta\,\|\nabla^2 v_t(\theta) - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2 \quad &\le \quad 2nt^4\,\Big(1 + \frac{10\,nt^4}{(n-1)^2}\Big)\,\mathbb{E}_\theta\,|l_t(\theta)|^2 \\ &\quad + 2\Lambda t^4 + \frac{20\,n}{(n-1)^2}\,\Lambda t^8 + \frac{100\,n^2}{(n-1)^4}\,(\Lambda t^{12} + 2t^{14}).\end{aligned}$$

Now, if $|t| \le An^{1/5}$, the coefficient in front of $\mathbb{E}_\theta\,|l_t(\theta)|^2$ does not exceed a multiple of nt^4. Similarly, in this region the last three terms can be bounded by Λt^4 up to a numerical factor (since $\Lambda \ge \frac{1}{2}$). Hence the above bound is simplified to

$$c\,\mathbb{E}_\theta\,\|\nabla^2 v_t(\theta) - a\,\mathbb{I}_n\|_{\mathrm{HS}}^2 \,\le\, nt^4\,\|l_t\|_{L^2}^2 + \Lambda t^4 \tag{5.13}$$

with some constant c depending A. Since $nt^4 < A^4 n^2$, by Proposition 4.1, we get

$$c\,\mathbb{E}_\theta\,|v_t(\theta)|^2 \,\le\, \mathbb{E}_\theta\,|l_t(\theta)|^2 + \frac{5}{(n-1)^2}\,\Lambda t^4.$$

In view of (5.7), this proves the inequality (5.8).

To get a bound for the ψ_1-norm, note that, by (5.10), the conditions of Proposition 4.2 (in its second part) are fulfilled with $-\frac{1}{2}\,f(t)$ in place of a for the function

$$u(\theta) = \frac{1}{2t^2}\,(f_\theta(t) - f(t)), \qquad \theta \in \mathbb{R}^n, \;\; t \ne 0.$$

Since (5.13) holds for u_t as well (provided that $|t| \le An^{1/5}$), this inequality may be rewritten as

$$c\,\mathbb{E}_\theta\,\Big\|\nabla^2 u(\theta) + \frac{1}{2}\,f(t)\,\mathbb{I}_n\Big\|_{\mathrm{HS}}^2 \,\le\, n\,\|l_t\|_{L^2}^2 + \Lambda.$$

The linear part of u is given by $l_t/(2t^2)$. Hence, the inequality (4.6) of Proposition 4.2 yields

$$c\,\Big\|\frac{1}{2t^2}\,(f_\theta(t)-f(t))\Big\|_{\psi_1} \,\le\, \frac{1+\Lambda}{n} + \|l_t\|_{L^2}^2 + \frac{1}{2t^2}\,\|l_t\|_{L^2}.$$

Using once more $\Lambda \ge \frac{1}{2}$, the above is simplified to

$$c\,\|f_\theta(t)-f(t)\|_{\psi_1} \,\le\, \frac{\Lambda t^2}{n} + \|l_t\|_{L^2} + \|l_t\|_{L^2}^2\,t^2. \tag{5.14}$$

Here, the last term on the right-hand side is dominated by the second last term in the smaller interval $|t| \le An^{1/6}$. Indeed, according to the concentration inequality (5.2),

$$\|l_t\|_{L^2}\,t^2 \,\le\, \|f_\theta(t)-f(t)\|_{L^2}\,t^2 \,\le\, \frac{|t|^3}{\sqrt{n-1}} \,\le\, 2A^3.$$

Hence $\|l_t\|_{L^2}^2\,t^2 \le 2A^3\,\|l_t\|_{L^2}$. As a result, (5.14) leads to the required form (5.9). □

6 Linear part of characteristic functions

In order to make the bounds (5.8)–(5.9) effective, we need to properly estimate the L^2-norm of the linear part $l_t(\theta)$ of $f_\theta(t)$ in $L^2(\mathbb{R}^n, \mathfrak{s}_{n-1})$. According to (4.10), it is described as

$$I(t) \,=\, \|l_t\|_{L^2}^2 \,=\, n\,\mathbb{E}_\theta\,\mathbb{E}_{\theta'}\,\langle\theta,\theta'\rangle\,f_\theta(t)\bar{f}_{\theta'}(t). \tag{6.1}$$

Let us find an asymptotically explicit expression for this function.

Proposition 6.1. *Let X be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}\,|X|^2 = n$. For any $t \in \mathbb{R}$, the characteristic function $f_\theta(t) = \mathbb{E}\,e^{it\langle X,\theta\rangle}$ as a function of θ on the sphere has a linear part, whose squared $L^2(\mathfrak{s}_{n-1})$-norm may be represented as*

$$I(t) = \frac{t^2}{n}\,\mathbb{E}\,\langle X,Y\rangle\,\Big(1 - \frac{(U^2+V^2)\,t^4 - 8R^2t^2}{4n}\Big)\,e^{-R^2t^2} + O(t^2n^{-2}), \tag{6.2}$$

where Y is an independent copy of X, and

$$R^2 = \frac{1}{2n}\,(|X|^2+|Y|^2), \quad U = \frac{1}{n}\,|X|^2, \quad V = \frac{1}{n}\,|Y|^2.$$

The remainder term may be improved to $O(t^2n^{-5/2})$, if X is isotropic.

Proof. Using an independent copy Y of X, one may rewrite (6.1) equivalently as

$$I(t) \,=\, n\sum_{k=1}^n |\,\mathbb{E}_\theta\,\theta_k f_\theta(t)|^2 \,=\, n\sum_{k=1}^n \mathbb{E}\,\mathbb{E}_\theta\,\mathbb{E}_{\theta'}\Big[\theta_k\theta_k'\,e^{it\langle X,\theta\rangle - it\langle Y,\theta'\rangle}\Big].$$

To compute the inner expectations, introduce the function

$$K_n(t) = J_n\big(\sqrt{tn}\big), \qquad t \ge 0,$$

where, as before, J_n denotes the characteristic function of the first coordinate of a point on the unit sphere $\mathbb{S}^{n-1}$ under the normalized Lebesgue measure $\mathfrak{s}_{n-1}$. By the definition,

$$\mathbb{E}_\theta\,e^{i\langle v,\theta\rangle} = J_n(|v|) = K_n\Big(\frac{|v|^2}{n}\Big), \qquad v = (v_1,\dots,v_n) \in \mathbb{R}^n.$$

Differentiating this equality with respect to the variable v_k, we obtain that

$$i\,\mathbb{E}_\theta\,\theta_k e^{i\langle v,\theta\rangle} \,=\, \frac{2v_k}{n}\,K_n'\Big(\frac{|v|^2}{n}\Big).$$

Let us multiply this by a similar equality

$$-i\,\mathbb{E}_\theta\,\theta_k e^{-i\langle w,\theta\rangle} = \frac{2w_k}{n}\,K_n'\Big(\frac{|w|^2}{n}\Big),$$

to get that, for all $v, w \in \mathbb{R}^n$,

$$\mathbb{E}_\theta\,\mathbb{E}_{\theta'}\Big[\theta_k\theta_k'\,e^{i\langle v,\theta\rangle - i\langle w,\theta'\rangle}\Big] = \frac{4v_k w_k}{n^2}\,K_n'\Big(\frac{|v|^2}{n}\Big)\,K_n'\Big(\frac{|w|^2}{n}\Big).$$

Hence, summing over all $k \le n$, we get

$$\sum_{k=1}^n \mathbb{E}_\theta\,\mathbb{E}_{\theta'}\Big[\theta_k\theta_k'\,e^{i\langle v,\theta\rangle - i\langle w,\theta'\rangle}\Big] = \frac{4\,\langle v,w\rangle}{n^2}\,K_n'\Big(\frac{|v|^2}{n}\Big)\,K_n'\Big(\frac{|w|^2}{n}\Big).$$

It remains to make the substitution $v = tX$, $w = tY$ and to take the expectation over (X,Y). Then we arrive at the following expression

$$I(t) = \frac{4t^2}{n}\,\mathbb{E}\,\langle X,Y\rangle\,K_n'\Big(\frac{t^2|X|^2}{n}\Big)K_n'\Big(\frac{t^2|Y|^2}{n}\Big). \tag{6.3}$$

In particular, if $|X| = \sqrt{n}$ a.s., then

$$I(t) = \frac{4t^2}{n}\,K_n'^{\,2}(t^2)\,\mathbb{E}\,\langle X,Y\rangle\,,$$

which is vanishing, as soon as X has mean zero. In fact, the property $I(t) = 0$ remains valid for more general random vectors. In particular, this is the case, where the conditional distribution of X given that $|X| = r$ has mean zero for any $r > 0$.

