Chapter 5
Edgeworth Corrections in Randomized Central Limit Theorems

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Abstract We consider rates of approximation of distributions of weighted sums of independent, identically distributed random variables by the Edgeworth correction of the 4-th order.

5.1 Introduction

Given independent, identically distributed random variables $X_1, \dots, X_n$ (for short - i.i.d.), we consider weighted sums

$$S_\theta = \theta_1 X_1 + \cdots + \theta_n X_n, \qquad \theta = (\theta_1, \dots, \theta_n),$$

with $\theta_1^2 + \cdots + \theta_n^2 = 1$, thus indexed by the points from the unit sphere S^{n-1} in $\mathbb{R}^n$ ($n \geq 2$). Throughout it is assumed that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$, so that $\mathbb{E}S_\theta = 0$, $\mathbb{E}S_\theta^2 = 1$. According to the central limit theorem, if all the coefficients θ_k's are small, the distribution function

$$F_\theta(x) = \mathbb{P}\{S_\theta \leq x\}, \qquad x \in \mathbb{R},$$

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is close to the normal distribution function $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy$ with density $\varphi(y) = \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}$. This property can be quantified in terms of the Kolmogorov distance

$$\rho(F_\theta, \Phi) = \sup_x |F_\theta(x) - \Phi(x)|,$$

by involving absolute moments $\beta_s = \mathbb{E}\,|X_1|^s$. In particular, if the 3-rd absolute moment β_3 is finite, then

$$\rho(F_\theta, \Phi) \leq c\beta_3 \sum_{k=1}^{n} |\theta_k|^3 \tag{5.1.1}$$

up to some absolute constant c (cf. [11]). As best, here the right-hand side is of order $1/\sqrt{n}$ which is optimal in general, including the case of equal coefficients; moreover, this rate may not be improved under higher order moment assumptions.

Nevertheless, the situation is different when one is concerned about the typical behavior of these distances for most of θ in the sense of the normalized Lebesgue measure $\mathfrak{s}_{n-1}$ on S^{n-1}. In particular, Klartag and Sodin [7] have showed that, under the 4-th moment condition, the value $\rho(F_\theta, \Phi)$ is actually at most of order $1/n$ on average. More precisely, with some absolute constants $c, r_0 > 0$, for any $r \geq r_0$, we have

$$\rho(F_\theta, \Phi) \leq \frac{cr}{n}\, \beta_4 \tag{5.1.2}$$

for all $\theta \in S^{n-1}$ except for a set of $\mathfrak{s}_{n-1}$-measure $\leq 2\exp\{-r^{1/2}\}$. This cannot be obtained on the basis of (5.1.1), since the average of $\sum_{k=1}^{n} |\theta_k|^3$ is proportional to $1/\sqrt{n}$.

As it turns out, under rather general conditions, the relation (5.1.2) admits a further refinement, by replacing Φ with the corrected normal "distribution" function

$$G(x) = \Phi(x) - \frac{\alpha}{n}\,(x^3 - 3x)\,\varphi(x), \qquad \alpha = \frac{\beta_4 - 3}{8}. \tag{5.1.3}$$

We will use $\mathbb{E}_\theta$ to denote integrals with respect to the measure $\mathfrak{s}_{n-1}$. Put $\alpha_3 = \mathbb{E}X_1^3$.

Theorem 5.1.1 *If $\alpha_3 = 0$, $\beta_5 < \infty$, then with some positive absolute constant c*

$$\mathbb{E}_\theta\, \rho(F_\theta, G) \leq \frac{c}{n^{3/2}}\, \beta_5. \tag{5.1.4}$$

Moreover, there exists an absolute constant $r_0 > 0$ such that for all $r \geq r_0$,

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, G) \geq \frac{cr}{n^{3/2}}\, \beta_5\Big\} \leq 2\, \exp\big\{-r^{1/2}\big\}. \tag{5.1.5}$$

Theorem 5.1.1 involves all symmetric probability distributions with finite 5-th absolute moment in which case $G = \Phi$. Moreover, this bound is optimal in the sense that it can be reversed in a typical situation, where the 4-th moment of X_1 is different than the 4-th moment of the standard normal law. The same is also true about (5.1.2) on average, when the 3-rd moment of X_1 is not zero. Denote by $\mathcal{G}$ the collection of all functions G of bounded variation on the real line such that $G(-\infty) = 0$ and $G(\infty) = 1$.

Theorem 5.1.2 *If $\alpha_3 \neq 0$, $\beta_4 < \infty$, then the inequality*

$$\inf_{G \in \mathcal{G}} \mathbb{E}_\theta\, \rho(F_\theta, G) \geq \frac{c}{n} \tag{5.1.6}$$

holds for all n with a constant $c > 0$ depending on α_3 and β_4 only. Moreover, if $\alpha_3 = 0$, $\beta_4 \neq 3$, $\beta_5 < \infty$, then

$$\inf_{G \in \mathcal{G}} \mathbb{E}_\theta\, \rho(F_\theta, G) \geq \frac{c}{n^{3/2}}, \tag{5.1.7}$$

where the constant $c > 0$ depends on β_4 and β_5.