Now, let us derive an asymptotic formula for the function K_n and its derivative. We know from Corollary 3.2 that

$$\frac{d}{dt}\,J_n(t\sqrt{n}) = -t\Big(1 - \frac{t^4 - 4t^2}{4n}\Big)\,e^{-t^2/2} + O\big(n^{-2}\min(1,|t|^3)\big).$$

Since $J_n(t\sqrt{n}) = K_n(t^2)$, after differentiation we find that

$$2tK_n'(t^2) = \frac{d}{dt}\,K_n(t^2) = -t\Big(1 - \frac{t^4 - 4t^2}{4n}\Big)\,e^{-t^2/2} + O\big(n^{-2}\min(1,|t|^3)\big).$$

Changing the variable, we arrive at

$$K_n'(t) = -\frac{1}{2}\,\Big(1 - \frac{t^2 - 4t}{4n}\Big)\,e^{-t/2} + O\big(n^{-2}\min(1,t)\big), \qquad t \ge 0.$$

From this,

$$K_n'(t)K_n'(s) \;=\; \frac{1}{4}\,\Big(1 - \frac{(t^2+s^2) - 4(t+s)}{4n}\Big)\,e^{-(t+s)/2} + O\big(n^{-2}\big)$$

uniformly over all $t, s \ge 0$, so,

$$\begin{aligned} 4K_n'\Big(\frac{t^2|X|^2}{n}\Big)K_n'\Big(\frac{t^2|Y|^2}{n}\Big) &= \Big(1 - \frac{t^4\,\big(\frac{|X|^4}{n^2} + \frac{|Y|^4}{n^2}\big) - 4t^2\,\big(\frac{|X|^2}{n} + \frac{|Y|^2}{n}\big)}{4n}\Big)\,e^{-\frac{t^2(|X|^2+|Y|^2)}{2n}} + \varepsilon \\ &= \Big(1 - \frac{(U^2+V^2)\,t^4 - 8R^2t^2}{4n}\Big)\,e^{-R^2t^2} + \varepsilon \end{aligned}$$

with a remainder term satisfying $|\varepsilon| \le \frac{c}{n^2}$ up to some absolute constant c. The latter yields

$$\frac{t^2}{n}\,\mathbb{E}\,|\,\langle X,Y\rangle\,|\,|\varepsilon| \;\le\; \frac{ct^2}{n^2}\,\mathbb{E}\,\frac{|X|^2+|Y|^2}{2n} \;=\; \frac{ct^2}{n^2},$$

assuming that $\mathbb{E}\,|X|^2 = n$. Hence, recalling (6.3), we obtain (6.2).

In the isotropic case, we have $\mathbb{E}\,|\,\langle X,Y\rangle\,| \le \sqrt{n}$, which leads to the corresponding improvement of the remainder term. □

7 Berry-Esseen bounds

The Kolmogorov distances between the distribution functions F_θ of the weighted sums $S_\theta = \langle X, \theta \rangle$ and the standard normal distribution function Φ can be explored by means of the Berry-Esseen-type bounds. They involve the characteristic functions

$$f_\theta(t) = \mathbb{E}\, e^{itS_\theta} = \int_{-\infty}^{\infty} e^{itx}\, dF_\theta(x), \qquad f(t) = \mathbb{E}_\theta f_\theta(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{7.1}$$

associated to $F_\theta(x)$ and the average distribution function $F(x) = \mathbb{E}_\theta F(x)$. Using the Λ-functional, let us state a few preliminary relations.

Lemma 7.1. *Given a random vector X in $\mathbb{R}^n$ such that $\mathbb{E}\,|X|^2 = n$, we have, for all $T \geq T_0 \geq 1$ and $\theta \in \mathbb{S}^{n-1}$,*

$$\begin{aligned} c\,\rho(F_\theta, \Phi) \ &\leq \ \int_0^{T_0} \frac{|f_\theta(t) - f(t)|}{t}\, dt \\ &\quad + \int_{T_0}^{T} \frac{|f_\theta(t)|}{t}\, dt + \frac{\Lambda}{n}\Big(1 + \log\frac{T}{T_0}\Big) + \frac{1}{T} + e^{-T_0^2/4}. \end{aligned} \tag{7.2}$$

The idea to involve two parameters T and T_0 stems upon the observation that the first integrand in (7.2) is small on a relatively moderate sized interval $[0, T_0]$ only, due to the concentration property of $f_\theta(t)$ about $f(t)$ as a function of θ (as discussed in Section 5). On the other hand, for $T_0 \leq t \leq T$ with a sufficiently large T, one may hope that both $f_\theta(t)$ and $f(t)$ will be just small in absolute value (in analogy with the case of independent components).

Proof. One can apply a general Berry-Esseen-type bound

$$c\,\rho(U, V) \ \leq \ \int_0^T \frac{|\hat{U}(t) - \hat{V}(t)|}{t}\, dt + \frac{1}{T}\int_0^T |\hat{V}(t)|\, dt \qquad (T > 0),$$

where U and V are arbitrary distribution functions with characteristic functions $\hat{U}$ and $\hat{V}$, respectively (cf. e.g. [7], [31], [32]). In particular, for all $\theta \in \mathbb{S}^{n-1}$,

$$c\,\rho(F_\theta, F) \ \leq \ \int_0^T \frac{|f_\theta(t) - f(t)|}{t}\, dt + \frac{1}{T}\int_0^T |f(t)|\, dt.$$

Splitting the integration in the first integral to the subintervals $[0, T_0]$ and $[T_0, T]$, $T \geq T_0 > 0$, we then have

$$\begin{aligned} c\,\rho(F_\theta, F) \ &\leq \ \int_0^{T_0} \frac{|f_\theta(t) - f(t)|}{t}\, dt \\ &\quad + \int_{T_0}^{T} \frac{|f_\theta(t)|}{t}\, dt + \int_{T_0}^{T} \frac{|f(t)|}{t}\, dt + \frac{1}{T}\int_0^T |f(t)|\, dt. \end{aligned} \tag{7.3}$$

The decay of the characteristic function $f(t)$ for large t can be controlled in terms of the variance-type functional $\sigma_4^2 = \frac{1}{n}\,\mathrm{Var}(|X|^2)$, which in turn satisfies $\sigma_4^2 \leq \Lambda$ according to the inequality (1.3) applied with coefficients $a_{ij} = 1$. Namely, write the definition (7.1) as

$$f(t) = \mathbb{E}\, J_n(t|X|), \qquad t \in \mathbb{R}.$$

Here, one may split the expectation into the event $A = \{|X|^2 \leq \frac{1}{2}\, n\}$ and its complement B. By the upper bound (3.1),

$$\mathbb{E}\,|J_n(t|X|)|\, 1_B \ \leq \ \mathbb{E}\,\Big(5\, e^{-t^2|X|^2/2n} + 4\, e^{-n/12}\Big)\, 1_B \ \leq \ 5\, e^{-t^2/4} + 4\, e^{-n/12}.$$

On the other hand, by Chebyshev's inequality,

$$\mathbb{P}(A) = \mathbb{P}\Big\{n - |X|^2 \geq \frac{1}{2}\,n\Big\} \ \leq\ \frac{\mathrm{Var}(|X|^2)}{(\frac{1}{2}\,n)^2} \ =\ \frac{4\sigma_4^2}{n} \ \leq\ \frac{4\Lambda}{n}. \tag{7.4}$$

Since $|J_n(s)| \leq 1$ for all $s \in \mathbb{R}$, we get

$$\mathbb{E}\,|J_n(t|X|)|\,1_A \ \leq\ \frac{4\Lambda}{n},$$

thus implying that $c\,|f(t)| \leq e^{-t^2/4} + \frac{\Lambda}{n}$ for all $t \in \mathbb{R}$, and therefore

$$\frac{c}{T}\int_0^T |f(t)|\,dt \ \leq\ \frac{\Lambda}{n} + \frac{1}{T}. \tag{7.5}$$

If $T_0 \geq 1$, then also

$$c\int_{T_0}^T \frac{|f(t)|}{t}\,dt \ \leq\ e^{-T_0^2/4} + \frac{\Lambda}{n}\,\log(T/T_0). \tag{7.6}$$

Using these bounds in the inequality (7.3), it is simplified to

$$\begin{aligned} c\,\rho(F_\theta, F) \ \ &\leq\ \ \int_0^{T_0} \frac{|f_\theta(t) - f(t)|}{t}\,dt \\ &\quad + \int_{T_0}^T \frac{|f_\theta(t)|}{t}\,dt + \frac{\Lambda}{n}\Big(1 + \log\frac{T}{T_0}\Big) + \frac{1}{T} + e^{-T_0^2/4}. \end{aligned}$$

The variance functional may also be used to quantify closeness of F to the standard normal distribution function via the inequality (cf. [11])

$$c\,\rho(F, \Phi) \leq \frac{1}{n}\,(1 + \sigma_4^2).$$

Since $\sigma_4^2 \leq \Lambda$, (7.2) immediately follows in view of the triangle inequality for the Kolmogorov metric. $\square$

Lemma 7.1 may be used to derive the following upper bound on average which represents a generalization of the inequality (1.4).

Lemma 7.2. *Given an isotropic random vector X in $\mathbb{R}^n$, with $T_0 = 4\sqrt{\log n}$ we have*

$$c\,\mathbb{E}_\theta\,\rho(F_\theta, \Phi) \leq \frac{\log n}{n}\,\Lambda + \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\,dt, \tag{7.7}$$

where $I(t)$ denotes the squared L^2-norm of the linear part of $f_\theta(t)$ in $L^2(\mathfrak{s}_{n-1})$.

Proof. When bounding $\rho(F_\theta, \Phi)$ on average with respect to $\mathfrak{s}_{n-1}$, the inequality (7.6) is actually not needed. Using Jensen's inequality $|f(t)| \leq \mathbb{E}_\theta\,|f_\theta(t)|$, from (7.3) and (7.5) we obtain that, for all $T \geq T_0 \geq 1$,

$$c\,\mathbb{E}_\theta\,\rho(F_\theta, F) \ \leq\ \int_0^{T_0} \frac{\mathbb{E}_\theta\,|f_\theta(t) - f(t)|}{t}\,dt + \int_{T_0}^T \frac{\mathbb{E}_\theta\,|f_\theta(t)|}{t}\,dt + \frac{1}{T} + \frac{\Lambda}{n}. \tag{7.8}$$

Now, as was shown in [12] (Lemma 5.2 specialized to the parameter $p = 2$), for all $t \in \mathbb{R}$,

$$c\,\mathbb{E}_\theta\,|f_\theta(t)| \ \leq\ \frac{m_4^2 + \sigma_4^2}{n} + e^{-t^2/16}, \quad m_4 = \frac{1}{\sqrt{n}}\,\big(\mathbb{E}\,\langle X, Y\rangle^4\big)^{1/4}, \tag{7.9}$$

where Y is an independent copy of X. Using a simple relation $m_4 \leq M_4^2$ (Corollary 2.3 in [12]), one may also involve the functional

$$M_4 = \sup_{\theta \in \mathbb{S}^{n-1}} (\mathbb{E}\, \langle X, \theta \rangle^4)^{1/4}.$$

It may be bounded in terms of Λ as well as σ_4^2. Indeed, applying (1.3) with $a_{ij} = \theta_i \theta_j$, we get

$$\mathrm{Var}(\langle X, \theta \rangle^2) \leq \Lambda, \quad \theta \in S^{n-1},$$

which implies $M_4^4 \leq 1 + \Lambda \leq 3\Lambda$ in the isotropic case. This allows us to replace (7.9) with

$$c\, \mathbb{E}_\theta\, |f_\theta(t)| \;\leq\; \frac{\Lambda}{n} + e^{-t^2/16}.$$

Applying the latter in (7.8), this inequality is simplified to

$$c\, \mathbb{E}\, \rho(F_\theta, \Phi) \;\leq\; \int_0^{T_0} \frac{\mathbb{E}\, |f_\theta(t) - f(t)|}{t}\, dt + \frac{\Lambda}{n} \Big(1 + \log \frac{T}{T_0}\Big) + \frac{1}{T} + e^{-T_0^2/16}. \tag{7.10}$$

Here, the integral can be bounded by virtue of the L^2-bound (5.8) which yields

$$c\, \mathbb{E}_\theta\, |f_\theta(t) - f(t)| \;\leq\; \sqrt{I(t)} + \frac{t^2}{n} \sqrt{\Lambda}$$

for $|t| \leq A n^{1/5}$ with a prescribed constant $A > 0$. This gives

$$c \int_0^{T_0} \frac{\mathbb{E}\, |f_\theta(t) - f(t)|}{t}\, dt \;\leq\; \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\, dt + \frac{T_0^2}{2n} \sqrt{\Lambda},$$

as long as $T_0 \leq A n^{1/5}$. Applying this in (7.10), we arrive at

$$c\, \mathbb{E}_\theta\, \rho(F_\theta, \Phi) \;\leq\; \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\, dt + \frac{\Lambda}{n} \Big(1 + \log \frac{T}{T_0}\Big) + \frac{1}{T} + \frac{T_0^2}{n} \sqrt{\Lambda} + e^{-T_0^2/16}.$$

Finally, choosing $T = 4n$, $T_0 = 4\sqrt{\log n}$, we obtain (7.7). □

8 Large deviations related to moderate sized and long intervals

A similar argument can be used when bounding the ψ_1-Orlicz norm of $\rho(F_\theta, \Phi)$. As a preliminary step, let us start with the first integral in (7.2) over the moderate interval. Applying now the inequality (5.9), we have

$$\begin{aligned} c \Big\| \int_0^{T_0} |f_\theta(t) - f(t)|\, \frac{dt}{t} \Big\|_{\psi_1} &\leq c \int_0^{T_0} \|f_\theta(t) - f(t)\|_{\psi_1} \frac{dt}{t} \\ &\leq \int_0^{T_0} \Big(\sqrt{I(t)} + \frac{\Lambda t^2}{n}\Big) \frac{dt}{t} \\ &= \frac{\Lambda}{2n}\, T_0^2 + \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\, dt, \end{aligned}$$

which is used with the same parameter T_0 as in Lemma 7.2. In general, by Markov's inequality,

$$\mathfrak{s}_{n-1}\big\{|\xi| \geq r \|\xi\|_{\psi_1}\big\} \leq 2\, e^{-r}, \quad r > 0.$$

Hence, we get:

Lemma 8.1. *Let X be an isotropic random vector in $\mathbb{R}^n$. For all $r > 0$, with $T_0 = 4\sqrt{\log n}$,*

$$\mathfrak{s}_{n-1}\bigg\{c\int_0^{T_0} \frac{|f_\theta(t) - f(t)|}{t}\,dt \geq \frac{\Lambda \log n}{n}\,r + r\int_0^{T_0} \frac{\sqrt{I(t)}}{t}\,dt\bigg\} \,\leq\, 2\,e^{-r}.$$

Outside the moderate sized interval, that is, on the long interval $[T_0, T]$, both $|f(t)|$ and $|f_\theta(t)|$ are expected to be small for most of θ. To study this property, let us consider the growth of the moments of the integral

$$L(\theta) = \int_{T_0}^T \frac{|f_\theta(t)|}{t}\,dt. \tag{8.1}$$

Lemma 8.2. *Let $X^{(k)}$, $Y^{(k)}$ $(k = 1, \dots, p)$ be independent copies of a random vector X in $\mathbb{R}^n$. For the integral in* (8.1) *with parameters $T_0 = 4\sqrt{\log n}$ and $T = T_0 n$, we have*

$$\mathbb{E}_\theta\, L(\theta)^{2p} \,\leq\, (c\log n)^{2p}\,\big(p^{2p}\,n^{-2p} + \mathbb{P}(A)\big), \tag{8.2}$$

where

$$A = \Big\{|\Sigma_p|^2 \leq \frac{np}{2}\Big\}, \quad \Sigma_p = \sum_{k=1}^p (X^{(k)} - Y^{(k)}).$$

Proof. By Hölder's inequality,

$$L(\theta)^{2p} \,\leq\, \log^{2p-1}\Big(\frac{T}{T_0}\Big)\int_{T_0}^T \frac{|f_\theta(t)|^{2p}}{t}\,dt,$$

so that

$$\mathbb{E}_\theta\, L(\theta)^{2p} \,\leq\, \log^{2p-1}\Big(\frac{T}{T_0}\Big)\int_{T_0}^T \frac{\mathbb{E}_\theta\,|f_\theta(t)|^{2p}}{t}\,dt.$$