The paper is organized as follows. First, we recall a general scheme of Edgeworth corrections. Being specialized to the weighted sums, the corresponding asymptotic expansions contain as parameters special functions on the sphere, which we discuss in Sect. 5.3. The behavior of characteristic functions of the weighted sums on large intervals is analyzed separately in Sect. 5.4. These preparations are sufficient for the proof of Theorem 5.1.1, cf. Sect. 5.5 (where we also give a slight refinement of Klartag–Sodin's theorem in the i.i.d. situation). Sections 5.6 and 5.7 deal with lower bounds on the Kolmogorov distance, which are used to prove Theorem 5.1.2 (Sect. 5.8).

In the sequel, we use c, C to denote positive absolute constants, in general different in different places; similarly, c_q, C_q denote constants depending on a parameter q.

5.2 Construction of Asymptotic Expansions

Let $\xi_1, \dots, \xi_n$ be independent, not necessarily identically distributed random variables such that $\mathbb{E}\xi_k = 0$ and $\sum_{k=1}^n \mathbb{E}\xi_k^2 = 1$. Consider the sum $S_n = \xi_1 + \dots + \xi_n$, which thus has mean zero and variance one. An asymptotic behaviour of the distribution of S_n in a weak sense is usually analyzed in terms of its characteristic function $f_n(t) = \mathbb{E}\, e^{itS_n}$. In turn, the behaviour of $f_n(t)$ on large t-intervals is controlled by the Lyapunov coefficients

$$L_s = \sum_{k=1}^n \mathbb{E}\,|\xi_k|^s, \qquad s \geq 2.$$

Note that $L_s \geq n^{-\frac{s-2}{2}}$. In fact, these quantities are often of the order $n^{-\frac{s-2}{2}}$. For example,

$$L_s = n^{-\frac{s-2}{2}}\, \mathbb{E}\,|X_1|^s \quad \text{in case} \;\; \xi_k = \frac{1}{\sqrt{n}}\, X_k$$

with identically distributed X_k. Since the function $s \to L_s^{\frac{1}{s-2}}$ is non-decreasing on the half-axis $s > 2$ (due to $L_2 = 1$), we have $L_3 \leq L_4^{1/2} \leq L_5^{1/3}$.

If L_s is finite for a fixed integer $s \geq 2$, the cumulants

$$\gamma_p(\xi_k) = \frac{d^p}{i^p\, dt^p} \log \mathbb{E}\, e^{it\xi_k}\big|_{t=0}$$

are well-defined and finite for all $p = 1, \dots, s$. Every cumulant $\gamma_p(\xi_k)$ is determined by the first p moments $\alpha_{r,k} = \mathbb{E}\, \xi_k^r$, $r = 1, \dots, p$. The first cumulants are

$$\gamma_1(\xi_k) = \alpha_{1,k} = 0, \quad \gamma_2(\xi_k) = \alpha_{2,k}^2, \quad \gamma_3(\xi_k) = \alpha_{3,k}, \quad \gamma_4(\xi_k) = \alpha_{4,k} - 3\alpha_{2,k}^2.$$

A result of Bikjalis asserts that $|\gamma_p(\xi_k)| \leq (p-1)!\, \mathbb{E}\,|\xi_k|^p$ (cf. [1, 3]). The cumulants of S_n exist for the same values of p and have an additive structure:

$$\gamma_p(S_n) = \frac{d^p}{i^p\, dt^p} \log \mathbb{E}\, e^{itS_n}\Big|_{t=0} = \sum_{k=1}^n \gamma_p(\xi_k).$$

Hence, they admit a similar upper estimate

$$|\gamma_p(S_n)| \leq (p-1)!\, L_p. \tag{5.2.1}$$

The Lyapunov coefficients may also be used to bound absolute moments of S_n. The well-known Rosenthal inequality indicates that $\mathbb{E}\,|S_n|^p \leq C_p\,(1 + L_p)$ for $p \geq 2$.

We refer an interested reader to [2, 3, 11] for more references and here only mention a few definitions and basic results.

Definition 5.2.1 Let L_s be finite for an integer $s \geq 3$. An Edgeworth approximation of order $s-1$ for the characteristic function $f_n(t) = \mathbb{E}\, e^{itS_n}$ is given by

$$g_{s-1}(t) = e^{-t^2/2} + e^{-t^2/2} \sum \frac{1}{k_1! \dots k_{s-3}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{s-1}}{(s-1)!}\Big)^{k_{s-3}} (it)^k, \quad t \in \mathbb{R}.$$

Here $\gamma_p = \gamma_p(S_n)$, $k = 3k_1 + \dots + (s-1)k_{s-3}$, and the summation is performed over all tuples $(k_1, \dots, k_{s-3})$ of non-negative integers, not all zero, such that $k_1 + 2k_2 + \dots + (s-3)k_{s-3} \leq s-3$.