Since $|f_\theta(t)|^{2p} = \mathbb{E}\,e^{it\langle\Sigma_p,\theta\rangle}$, we may write

$$\mathbb{E}_\theta\,|f_\theta(t)|^{2p} \,=\, \mathbb{E} J_n(t\,|\Sigma_p|).$$

Thus,

$$\mathbb{E}_\theta\, L(\theta)^{2p} \,\leq\, \log^{2p-1}\Big(\frac{T}{T_0}\Big)\int_{T_0}^T \mathbb{E} J_n(t\,|\Sigma_p|)\,\frac{dt}{t}.$$

Next, we split the expectation to the event A and its complement $B = \big\{|\Sigma_p|^2 > \frac{np}{2}\big\}$. Applying the upper bound (3.1), we get

$$\begin{aligned}\int_{T_0}^T \mathbb{E} J_n(t\,|\Sigma_p|)\,1_B\,\frac{dt}{t} \quad &\leq \quad \int_{T_0}^T \frac{5\,e^{-pt^2/4} + 4\,e^{-n/12}}{t}\,dt \\ &\leq \quad (5\,e^{-pT_0^2/4} + 4\,e^{-n/12})\log\Big(\frac{T}{T_0}\Big),\end{aligned}$$

while

$$\int_{T_0}^T \mathbb{E} J_n(t|\Sigma_p|)\,1_A\,\frac{dt}{t} \,\leq\, \mathbb{P}(A)\log\Big(\frac{T}{T_0}\Big)$$

(since $|J(s)| \leq 1$ for all $s \in \mathbb{R}$). Hence,

$$\mathbb{E}_\theta\, L(\theta)^{2p} \,\leq\, c\log^{2p}\Big(\frac{T}{T_0}\Big)\Big(e^{-pT_0^2/4} + e^{-n/12} + \mathbb{P}(A)\Big).$$

For the choice $T_0 = 4\sqrt{\log n}$, $T = T_0 n$, this leads to

$$\mathbb{E}_\theta\, L(\theta)^{2p} \,\leq\, c\,(\log n)^{2p}\,\big(n^{-4p} + e^{-n/12} + \mathbb{P}(A)\big).$$

Using the inequality $x^{2p}\,e^{-x} \leq p^{2p}$ $(x \geq 0)$, we have $e^{-n/12} \leq (12\,p)^{2p}\,n^{-2p}$, and the above bound is simplified to (8.2). $\square$

9 Concentration in presence of Poincaré-type inequalities

In order to simplify the bounds in Lemma 7.2 and Lemmas 8.1–8.2, we need more information about the distribution of X, which would allow us to say more on the involved function $I(t)$ and the probability of the event A as in Lemma 8.2. To this aim, our starting hypothesis will be described by Poincaré-type inequalities.

Let us first recall several concentration results, assuming that the random vector $X = (X_1, \dots, X_n)$ in $\mathbb{R}^n$ admits the Poincaré-type inequality

$$\lambda_1 \operatorname{Var}(u(X)) \le \mathbb{E}\, |\nabla u(X)|^2 \tag{9.1}$$

for all smooth functions u on $\mathbb{R}^n$ with a positive constant λ_1. As was discovered by Gromov and Milman [20] and by Borovkov and Utev [14], deviations of random variables $u(X)$ from their means are subexponential, as long as u is a Lipschitz function on $\mathbb{R}^n$ (cf. also [1], [28]). In a somewhat optimal way, worst possible deviations of $u(X)$ are described in the following assertion proved in [4].

Proposition 9.1. *If the function $u : \mathbb{R}^n \to \mathbb{R}$ has a Lipschitz semi-norm $\|u\|_{\mathrm{Lip}} \le 1$, then, for any $r \ge 0$,*

$$\mathbb{P}\big\{u(X) - \mathbb{E}\, u(X) \ge r\big\} \le 3\, e^{-2\sqrt{\lambda_1}\, r}. \tag{9.2}$$

Using a smoothing argument, the inequality (9.1) may be extended to all locally Lipschitz functions, in which case the modulus of the gradient may be understood as an upper semi-continuous function

$$|\nabla u(x)| = \limsup_{y_1, y_2 \to x} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|}, \quad x \in \mathbb{R}.$$

In terms of partial derivatives, it leads to the usual expression $\big(\sum_{k=1}^n (\partial_{x_k} u(x))^2\big)^{1/2}$ assuming that u is differentiable at the point x.

If the function u is not Lipschitz (for example, a polynomial), the bound (9.2) is no longer true, and a more general variant of Proposition 9.1 is needed, which would allow us to control probabilities of large deviations. To this aim, proper bounds on the L^p-norms of u in terms of the L^p-norms of the modulus of the gradient are useful.

Proposition 9.2. *Given a locally Lipschitz function u on $\mathbb{R}^n$, suppose that the moment $\mathbb{E}\, |\nabla u(X)|^p$ is finite for $p \ge 2$. Then, $u(X)$ has finite absolute moments up to order p, and*

$$\mathbb{E}\, |u(X) - \mathbb{E}\, u(X)|^p \le \Big(\frac{p}{\sqrt{2\lambda_1}}\Big)^p\, \mathbb{E}\, |\nabla u(X)|^p. \tag{9.3}$$

Proof. Let us include a simple argument, assuming that the function u is C^1-smooth. By the subadditivity property of the variance functional (cf. [27]), the Poincaré-type inequality (9.1) for the distribution μ of X on $\mathbb{R}^n$ is extended to the same relation on $\mathbb{R}^n \times \mathbb{R}^n$

$$\lambda_1 \operatorname{Var}_{\mu \otimes \mu}(f) \le \iint |\nabla f(x,y)|^2 \, d\mu(x) d\mu(y) \tag{9.4}$$

with respect to the product measure $\mu^2 = \mu \otimes \mu$. Here, for any C^1-smooth function $f = f(x,y)$, the modulus of the gradient is given by

$$|\nabla f(x,y)|^2 = |\nabla_x f(x,y)|^2 + |\nabla_y f(x,y)|^2.$$

Let us apply this $2n$-dimensional Poincaré-type inequality to the function

$$f(x,y) = |u(x) - u(y)|^{\frac{p}{2}} \operatorname{sign}(u(x) - u(y)),$$

which is C^1-smooth in the case $p > 2$. Its modulus of the gradient is given by

$$|\nabla f(x,y)| \,=\, \frac{p}{2}\,|u(x) - u(y)|^{\frac{p}{2}-1}\sqrt{|\nabla u(x)|^2 + |\nabla u(y)|^2}.$$

Since f has a symmetric distribution under μ^2, applying (9.4) together with Hölder's inequality, we conclude that

$$\begin{aligned}
&\lambda_1 \iint |u(x) - u(y)|^p\,d\mu^2(x,y) \\
&\le\, \frac{p^2}{4}\,\iint |u(x) - u(y)|^{p-2}\Big(|\nabla u(x)|^2 + |\nabla u(y)|^2\Big)\,d\mu^2(x,y) \\
&\le\, \frac{p^2}{4}\,\bigg(\iint |u(x) - u(y)|^p\,d\mu^2(x,y)\bigg)^{\frac{p-2}{p}}\,\bigg(\iint \Big(|\nabla u(x)|^2 + |\nabla u(y)|^2\Big)^{\frac{p}{2}}\,d\mu^2(x,y)\bigg)^{\frac{2}{p}}.
\end{aligned}$$

By Jensen's inequality, the last double integral does not exceed

$$2^{\frac{p}{2}-1}\iint \big(|\nabla u(x)|^p + |\nabla u(y)|^p\big)\,d\mu^2(x,y) \,=\, 2^{\frac{p}{2}}\int |\nabla u|^p\,d\mu,$$

and hence

$$\lambda_1\bigg(\iint |u(x) - u(y)|^p\,d\mu^2(x,y)\bigg)^{\frac{2}{p}} \,\le\, \frac{p^2}{2}\,\bigg(\int |\nabla u|^p\,d\mu\bigg)^{\frac{2}{p}}.$$

Equivalently,

$$\iint |u(x) - u(y)|^p\,d\mu^2(x,y) \,\le\, \Big(\frac{p}{\sqrt{2\lambda_1}}\Big)^p\,\int |\nabla u|^p\,d\mu.$$

If the right integral is finite, so is the left one, thus u is integrable. Moreover, the left integral is greater than or equal to $\int |u(x) - \mathbb{E}\,u(X)|^p\,d\mu(x)$ (by Jensen's inequality). □

Let us now connect the Poincaré constant with small ball probabilities.

Corollary 9.3. *If* $\mathbb{E}\,|X|^2 = n$*, then*

$$\mathbb{P}\Big\{|X|^2 \le \frac{1}{4}\,n\Big\} \,\le\, 3\,e^{-\frac{1}{2}\sqrt{\lambda_1 n}}. \tag{9.5}$$

Proof. Applying (9.2) to the function $u(x) = -|x|$, we have

$$\mathbb{P}\big\{|X| - \mathbb{E}\,|X| \le -r\big\} \,\le\, 3\,e^{-2\sqrt{\lambda_1}\,r}, \quad r \ge 0. \tag{9.6}$$

One can bound $\mathbb{E}\,|X|$ from below by virtue of the Poincaré-type inequality (9.1) which gives

$$n - (\mathbb{E}\,|X|)^2 = \mathrm{Var}(|X|) \le \frac{1}{\lambda_1}.$$

In the case $\lambda_1 n \ge \frac{4}{3}$, this implies $\mathbb{E}\,|X| \ge \sqrt{n - \frac{1}{\lambda_1}} \ge \frac{1}{2}\sqrt{n}$. Hence, applying (9.6) with $r = \mathbb{E}\,|X| - \frac{1}{2}\sqrt{n}$, we get

$$\mathbb{P}\Big\{|X| \le \frac{1}{2}\sqrt{n}\Big\} \le 3\,e^{-2\sqrt{\lambda_1}\,r}.$$

Here $r \ge \sqrt{n - \frac{1}{\lambda_1}} - \frac{1}{2}\sqrt{n} \ge \frac{1}{4}\sqrt{n}$ under a stronger assumption $\lambda_1 n \ge \frac{16}{7}$, in which case the above bound yields the desired inequality (9.5).

It remains to note that (9.5) is fulfilled automatically when $\lambda_1 n < \frac{16}{7}$, since then the right-hand side is greater than 1. □

Let us give another version of this statement for convolutions, namely, for sums

$$\Sigma_p = \sum_{k=1}^{p} (X^{(k)} - Y^{(k)}),$$

where $X^{(k)}$, $Y^{(k)}$ ($1 \le k \le p$) are independent copies of X.

Corollary 9.4. *If X has mean zero, and $\mathbb{E}\,|X|^2 = n$, then*

$$\mathbb{P}\Big\{|\Sigma_p|^2 \le \frac{np}{2}\Big\} \,\le\, 3\,e^{-\frac{1}{3}\sqrt{\lambda_1 n}}.$$

Proof. One may use the property that the product measure $\mu^{\otimes 2p}$ on $(\mathbb{R}^n)^{2p} = \mathbb{R}^{2pn}$ has the same Poincaré constant λ_1 as the distribution μ of X. The function

$$u(x_1,\dots,x_p,y_1,\dots,y_p) \,=\, -\Big|\sum_{k=1}^{p} (x_k - y_k)\Big|, \quad x_k, y_k \in \mathbb{R}^n,$$

has Lipschitz semi-norm $\sqrt{2p}$ with respect to the Euclidean distance on $\mathbb{R}^{2pn}$. Therefore, according to Proposition 9.1, it admits an exponential inequality

$$\mu^{\otimes 2p}\{u - m \ge r\} \,\le\, 3e^{-2\sqrt{\lambda_1}\, r/\sqrt{2p}} \qquad (r > 0),$$

where m is the $\mu^{\otimes 2p}$-mean of u. That is,

$$\mathbb{P}\big\{|\Sigma_p| - \mathbb{E}\,|\Sigma_p| \le -r\big\} \,\le\, 3e^{-2\sqrt{\lambda_1}\, r/\sqrt{2p}}. \tag{9.7}$$

By the Poincaré-type inequality on the product space, and using $\mathbb{E}\,|\Sigma_p|^2 = 2pn$, we have

$$2pn - (\mathbb{E}\,|\Sigma_p|)^2 \le \frac{2p}{\lambda_1} \le pn,$$

where the last inequality holds true when $\lambda_1 n \ge 2$. In this case, $\mathbb{E}\,|\Sigma_p| \ge \sqrt{pn}$, and applying (9.7) with $r = (1 - \frac{1}{\sqrt{2}})\sqrt{pn}$, we obtain that

$$\mathbb{P}\Big\{|\Sigma_p| \le \frac{1}{\sqrt{2}}\sqrt{np}\Big\} \,\le\, 3\,e^{-(\sqrt{2}-1)\sqrt{\lambda_1 n}} < 3\,e^{-\frac{1}{3}\sqrt{\lambda_1 n}}.$$

In the case $\lambda_1 n \le 2$, the inequality of the corollary is fulfilled automatically. $\square$

Remark 9.5. If the random vector X in $\mathbb{R}^n$ ($n \ge 2$) is isotropic, then necessarily $\lambda_1 \le 1$. Indeed, applying (9.1) with linear functions $u(x) = \langle x, \theta\rangle$, we get

$$\lambda_1 \left(1 - \langle a, \theta\rangle^2\right) \le 1, \quad \theta \in \mathbb{S}^{n-1},$$

where $a = \mathbb{E}X$. Since one may choose θ to be orthogonal to the vector a, the conclusion follows. The upper bound $\lambda_1 \le 1$ is also valid in dimension $n = 1$, as long as $\mathbb{E}X = 0$ (however, we only have $\lambda_1 \le 1/\mathrm{Var}(X)$ without the mean zero assumption).

10 The case of non-symmetric distributions

In order to extend the bound

$$\mathbb{E}_\theta\, \rho(F_\theta, \Phi) \le \frac{c\, \log n}{n}\, \Lambda \tag{10.1}$$

to the case where the distribution of X is not necessarily symmetric about the origin, we need to employ more sophisticated results reflecting the size of the linear part of the characteristic functions $f_\theta(t)$ in $L^2(\mathfrak{s}_{n-1})$ with respect to the variable θ. This may be achieved at the expense of a certain term that has to be added to the right-hand side in (10.1). More precisely, we derive the following:

Proposition 10.1. *Given an isotropic random vector $X = (X_1, \dots, X_n)$ in $\mathbb{R}^n$,*

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi)\ \le\ \frac{\log n}{n}\,\Lambda + \Big(\frac{\log n}{n}\Big)^{1/4}\bigg(\mathbb{E}\,\frac{\langle X,Y\rangle}{\sqrt{|X|^2+|Y|^2}}\bigg)^{1/2}, \tag{10.2}$$

where Y is an independent copy of X.

The ratio $\langle X,Y\rangle/\sqrt{|X|^2+|Y|^2}$ is understood to be zero in the case $X = Y = 0$. Note that the last expectation in (10.2) is non-negative which follows from the representation

$$\mathbb{E}\,\frac{\langle X,Y\rangle}{\sqrt{|X|^2+|Y|^2}}\ =\ \frac{2}{\sqrt{\pi}}\int_0^\infty \sum_{k=1}^n \Big(\mathbb{E}X_k\,e^{-|X|^2r^2}\Big)^2\,dr.$$

If the distribution of X is symmetric, this expectation is vanishing, and in (10.2) we return to (10.1).

Returning to Proposition 6.1, define the random variables

$$R^2 = \frac{|X|^2+|Y|^2}{2n}\quad (R\ge 0),\qquad U = \frac{|X|^2}{n},\ V = \frac{|Y|^2}{n},$$

and recall that in the isotropic case the squared L^2-norm of the linear part of the characteristic function $f_\theta(t)$ of the weighted sums $\langle X,\theta\rangle$ admits an asymptotic representation

$$I(t) = \frac{t^2}{n}\,\mathbb{E}\,\langle X,Y\rangle\,\Big(1-\frac{(U^2+V^2)\,t^4-8R^2t^2}{4n}\Big)\,e^{-R^2t^2} + O(t^2n^{-5/2}). \tag{10.3}$$

Lemma 10.2. *If X is isotropic, then, putting $T_0 = 4\sqrt{\log n}$, we have*

$$\int_0^{T_0}\frac{I(t)}{t^2}\,dt\ \le\ \frac{c}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} + O\big(\Lambda^2 n^{-2}\big). \tag{10.4}$$

Proof. Introduce the events $A = \{R\le\frac12\}$ and $B = \{R>\frac12\}$. Starting from (10.3), we have

$$\begin{aligned}\int_0^{T_0}\frac{I(t)}{t^2}\,dt\ &=\ \frac{1}{n}\,\mathbb{E}\,\langle X,Y\rangle\int_0^{T_0} e^{-R^2t^2}\,dt\\ &\quad+\frac{2}{n^2}\,\mathbb{E}\,\langle X,Y\rangle\int_0^{T_0} R^2t^2e^{-R^2t^2}\,dt\\ &\quad-\frac{1}{4n^2}\,\mathbb{E}\,\langle X,Y\rangle\int_0^{T_0}(U^2+V^2)\,t^4e^{-R^2t^2}\,dt + O(n^{-2}).\end{aligned}$$