The function g_{s-1} is also called the corrected normal characteristic function (although it is not a characteristic function in the usual sense). The index $s-1$ indicates that the cumulants up to γ_{s-1} participate in the constructions. Note that the above sum represents a polynomial of degree at most $3(s-3)$ in variable t.

When $s=3$, we have $g_2(t)=e^{-t^2/2}$ which is the standard normal characteristic function. The next Edgeworth correction is given by

$$g_3(t) = e^{-t^2/2}\Big(1+\gamma_3\frac{(it)^3}{3!}\Big), \qquad \gamma_3 = \sum_{k=1}^n \mathbb{E}\xi_k^3. \tag{5.2.2}$$

For $s=5$, if $\gamma_3=0$, we have

$$g_4(t) = \Big(1+\gamma_4\frac{(it)^4}{4!}\Big)\, e^{-t^2/2}, \quad \gamma_4 = \sum_{k=1}^n \Big(\mathbb{E}\xi_k^4 - 3\,(\mathbb{E}\xi_k^2)^2\Big). \tag{5.2.3}$$

We will need the following general statement about the Edgeworth approximations.

Proposition 5.2.2 *Let $L_s<\infty$ for an integer $s\ge 3$. Then in the interval $|t|\le \frac{1}{L_3}$,*

$$\big|f_n(t)-g_{s-1}(t)\big| \,\le\, C_s L_s \min\big\{1,|t|^s\big\}\, e^{-t^2/8}. \tag{5.2.4}$$

When $s=3$, (5.2.4) leads to the popular inequality

$$\big|f_n(t)-e^{-t^2/2}\big| \,\le\, C L_3 \min\{1,|t|^3\}\, e^{-t^2/8}.$$

By (5.2.1), the cumulants of S_n satisfy $|\gamma_p| \le (p-1)!\,L_p \le (p-1)!\,L_s^{\frac{p-2}{s-2}}$ implying that

$$\Big|\Big(\frac{\gamma_3}{3!}\Big)^{k_1}\dots\Big(\frac{\gamma_{s-1}}{(s-1)!}\Big)^{k_{s-3}}\Big| \le \frac{L_s^{k/(s-2)}}{3^{k_1}\dots(s-1)^{k_{s-3}}} \tag{5.2.5}$$

with some $1\le k\le s-3$. Applying this bound in Definition 5.2.1, it readily follows that

$$\big|g_{s-1}(t)-e^{-t^2/2}\big| \,\le\, C_s\, \max\{|t|^3,|t|^{3(s-3)}\}\, e^{-t^2/2}\, \max\big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{s-3}{s-2}}\big\}.$$

In particular,

$$\int_{-\infty}^{\infty} \big|g_{s-1}(t)-e^{-t^2/2}\big|\,dt \,\le\, C_s\, \max\{1,L_s\}.$$

Being integrable, the function g_{s-1} appears as the Fourier–Stieltjes transform of a certain signed Borel measure μ_{s-1} on the real line with density

$$\varphi_{s-1}(x) = \varphi(x) + \varphi(x) \sum \frac{1}{k_1! \dots k_{s-3}!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_{s-1}}{(s-1)!}\right)^{k_{s-3}} H_k(x), \quad x \in \mathbb{R}.$$

Here, the summation is as before, and H_k are the Chebyshev-Hermite polynomials of degrees $k = 3k_1 + \dots + (s-1)k_{s-3}$ with leading coefficient 1. By the construction, $g_{s-1}(0) = 1$, that is, μ_{s-1} has total mass 1. Moreover, g_{s-1} and f_n have equal derivatives at zero up to order $s-1$, which is equivalent to

$$\mathbb{E}\, S_n^p = \int_{-\infty}^{\infty} x^p \, d\mu_{s-1}(x) = \int_{-\infty}^{\infty} x^p \, \varphi_{s-1}(x)\, dx, \qquad p = 1, \dots, s-1.$$

Using (5.2.5), we can also see that

$$\sup_x |\varphi_{s-1}(x)| \le C_s \max\{1, L_s\}, \qquad \int_{-\infty}^{\infty} |\varphi_{s-1}(x)|\, dx \le C_s \max\{1, L_s\}. \tag{5.2.6}$$

The associated "distribution" function

$$\Phi_{s-1}(x) = \mu_{s-1}\big((-\infty, x]\big) = \int_{-\infty}^{x} \varphi_{s-1}(y)\, dy, \qquad x \in \mathbb{R},$$

has a similar description

$$\Phi_{s-1}(x) = \Phi(x) - \varphi(x) \sum \frac{1}{k_1! \dots k_{s-3}!} \left(\frac{\gamma_3}{3!}\right)^{k_1} \dots \left(\frac{\gamma_{s-1}}{(s-1)!}\right)^{k_{s-3}} H_{k-1}(x).$$

This function has bounded total variation and satisfies $\Phi_{s-1}(-\infty) = 0$ and $\Phi_{s-1}(\infty) = 1$.