After the change of the variable $Rt = s$ (assuming without loss of generality that $R>0$) and putting $T_1 = RT_0$, the above is simplified to

$$\begin{aligned}\int_0^{T_0}\frac{I(t)}{t^2}\,dt\ &=\ \frac{1}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\int_0^{T_1} e^{-s^2}\,ds\\ &\quad+\frac{2}{n^2}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\int_0^{T_1} s^2e^{-s^2}\,ds\\ &\quad-\frac{1}{4n^2}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\,\frac{U^2+V^2}{R^4}\int_0^{T_1} s^4e^{-s^2}\,ds + O(n^{-2}).\end{aligned}$$

At the expense of a small error, integration here may be extended from the interval $[0,T_1]$ to the whole half-axis $(0,\infty)$. To see this, one can use the estimates

$$\int_{T_1}^\infty e^{-s^2}\,ds < \int_{T_1}^\infty s^2\,e^{-s^2}\,ds < \int_{T_1}^\infty s^4e^{-s^2}\,ds < c\,e^{-T_1^2/2}\qquad (T_1>1),$$

together with

$$\Big|\frac{\langle X,Y\rangle}{R}\Big| \le \frac{|X|\,|Y|}{R} \le \frac{|X|^2+|Y|^2}{2R} = Rn. \tag{10.5}$$

As was already noted in (7.4),

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\Big\{|X|^2+|Y|^2 \le \frac{n}{2}\Big\} \\ &\le \mathbb{P}\Big\{|X|^2 \le \frac{n}{2}\Big\}\,\mathbb{P}\Big\{|Y|^2 \le \frac{n}{2}\Big\} \,\le\, \frac{16\Lambda^2}{n^2}. \end{aligned} \tag{10.6}$$

Since on the set B, we have $T_1^2 = 16R^2\log n > 4\,\log n$, and due to $\mathbb{E}R^2 = 1$, it follows that

$$\begin{aligned} \mathbb{E}\,R\,e^{-T_1^2/2} &= \mathbb{E}\,R\,e^{-T_1^2/2}\,1_A + \mathbb{E}\,R\,e^{-T_1^2/2}\,1_B \\ &\le \frac{1}{2}\,\mathbb{P}(A) + \frac{1}{n^2}\,\mathbb{E}R \,\le\, \frac{c\Lambda^2}{n^2}, \end{aligned}$$

where we used the lower bound $\Lambda \ge \frac{1}{2}$. Hence

$$\mathbb{E}\,\frac{|\,\langle X,Y\rangle\,|}{R}\int_{T_1}^\infty e^{-s^2}\,ds \,\le\, n\,\mathbb{E}Re^{-T_1^2/2} \,\le\, \frac{c\Lambda^2}{n}.$$

By a similar argument,

$$\mathbb{E}\,\frac{|\,\langle X,Y\rangle\,|}{R}\int_{T_1}^\infty s^2e^{-s^2}\,ds \,\le\, cn\,\mathbb{E}R\,e^{-T_1^2/2} \,\le\, \frac{c\Lambda^2}{n}.$$

Using

$$\frac{U^2+V^2}{R^4} = \frac{4\,(U^2+V^2)}{(U+V)^2} \,\le\, 4,$$

we also have

$$\mathbb{E}\,\frac{|\,\langle X,Y\rangle\,|}{R}\,\frac{U^2+V^2}{R^4}\int_{T_1}^\infty s^4e^{-s^2}\,ds \,\le\, cn\,\mathbb{E}R\,e^{-T_1^2/2} \,\le\, \frac{c\Lambda^2}{n}.$$

Thus, extending the integration to the positive half-axis, we get

$$\begin{aligned} \int_0^{T_0}\frac{I(t)}{t^2}\,dt &= \frac{c_1}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} + \frac{c_2}{n^2}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \\ &\quad - \frac{c_3}{n^2}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\,\frac{U^2+V^2}{R^4} + O\big(\Lambda^2 n^{-2}\big) \end{aligned}$$

with some absolute constants $c_j > 0$. Moreover, using the identity

$$\frac{U^2+V^2}{R^4} = 2 + \frac{(U-V)^2}{2R^4} = 2 + 2\,\frac{(U-V)^2}{(U+V)^2}$$

and recalling that $\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \ge 0$, it follows that, with some other positive absolute constants

$$\int_0^{T_0}\frac{I(t)}{t^2}\,dt \,\le\, \frac{c_1}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} - \frac{c_2}{n^2}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\,\frac{(U-V)^2}{(U+V)^2} + O\big(\Lambda^2 n^{-2}\big). \tag{10.7}$$

To get rid of the last expectation (by showing that it is bounded by a dimension free quantity), first note that, by (10.5), the expression under this expectation is bounded in absolute value by Rn. Hence, applying Cauchy's inequality and using $\mathbb{E}R^2 = 1$, from (10.6) we obtain that

$$\begin{aligned} \mathbb{E}\,\Big|\frac{\langle X,Y\rangle}{R}\Big|\,\frac{(U-V)^2}{(U+V)^2}\,1_A &\le \mathbb{E}\,\Big|\frac{\langle X,Y\rangle}{R}\Big|\,1_A \\ &\le n\,\mathbb{E}R\,1_A \,\le\, n\sqrt{\mathbb{P}(A)} \,\le\, 4\Lambda. \end{aligned} \tag{10.8}$$

Turning to the complementary set, note that on B, we have $|\frac{\langle X,Y\rangle}{R}| \leq 2\,|\langle X,Y\rangle|$, while

$$\frac{(U-V)^2}{(U+V)^2} \leq \frac{|U-V|}{U+V} = \frac{|U-V|}{2R^2} \leq 2\,|U-V|.$$

Hence, by Cauchy's inequality, and using $\mathbb{E}\,\langle X,Y\rangle^2 = n$, we get

$$\begin{aligned}
\mathbb{E}\Big|\frac{\langle X,Y\rangle}{R}\Big|\,\frac{(U-V)^2}{(U+V)^2}\,1_B &\leq 4\,\mathbb{E}\,|\langle X,Y\rangle|\,|U-V| \\
&\leq 4\sqrt{n}\,\sqrt{\mathbb{E}\,(U-V)^2} = 4\sqrt{2}\,\sigma_4 \leq 4\sqrt{2\Lambda}.
\end{aligned}$$

Combining this bound with (10.8), we finally obtain that

$$\mathbb{E}\Big|\frac{\langle X,Y\rangle}{R}\Big|\,\frac{(U-V)^2}{(U+V)^2} \leq c\Lambda.$$

As a result, we arrive in (10.7) at the bound (10.4). □

Proof of Proposition 10.1. We employ the bound (7.7) of Lemma 7.2 which was stated with $T_0 = 4\sqrt{\log n}$. Using Cauchy's inequality and applying (10.4), it gives

$$\begin{aligned}
c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi) &\leq \frac{\log n}{n}\,\Lambda + \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\,dt \\
&\leq \frac{\log n}{n}\,\Lambda + \sqrt{T_0}\,\bigg(\int_0^{T_0} \frac{I(t)}{t^2}\,dt\bigg)^{1/2} \\
&\leq \frac{\log n}{n}\,\Lambda + c'\sqrt{T_0}\,\bigg(\frac{1}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} + \frac{\Lambda^2}{n^2}\bigg)^{1/2}.
\end{aligned}$$

Simplifying the expression on the right-hand side, we arrive at (10.2). □

11 The estimate on average

Let us rewrite the bound (10.2) as

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi) \leq \frac{\log n}{n}\,\Lambda + \frac{(\log n)^{1/4}}{\sqrt{n}}\Big(\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\Big)^{1/2}, \tag{11.1}$$

where $R^2 = \frac{1}{2n}\,(|X|^2 + |Y|^2)$, $R \geq 0$, and where Y is an independent copy of X. In the next step, we are going to simplify the last expectation in terms of λ_1. Note that, under our standard assumptions as in Proposition 10.1,

$$\mathbb{E}R^2 = 1, \quad \mathrm{Var}(R^2) = \frac{\sigma_4^2}{2n} \leq \frac{\Lambda}{2n}.$$

Hence, with high probability the ratio $\frac{\langle X,Y\rangle}{R}$ is almost $\langle X,Y\rangle$ which in turn has zero expectation, as long as X has mean zero. However, in general it is not clear whether or not this approximation is sufficient to make further simplification. Nevertheless, the approximation $R^2 \sim 1$ is indeed sufficiently strong, for example, in the case where the distribution μ of X satisfies the Poincaré-type inequality (1.5).