The measure μ_2 is just the standard Gaussian measure with distribution function $\Phi_2 = \Phi$. The Edgeworth correction g_3 corresponds to the signed measure with "distribution" function

$$\Phi_3(x) = \Phi(x) - \frac{\gamma_3}{3!}\,(x^2 - 1)\varphi(x). \tag{5.2.7}$$

If $\gamma_3 = 0$, the next Edgeworth correction g_4 corresponds to the "distribution" function

$$\Phi_4(x) = \Phi(x) - \frac{\gamma_4}{4!}\,(x^3 - 3x)\,\varphi(x). \tag{5.2.8}$$

Since Proposition 5.2.2 quantifies closeness of $f_n(t)$ to $g_{s-1}(t)$ on large t-intervals (when L_s is small), one may hope that, under some additional assumptions, the distribution function F_n will be properly approximated by Φ_{s-1} in Kolmogorov distance. This may be achieved by applying Berry–Esseen-type theorems such as the following:

Proposition 5.2.3 *If L_s is finite for an integer $s \geq 3$, then*

$$c_s\,\rho(F_n, \Phi_{s-1}) \;\leq\; L_s + 1_{\{L_s \leq L_3 \leq 1\}} \int_{1/L_3}^{1/L_s} \frac{|f_n(t)|}{t}\,dt. \tag{5.2.9}$$

Proof A classical theorem due to Esseen asserts the following: Let F be a non-decreasing bounded function, and G be a differentiable function of bounded variation such that $F(-\infty) = G(-\infty) = 0$. If $|G'(x)| \leq M$ for all x, then for any $T > 0$,

$$c\,\rho(F, G) \leq \int_0^T \Big|\frac{f(t) - g(t)}{t}\Big|\,dt + \frac{M}{T}. \tag{5.2.10}$$

Here,

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\,dF(x), \qquad g(t) = \int_{-\infty}^{\infty} e^{itx}\,dG(x)$$

denote the Fourier–Stieltjes transforms of F and G, respectively.

First, assume that $L_s \leq 1$. Necessarily $L_3 \leq L_s^{\frac{1}{s-2}} \leq 1$. Assuming moreover that $L_s \leq L_3$, we choose $T = 1/L_s$ and apply the bound (5.2.9) with $F = F_n$ and $G = \Phi_{s-1}$, in which case we have $|G'(x)| = |\varphi_{s-1}(x)| \leq C_s$, by (5.2.6). Then, applying (5.2.4) in (5.2.10), we get

$$c_s\,\rho(F_n, \Phi_{s-1}) \leq L_s + \int_{1/L_3}^{1/L_s} \frac{|f_n(t)|}{t}\,dt + \int_{1/L_3}^{1/L_s} \frac{|g_{s-1}(t)|}{t}\,dt. \tag{5.2.11}$$

To estimate the last integral, one may use the bound (cf. [3], Proposition 17.1)

$$|g_{s-1}(t)| \leq C_s L_s\, e^{-t^2/8}, \quad \text{if } |t| \max\Big\{L_s^{\frac{1}{s-2}}, L_s^{\frac{1}{3(s-2)}}\Big\} \geq \frac{1}{8}.$$

Since $L_3 \leq L_s^{\frac{1}{s-2}}$, it holds for $t \geq 1/(8L_3)$, and (5.2.11) thus yields (5.2.9).

Now, suppose that $L_3 \leq L_s \leq 1$. Then we choose $T = 1/L_3$ in (5.2.10) and apply (5.2.4) again, which leads to $c_s\,\rho(F_n, \Phi_{s-1}) \leq L_3$. Finally, if $L_s > 1$, one may use the second bound (5.2.6) which immediately implies that

$$\rho(F_n, \Phi_{s-1}) \leq \rho(F_n, \Phi) + \rho(\Phi, \Phi_{s-1}) \leq C_s L_s.$$

□

5.3 Moments and Deviations of Lyapunov Coefficients

Let us return to the scheme of the weighted sums. In the rest of the paper, we assume that

$$S_\theta = \theta_1 X_1 + \cdots + \theta_n X_n, \qquad \theta = (\theta_1, \dots, \theta_n) \in S^{n-1},$$

where X_k's are i.i.d. random variables such that $\mathbb{E} X_1 = 0$, $\mathbb{E} X_1^2 = 1$, and $\beta_s = \mathbb{E}\,|X_1|^s < \infty$ for an integer $s \geq 3$. First, we will be focusing on the application of Proposition 5.2.3 to the approximation of the distribution functions F_θ of S_θ for most of θ's by the corresponding Edgeworth corrections $\Phi_{s-1} = \Phi_{s-1,\theta}$, especially with $s = 4$ and $s = 5$.

According to (5.2.9), in order to control the Kolmogorov distance from F_θ to $\Phi_{s-1,\theta}$, one should estimate the Lyapunov coefficients $L_s = L_s(\theta)$; we also need information about the magnitude of the characteristic functions $f_\theta(t) = \mathbb{E}\, e^{itS_\theta}$ on large t-intervals such as $|t| \leq 1/L_s(\theta)$. Note that the Lyapunov coefficients take the form

$$L_p(\theta) \,=\, \beta_p\, l_p(\theta) \quad \text{where} \ \ l_p(\theta) = \sum_{k=1}^n |\theta_k|^p \quad (2 \leq p \leq s).$$

On the other hand, according to Definition 5.2.1, the construction of the functions $\Phi_{s-1,\theta}$ is based on the cumulants $\gamma_p(\theta) = \gamma_p(S_\theta)$ for $p \leq s-1$, which are given in terms of the cumulants $\gamma_p = \gamma_p(X_1)$ of the underlying distribution by

$$\gamma_p(\theta) \,=\, \gamma_p\, \alpha_p(\theta), \quad \alpha_p(\theta) = \sum_{k=1}^n \theta_k^p.$$

In particular, $\gamma_1(\theta) = 0$, $\gamma_2(\theta) = 1$, and

$$\gamma_3(\theta) = \gamma_3\, \alpha_3(\theta) \,=\, \alpha_3 \sum_{k=1}^n \theta_k^3 \qquad (\alpha_3 = \mathbb{E} X_1^3),$$

$$\gamma_4(\theta) = \gamma_4\, l_4(\theta) \,=\, \big(\beta_4 - 3\big) \sum_{k=1}^n \theta_k^4 \qquad (\beta_4 = \mathbb{E} X_1^4).$$

Thus, in order to study the typical behaviour of distances $\rho(F_\theta, \Phi_{s-1,\theta})$, we have to explore the distribution of the functionals l_p and α_p under the measure $\mathfrak{s}_{n-1}$ (note that $\alpha_p = l_p$ for even p). The behaviour of distributions of l_p for large n is mainly described by their means and variances. Since the distribution of the first coordinate

θ_1 under $\mathfrak{s}_{n-1}$ has density

$$c_n (1-x^2)^{\frac{n-3}{2}} \quad (|x| \le 1), \qquad c_n = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\ \Gamma(\frac{n-1}{2})},$$

we get

$$\mathbb{E}_\theta\, |\theta_1|^p \ = \ 2c_n \int_0^1 x^p\, (1-x^2)^{\frac{n-3}{2}}\, dx \ = \ \frac{\Gamma(\frac{p+1}{2})\, \Gamma(\frac{n}{2})}{\sqrt{\pi}\ \Gamma(\frac{p+n}{2})}.$$

In particular, since $\mathbb{E}_\theta\, l_p(\theta) = n\, \mathbb{E}_\theta |\theta_1|^p$, we have

$$\mathbb{E}_\theta\, l_{2k}(\theta) \ = \ \mathbb{E}_\theta\, \alpha_{2k}(\theta) \ = \ \frac{(2k-1)\,!!}{(n+2)\dots(n+2k-2)} \tag{5.3.1}$$

for even powers $p = 2k$, $k = 2, 3, \dots$ Hence,

$$\mathbb{E}_\theta\, l_{2k} < 2^k k!\, n^{-(k-1)} < p^{p/2}\, n^{-\frac{p-2}{2}}.$$

Here, the resulting bound also holds for $p = 2k-1$. Indeed, using the property that the function $p \to l_p^{\frac{1}{p-2}}$ is non-decreasing in $p > 2$, we have $l_{2k-1} \le l_{2k}^{\frac{2k-3}{2k-2}}$ and $\mathbb{E}_\theta\, l_{2k-1} \le (\mathbb{E}_\theta\, l_{2k})^{\frac{2k-3}{2k-2}}$. Therefore

$$\begin{aligned}
\mathbb{E}_\theta\, |\theta_1|^{2k-1} &\le \big(\mathbb{E}_\theta\, \theta_1^{2k}\big)^{\frac{2k-3}{2k-2}} \\
&< (2^k k!\, n^{-(k-1)})^{\frac{2k-3}{2k-2}} \ = \ (2^k k!)^{\frac{2k-3}{2k-2}}\, n^{-\frac{p-2}{2}} \\
&< (2k-1)^{\frac{k(2k-3)}{2k-2}}\, n^{-\frac{p-2}{2}} \ < \ (2k-1)^{\frac{2k-1}{2}}\, n^{-\frac{p-2}{2}},
\end{aligned}$$

where we used a simple inequality $2^k k! < (2k-1)^k$. That is, we have:

Lemma 5.3.1 *For all integers $p \ge 3$, we have $\mathbb{E}_\theta\, l_p < p^{p/2}\, n^{-\frac{p-2}{2}}$.*

For the first Lyapunov coefficients, the p-dependent constant can slightly be improved. For example,

$$\mathbb{E}_\theta\, l_3 \le \mathbb{E}_\theta\, l_4^{1/2} \le \big(\mathbb{E}_\theta\, l_4\big)^{1/2} = \Big(\frac{3}{n+2}\Big)^{1/2} < \frac{2}{n^{1/2}}.$$