Lemma 11.1. *Let X be an isotropic random vector in $\mathbb{R}^n$ with mean zero and a positive Poincaré constant λ_1, and let Y be an independent copy of X. Then*

$$\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \leq \frac{c}{\lambda_1^2\,n}. \tag{11.2}$$

Applying (11.2) in (11.1) and using $\Lambda \leq 4/\lambda_1$ (cf. [13], Proposition 3.4), we get an estimate on average

$$c\,\mathbb{E}_\theta\,\rho(F_\theta,\Phi)\ \leq\ \frac{\log n}{n}\,\frac{1}{\lambda_1} + \frac{(\log n)^{1/4}}{\sqrt{n}}\,\frac{1}{\lambda_1\sqrt{n}},$$

thus proving the relation (1.6).

Proof of Lemma 11.1. Without loss of generality, assume that $R > 0$ a.s. Put $\delta_n = \frac{1}{\lambda_1 n}$.

We apply the Poincaré-type inequality for the product measure $\mu \otimes \mu$,

$$\iint |u(x,y)|^2\,d\mu(x)\,d\mu(y)\ \leq\ \frac{1}{\lambda_1}\iint |\nabla u(x,y)|^2\,d\mu(x)\,d\mu(y), \tag{11.3}$$

which holds true for any smooth function u on $\mathbb{R}^n \times \mathbb{R}^n$ with $(\mu\otimes\mu)$-mean zero. Moreover, according to the inequality (9.3), for any $p \geq 2$,

$$\iint |u(x,y)|^p\,d\mu(x)\,d\mu(y)\ \leq\ \frac{p^p}{(2\lambda_1)^{p/2}}\iint |\nabla u(x,y)|^p\,d\mu(x)\,d\mu(y). \tag{11.4}$$

By Corollary 9.3 applied in $\mathbb{R}^{2n}$ to the random vector (X,Y), it also follows that the event $A = \{R \leq \frac{1}{2}\}$ has probability

$$\mathbb{P}(A) \leq 3e^{-\sqrt{\lambda_1 n/2}}.$$

Using

$$|\,\langle X,Y\rangle\,| \leq R^2 n, \tag{11.5}$$

cf. (10.5), we have

$$\mathbb{E}\,\frac{|\,\langle X,Y\rangle\,|}{R}\,1_A\ \leq\ n\,\mathbb{E}R\,1_A\ \leq\ \frac{n}{2}\,\mathbb{P}(A)\ \leq\ \frac{3n}{2}\,e^{-\sqrt{\lambda_1 n/2}}\ \leq\ \frac{c}{\lambda_1^2\,n}. \tag{11.6}$$

Similarly,

$$\mathbb{E}\,|\,\langle X,Y\rangle\,|\,1_A\ \leq\ \frac{n}{4}\,\mathbb{P}(A)\ \leq\ \frac{c}{\lambda_1^2\,n},$$

and since X has mean zero, for the complementary set $B = \{R > \frac{1}{2}\}$ we have the same bound

$$\big|\,\mathbb{E}\,\langle X,Y\rangle\,1_B\big| \leq \frac{c}{\lambda_1^2\,n}.$$

Using once more (11.5), on the set A we also have

$$\mathbb{E}\,|\,\langle X,Y\rangle\,|\,R^2\,1_A\ \leq\ \frac{n}{4}\,\mathbb{P}(A)\ \leq\ \frac{c}{\lambda_1^2 n}$$

and

$$\mathbb{E}\,|\,\langle X,Y\rangle\,|\,R^4\,1_A\ \leq\ \frac{n}{16}\,\mathbb{P}(A)\ \leq\ \frac{c}{\lambda_1^2\,n}.$$

Now, consider the function $w(\varepsilon) = (1+\varepsilon)^{-1/2}$ on the half-axis $\varepsilon \geq -\frac{3}{4}$. By Taylor's formula, for some point ε_1 between $-\frac{3}{4}$ and ε,

$$w(\varepsilon)\ =\ 1 - \frac{1}{2}\,\varepsilon + \frac{3}{8}\,\varepsilon^2 - \frac{5}{16}\,(1+\varepsilon_1)^{-7/2}\,\varepsilon^3\ =\ 1 - \frac{1}{2}\,\varepsilon + \frac{3}{8}\,\varepsilon^2 - \beta\varepsilon^3$$

with some $0 \leq \beta \leq 40$. Putting $\varepsilon = R^2 - 1$, we then get on the set B

$$\begin{aligned}
\frac{\langle X,Y\rangle}{R} &= \langle X,Y\rangle - \frac{1}{2}\,\langle X,Y\rangle\,(R^2-1) + \frac{3}{8}\,\langle X,Y\rangle\,(R^2-1)^2 - \beta\,\langle X,Y\rangle\,(R^2-1)^3 \\
&= \frac{15}{8}\,\langle X,Y\rangle - \frac{5}{4}\,\langle X,Y\rangle\,R^2 + \frac{3}{8}\,\langle X,Y\rangle\,R^4 - \beta\,\langle X,Y\rangle\,(R^2-1)^3.
\end{aligned}$$

By the independence of X and Y, and due to the mean zero assumption, $\mathbb{E}\,\langle X,Y\rangle = \mathbb{E}\,\langle X,Y\rangle\,R^2 = 0$. Hence, writing $1_B = 1 - 1_A$, we have

$$\begin{aligned}\mathbb{E}\,\frac{\langle X,Y\rangle}{R}\,1_B \;&=\; -\frac{15}{8}\,\mathbb{E}\,\langle X,Y\rangle\,1_A + \frac{5}{4}\,\mathbb{E}\,\langle X,Y\rangle\,R^2\,1_A - \frac{3}{8}\,\mathbb{E}\,\langle X,Y\rangle\,R^4\,1_A \\ &\quad + \frac{3}{8}\,\mathbb{E}\,\langle X,Y\rangle\,R^4 - \beta\,\mathbb{E}\,\langle X,Y\rangle\,(R^2-1)^3\,1_B.\end{aligned}$$

Here, the first three expectations on the right-hand side do not exceed in absolute value a multiple of $\frac{1}{\lambda_1^2 n}$. Hence, using the previous bound (11.6), we get

$$\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \;=\; \frac{c_1}{\lambda_1^2\,n} + \frac{3}{8}\,\mathbb{E}\,\langle X,Y\rangle\,R^4 + c_2\,\mathbb{E}\,|\,\langle X,Y\rangle\,|\,|R^2-1|^3, \tag{11.7}$$

where the quantities c_1 and c_2 are bounded by an absolute constant.

By Cauchy's inequality, the square of the last expectation does not exceed,

$$\mathbb{E}\,\langle X,Y\rangle^2\;\mathbb{E}\,(R^2-1)^6 \;=\; n\,\mathbb{E}\,(R^2-1)^6.$$

In turn, the latter expectation may be bounded by virtue of the inequality (11.4) applied with $p=6$ to the function $u(x,y) = \frac{1}{2n}\,(|x|^2+|y|^2) - 1$. Since

$$|\nabla u(x,y)|^2 = |\nabla_x u(x,y)|^2 + |\nabla_y u(x,y)|^2 = \frac{|x|^2+|y|^2}{n^2},$$

it gives

$$\mathbb{E}\,(R^2-1)^6 \;\le\; \frac{c}{\lambda_1^3\,n^3}\,\mathbb{E}R^6. \tag{11.8}$$

On the other hand, the Poincaré-type inequality easily yields the bound $\mathbb{E}R^6 \le c/\lambda_1^3$. However, in this step a more accurate estimation is required. Write

$$R^6 = (R^2-1)^3 + 3\,(R^2-1)^2 + 3\,(R^2-1) + 1,$$

so that

$$\mathbb{E}R^6 = \mathbb{E}\,(R^2-1)^3 + 3\,\mathbb{E}\,(R^2-1)^2 + 1. \tag{11.9}$$

By (11.3) with the same function u, we have

$$\mathbb{E}\,(R^2-1)^2 \le \frac{2}{\lambda_1 n}\,\mathbb{E}R^2 = 2\delta_n,$$

while (11.4) with $p=3$ gives

$$\mathbb{E}\,|R^2-1|^3 \;\le\; 27\,\delta_n^{3/2}\,\mathbb{E}\,|R|^3.$$

Putting $x^2 = \mathbb{E}R^6$ ($x>0$) and using $\mathbb{E}\,|R|^3 \le x$, we therefore get from (11.9) that

$$x^2 \;\le\; 27\,\delta_n^{3/2} x + 6\delta_n + 1.$$

This quadratic inequality is easily solved to yield $x^2 \le c\,(\delta_n+1)^3$. One can now apply this bound in (11.8) to conclude that

$$\mathbb{E}\,(R^2-1)^6 \;\le\; \frac{c}{\lambda_1^3\,n^3}\,(\delta_n+1)^3.$$

This implies

$$\mathbb{E}\,\langle X,Y\rangle^2\;\mathbb{E}\,(R^2-1)^6 \;\le\; \frac{c}{\lambda_1^3\,n^2}\,(\delta_n+1)^3,$$

which allows us to simplify the representation (11.7) to the form

$$\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \;=\; \frac{c_1}{\lambda_1^2\,n} + \frac{c_2}{\lambda_1^{3/2}\,n}\,(\delta_n+1)^{3/2} + \frac{3}{8}\,\mathbb{E}\,\langle X,Y\rangle\,R^4, \tag{11.10}$$

where the new quantity c_2 is bounded by an absolute constant.