Similarly, since $l_5^{1/3} \le l_6^{1/4}$,

$$\mathbb{E}_\theta\, l_5 \le \mathbb{E}_\theta\, l_6^{3/4} \le \big(\mathbb{E}_\theta\, l_6\big)^{3/4} = \Big(\frac{15}{(n+2)(n+4)}\Big)^{3/4} < \frac{8}{n^{3/2}}.$$

Using (5.3.1) together with a similar formula

$$\mathbb{E}\,|Z|^{2k} = 2^k\,\frac{\Gamma(\frac{2k+n}{2})}{\Gamma(\frac{n}{2})} = n(n+2)\dots(n+2k-2), \qquad k = 1, 2, \dots,$$

where Z is a standard normal random vector in $\mathbb{R}^n$ (needed with $k = 4$), we also find that

$$\mathrm{Var}_\theta(\alpha_3) = \frac{15}{(n+2)(n+4)} < \frac{15}{n^2},$$
$$\mathrm{Var}_\theta(l_4) = \frac{24\,(n-1)}{(n+2)^2(n+4)(n+6)} < \frac{24}{n^3}.$$

This means that the deviations of α_3 are of order $1/n$, while the deviations of l_4 from its mean are of order $n^{-3/2}$.

With worse numerical constants these bounds can also be obtained by applying the spherical Poincaré inequality. However, by virtue of the (stronger) logarithmic Sobolev inequality on the unit sphere with an optimal constant [8, 10], namely

$$\int u^2\,\log u^2\,d\mathfrak{s}_{n-1} - \int u^2\,d\mathfrak{s}_{n-1}\,\log\int u^2\,d\mathfrak{s}_{n-1} \le \frac{2}{n-1}\int |\nabla u|^2\,ds_{n-1},$$

one can get more information, such as the bound on the growth of moments

$$\|u - \mathbb{E}_\theta\,u\|_p \le \frac{\sqrt{p-1}}{\sqrt{n-1}}\,\|\nabla u\|_p, \qquad p \ge 2. \tag{5.3.2}$$

Both inequalities hold true for any smooth function u on $\mathbb{R}^n$ with gradient ∇u, and with L^p-norms being understood with respect to the measure $\mathfrak{s}_{n-1}$ (cf. e.g. [4], Theorem 4.1).

Generalizing $\alpha_3(\theta)$ and $l_4(\theta)$, now consider the functions

$$Q_3(\theta) = \sum_{k=1}^n a_k\theta_k^3, \quad Q_4(\theta) = \sum_{k=1}^n a_k\theta_k^4.$$

Lemma 5.3.2 *Assume that* $\frac{1}{n}\sum_{k=1}^n a_k^2 = 1$ *and put* $\bar a = \frac{1}{n}\sum_{k=1}^n a_k$. *For all* $r > 0$,

$$\mathfrak{s}_{n-1}\{n\,|Q_3| \ge r\} \le 2\,\exp\Big\{-\frac{1}{23}\,r^{2/3}\Big\},$$
$$\mathfrak{s}_{n-1}\Big\{n^{3/2}\,\Big|Q_4 - \frac{3\,\bar a}{n+2}\Big| \ge r\Big\} \le 2\,\exp\Big\{-\frac{1}{38}\,r^{1/2}\Big\}.$$

Proof We apply (5.3.2) to the function $u = nQ_3$. Using $\frac{p-1}{n-1} \leq \frac{2p}{n}$, by Jensen's inequality, for any $p \geq 2$,

$$\begin{aligned}\|u\|_p^p &\leq n^p \Big(\frac{2p}{n}\Big)^{p/2} \|\nabla Q_3\|_p^p \\ &= n^{p/2}\,(2p)^{p/2}\,3^p \int \Big(\sum_{k=1}^n a_k^2\,\theta_k^4\Big)^{p/2} d\mathfrak{s}_{n-1}(\theta) \\ &\leq n^p\,(2p)^{p/2}\,3^p \cdot \frac{1}{n}\sum_{k=1}^n a_k^2 \int |\theta_k|^{2p}\,d\mathfrak{s}_{n-1}(\theta) \;=\; n^p\,(18\,p)^{p/2}\,\mathbb{E}_\theta|\theta_1|^{2p}.\end{aligned}$$

If $p = m$ is integer, applying the relation (5.3.1), we get

$$\begin{aligned}\|u\|_m^m &\leq n^m\,(18\,m)^{m/2}\,\mathbb{E}_\theta|\theta_1|^{2m} \\ &\leq (18\,m)^{m/2}\,(2m-1)!! \;\leq\; 18^{m/2}\,2^m\,m^{3m/2},\end{aligned}$$

where we used the bound $(2m-1)!! < (2m)^m$. Thus, $\|u\|_m \leq 6\sqrt{2}\,m^{3/2}$. At the expense of a larger absolute factor, this inequality can be extended to all real $p \geq 2$ in place of m. Indeed, pick up an integer m such that $m \leq p < m+1$. Then

$$\|u\|_p \leq \|u\|_{m+1} \leq 6\sqrt{2}\,(m+1)^{3/2} \leq 6\sqrt{2}\,(p+1)^{3/2} \leq 9\sqrt{3}\,p^{3/2},$$

i.e. $\|u\|_p^p \leq (bp)^{3p/2}$ with $b = (9\sqrt{3})^{2/3}$. By Markov's inequality, choosing $p = \frac{1}{2^{1/3}\,b}\,r^{2/3}$ $(r > 0)$, we get

$$\mathfrak{s}_{n-1}\{|u| \geq r\} \;\leq\; \frac{(bp)^{3p/2}}{r^p} \;=\; \exp\Big\{-\frac{p}{2}\,\log 2\Big\},$$

provided that $p \geq 2$. But, in the case $0 < p < 2$, the above right-hand side is greater than $1/2$, so that we have

$$\mathfrak{s}_{n-1}\{|u| \geq r\} \;\leq\; 2\,\exp\Big\{-\frac{p}{2}\,\log 2\Big\}$$

for all $p > 0$. It remains to note that $\frac{p}{2}\,\log 2 = \frac{\log 2}{2^{4/3}\,3^{5/3}}\,r^{2/3} > \frac{1}{23}\,r^{2/3}$ thus proving the first inequality of the lemma.

To derive the second one, let us apply (5.3.2) to the function $u = n^{3/2}\,(Q_4 - \frac{3\,\bar{a}}{n+2})$. Similarly, for any $p \geq 2$,

$$\begin{aligned}
\|u\|_p^p &\leq n^{3p/2}\,\Big(\frac{2p}{n}\Big)^{p/2}\,\|\nabla Q_4\|_p^p \\
&= n^{3p/2}\,\Big(\frac{2p}{n}\Big)^{p/2}\,4^p \int \Big(\sum_{k=1}^n a_k^2\,\theta_k^6\Big)^{p/2}\,d\mathfrak{s}_{n-1}(\theta) \\
&\leq n^{2p}\Big(\frac{2p}{n}\Big)^{p/2}\,4^p \cdot \frac{1}{n}\sum_{k=1}^n a_k^2 \int |\theta_k|^{3p}\,d\mathfrak{s}_{n-1}(\theta) = n^{3p/2}\,(32\,p)^{p/2}\,\mathbb{E}_\theta\,|\theta_1|^{3p}.
\end{aligned}$$

Let us replace p with $2m$ assuming that $m \geq 1$ is integer. By (5.3.1), we get

$$\begin{aligned}
\|u\|_{2m}^{2m} &\leq n^{3m}\,(64\,m)^m\,\mathbb{E}_\theta\,|\theta_1|^{6m} \\
&\leq (64\,m)^m\,(6m-1)!! \,\leq\, (48\sqrt{6}\,m^2)^{2m}.
\end{aligned}$$

Hence $\|u\|_{2m} \,\leq\, 48\sqrt{6}\,m^2$. To extend this inequality to real $p \geq 2$, pick up an integer m such that $2m \leq p < 2(m+1)$. Then

$$\|u\|_p \leq \|u\|_{2(m+1)} \leq 48\sqrt{6}\,(m+1)^2 \leq 48\sqrt{6}\,p^2 = (bp)^{2p}, \quad b = (48\sqrt{6})^{1/2}.$$

By Markov's inequality, given $r > 0$ and choosing $p = \frac{1}{2^{1/4}\,b}\sqrt{r}$, we get

$$\mathfrak{s}_{n-1}\{|u| \geq r\} \leq \frac{(bp)^{2p}}{r^p} = \exp\Big\{-\,\frac{p}{2}\,\log 2\Big\}$$

provided that $p \geq 2$. In the case $0 < p < 2$, the right-hand side is greater than $1/2$, so that

$$\mathfrak{s}_{n-1}\{|u| \geq r\} \,\leq\, 2\,\exp\Big\{-\,\frac{p}{2}\,\log 2\Big\}$$

for all $p > 0$. It remains to note that $\frac{p}{2}\,\log 2 > \frac{1}{38}\,r^{1/2}$. □

Let us now consider deviations of l_p above their means.

Lemma 5.3.3 *For all real $r \geq 1$ and integer $p > 2$,*

$$\mathfrak{s}_{n-1}\Big\{n^{\frac{p-2}{2}}\,l_p \geq c_p\,r\Big\} \,\leq\, \exp\big\{-\,(rn)^{2/p}\big\}, \tag{5.3.3}$$

where one may take $c_3 = 33$, $c_4 = 121$, and $c_p = (\sqrt{p}+2)^p$ in general.