We are left with the estimation of $\mathbb{E}\,\langle X,Y\rangle\,R^4$. Since $\mathbb{E}\,\langle X,Y\rangle\,|X|^4 = \mathbb{E}\,\langle X,Y\rangle\,|Y|^4 = 0$, it follows that

$$\mathbb{E}\,\langle X,Y\rangle\,R^4 \;=\; \frac{1}{2n^2}\,\mathbb{E}\,\langle X,Y\rangle\,|X|^2\,|Y|^2 \;=\; \frac{1}{2n^2}\,\big|\,\mathbb{E}\,|X|^2 X\,\big|^2.$$

The latter expectation is understood in the usual vector sense. That is, in terms of the components in $X = (X_1,\dots,X_n)$ defined on a probability space $(\Omega,\mathbb{P})$, we have

$$\mathbb{E}\,|X|^2 X = (a_1,\dots,a_n), \qquad a_k = \mathbb{E}\,|X|^2 X_k = \mathbb{E}\,(|X|^2-n)\,X_k.$$

Since the collection $\{X_1,\dots,X_n\}$ appears as an orthonormal system in the Hilbert space $L^2(\Omega,\mathbb{P})$, the numbers a_k represent the (Fourier) coefficients for the projection of the random variable $|X|^2-n$ onto the span of X_k's. Hence, by Bessel's inequality,

$$\big|\,\mathbb{E}\,|X|^2 X\,\big|^2 \;=\; \sum_{k=1}^n a_k^2 \;\le\; \big\|\,|X|^2-n\big\|_{L^2(\Omega,\mathbb{P})}^2 \;=\; \mathrm{Var}(|X|^2) \;=\; n\,\sigma_4^2(X) \;\le\; \frac{4n}{\lambda_1},$$

so that

$$\mathbb{E}\,\langle X,Y\rangle\,R^4 \;\le\; \frac{2}{\lambda_1 n}$$

In view of the upper bound $\lambda_1 \le 1$ (Remark 9.5), the expectation in (11.10) is thus dominated by the first term, and we arrive at

$$\mathbb{E}\,\frac{\langle X,Y\rangle}{R} \;\le\; \frac{c}{\lambda_1^2\,n} + \frac{c}{\lambda_1^{3/2}\,n}\,\Big(\frac{1}{\lambda_1 n}+1\Big)^{3/2}.$$

If $\lambda_1 \ge n^{-1}$, the first term on the right-hand side dominates the second one, and we arrive at the desired inequality (11.2). In the other case, we have $\frac{1}{\lambda_1^2\,n} \ge n$, and then (11.2) holds true as well, by (11.5), since $\mathbb{E}R \le 1$. □

12 Proof of Theorem 1.1

Let us now derive the stronger inequality (1.7). With parameters $T_0 = 4\sqrt{\log n}$ and $T = T_0 n$, the bound (7.2) of Lemma 7.1 is simplified to

$$c\,\rho(F_\theta,\Phi) \;\le\; \int_0^{T_0} \frac{|f_\theta(t)-f(t)|}{t}\,dt + L(\theta) + \frac{\log n}{n}\,\Lambda, \tag{12.1}$$

where $L(\theta) = \int_{T_0}^T \frac{|f_\theta(t)|}{t}\,dt$. Combining Corollary 9.4 with Lemma 8.2, we obtain that

$$\mathbb{E}_\theta\,L(\theta)^{2p} \;\le\; (c\log n)^{2p}\,\Big(p^{2p}\,n^{-2p} + e^{-\frac{1}{3}\sqrt{\lambda_1 n}}\Big)$$

for any integer $p \ge 1$. One can simplify this bound, by using the inequality $e^{-x} \le (\frac{4p}{ex})^{4p}$ $(x>0)$. Since $\lambda_1 \le 1$ (as was explained above), it follows that

$$\big(\mathbb{E}_\theta\,L(\theta)^{2p}\big)^{1/2p} \;\le\; \frac{c\log n}{n}\,\lambda_1^{-1}\,p^2.$$

This inequality is readily extended to all real $p \geq 1/2$. Replacing here $2p$ with p, we get a similar bound

$$\big(\mathbb{E}_\theta\, L(\theta)^p\big)^{1/p} \ \leq\ \frac{c\log n}{n}\,\lambda_1^{-1}\,p^2,$$

which holds for all real $p \geq 1$. Now, by Markov's inequality,

$$\mathfrak{s}_{n-1}\Big\{L(\theta) \geq \frac{ce\log n}{n}\,\lambda_1^{-1} r\Big\} \leq \frac{p^{2p}}{(er)^p}, \qquad r \geq 1.$$

Choosing $p = \sqrt{r}$, we thus have

$$\mathfrak{s}_{n-1}\Big\{L(\theta) \geq \frac{ce\log n}{n}\,\lambda_1^{-1} r\Big\} \ \leq\ e^{-\sqrt{r}}. \tag{12.2}$$

It is time to involve Lemma 8.1. First, from Lemmas 10.2 and 11.1, it follows that

$$\begin{aligned} \int_0^{T_0} \frac{\sqrt{I(t)}}{t}\,dt \ &\leq\ \sqrt{T_0}\,\bigg(\int_0^{T_0} \frac{I(t)}{t^2}\,dt\bigg)^{1/2} \\ &\leq\ c\sqrt{T_0}\,\bigg(\frac{1}{n}\,\mathbb{E}\,\frac{\langle X,Y\rangle}{R} + \frac{\Lambda^2}{n^2}\bigg)^{1/2} \ \leq\ \frac{c'}{\lambda_1 n}\,(\log n)^{1/4}, \end{aligned}$$

where on the last step we used $\Lambda \leq \frac{4}{\lambda_1}$. Hence, by Lemma 8.1,

$$\mathfrak{s}_{n-1}\bigg\{\int_0^{T_0} \frac{|f_\theta(t) - f(t)|}{t}\,dt \geq \frac{c\log n}{\lambda_1 n}\,r\bigg\} \ \leq\ 2\,e^{-r}.$$

Being combined with (12.2) and applied in (12.1), this bound leads to the desired inequality

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta,\Phi) \geq \frac{c\log n}{n}\,\lambda_1^{-1} r\Big\} \ \leq\ 3\,e^{-\sqrt{r}}, \tag{12.3}$$

which also holds for $r < 1$ (when the right-hand side is greater than 1). Here, the constant 3 may be replaced with 2 by rescaling the variable r, and then we arrive at (1.7). $\square$

Corollary 12.1. *Let X be an isotropic random vector in $\mathbb{R}^n$ with mean zero and a positive Poincaré constant λ_1. For any $\beta > 0$, with $\mathfrak{s}_{n-1}$-probability at most $3n^{-\beta}$ we have*

$$\rho(F_\theta,\Phi) \leq \frac{c\beta^2\,(\log n)^3}{n}\,\lambda_1^{-1}.$$

Proof. Indeed, although the estimate (1.7) implies the bound on average (1.6), it is only effective for $r \geq (\log n)^2$. For the values $r = (\beta\log n)^2$, (12.3) provides a polynomial bound

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta,\Phi) \geq \frac{c\beta^2\,(\log n)^3}{n}\,\lambda_1^{-1}\Big\} \ \leq\ 3n^{-\beta}.$$

In other words, for a sufficiently large number A, with high $\mathfrak{s}_{n-1}$-probability

$$\rho(F_\theta,\Phi) \leq \frac{A\,(\log n)^3}{n}\,\lambda_1^{-1}. \qquad \square$$

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