Proof If u is a function on S^{n-1} with Lipschitz semi-norm $\|u\|_{\rm Lip} \le 1$ with respect to the Euclidean distance, then (cf. e.g. [9])

$$\mathfrak{s}_{n-1}\big\{u \ge \mathbb{E}_\theta\, u + t\big\} \le e^{-nt^2/4}, \qquad t \ge 0.$$

As a partial case, one may consider the ℓ_p^n-norms $u(\theta) = l_p(\theta)^{1/p}$ on $\mathbb{R}^n$ with $p \ge 2$, for which we thus have that

$$\mathfrak{s}_{n-1}\big\{l_p^{1/p} \ge (\mathbb{E}_\theta\, l_p)^{1/p} + t\big\} \le e^{-nt^2/4}.$$

Using the bound $\mathbb{E}_\theta\, l_p \le A_p\, n^{-\frac{p-2}{2}}$ with $A_p = p^{p/2}$ as in Lemma 5.3.1, the choice $t = 2r^{1/p}\, n^{-\frac{p-2}{2p}}$ leads to

$$\mathfrak{s}_{n-1}\Big\{n^{\frac{p-2}{2p}}\, l_p^{1/p} \ge A_p^{1/p} + 2r\Big\} \ \le\ \exp\big\{-(rn)^{2/p}\big\}.$$

Hence, we obtain (3.3.4) with $c_p = (A_p^{1/p} + 2)^p \le (\sqrt{p} + 2)^p$. Using $A_3 = \sqrt{3}$ and $A_4 = 3$, one may take $c_3 = (A_3^{1/3} + 2)^3 < 33$ and $(A_4^{1/4} + 2)^4 < 121$. $\square$

5.4 Upper Bounds on Characteristic Functions

The property that the values of the characteristic functions $f_\theta(t) = \mathbb{E}\, e^{itS_\theta}$ are small in absolute value for most of $\theta \in S^{n-1}$ with large t may be seen from the following:

Lemma 5.4.1 *For all $t \in \mathbb{R}$,*

$$\mathbb{E}_\theta\, |f_\theta(t)|^2 \le 5\, e^{-t^2/2} + 5\, e^{-n/(12\,\beta_4)}. \tag{5.4.1}$$

Proof Using an independent copy $Y = (Y_1 \dots, Y_n)$ of the random vector $X = (X_1, \dots, X_n)$ in $\mathbb{R}^n$, write

$$|f_\theta(t)|^2 = \mathbb{E}\, e^{it\langle X-Y,\theta\rangle}, \qquad \theta \in S^{n-1},$$

and integrate over the sphere, which gives

$$\mathbb{E}_\theta\, |f_\theta(t)|^2 = \mathbb{E}\, J_n(t\, |X-Y|),$$

where by $J_n(t) = \mathbb{E}_\theta\, e^{it\theta_1}$ we denote the characteristic function of the first coordinate on the sphere under $\mathfrak{s}_{n-1}$. One may split the last expectation to the event $|X-Y| \le \sqrt{n}$ and to the opposite one, which implies

$$\mathbb{E}_\theta\, |f_\theta(t)|^2 \ \le \sup_{u \ge t\sqrt{n}} |J_n(u)| + \mathbb{P}\{|X-Y|^2 \le n\}.$$

To proceed, we employ the bound $|J_n(u)| \leq 5\, e^{-u^2/2n} + 4\, e^{-n/12}$ derived in [5], cf. Proposition 3.3. Consequently, since $\beta_4 \geq 1$, the inequality (5.4.1) would follow from

$$\mathbb{P}\{|X-Y|^2 \leq n\} \leq e^{-n/(16\,\beta_4)} \tag{5.4.2}$$

But, this bound is a particular case of the following well-known observation: Given i.i.d. random variables $\xi_k \geq 0$ such that $\mathbb{E}\xi_1 = 1$, the sum $U_n = \xi_1 + \dots + \xi_n$ satisfies

$$\mathbb{P}\{U_n \leq \lambda n\} \leq \exp\Big\{-\frac{(1-\lambda)^2}{2\,\mathbb{E}\xi_1^2}\,n\Big\}, \qquad 0 < \lambda < 1. \tag{5.4.3}$$

To recall a standard argument, note that

$$\mathbb{E}\,e^{-rU_n} \geq e^{-\lambda r n}\,\mathbb{P}\{U_n \leq \lambda n\}, \qquad r \geq 0. \tag{5.4.4}$$

The function $\psi(r) = \mathbb{E}\,e^{-r\xi_1}$ is positive and admits Taylor's expansion near zero up to the quadratic form, which implies that

$$\psi(r) \ \leq\ 1 - r\,\mathbb{E}\xi_1 + \frac{r^2}{2}\,\mathbb{E}\xi_1^2 \ \leq\ \exp\Big\{-r\,\mathbb{E}\xi_1 + \frac{r^2}{2}\,\mathbb{E}\xi_1^2\Big\}.$$

Hence

$$\mathbb{E}\,e^{-rU_n} = \psi(r)^n \leq \exp\Big\{-rn + \frac{nr^2}{2}\,\mathbb{E}\xi_1^2\Big\}.$$

In view of (5.4.4), this bound yields

$$\mathbb{P}\{U_n \leq \lambda n\} \leq \exp\Big\{-(1-\lambda)nr + \frac{nr^2}{2}\,\mathbb{E}\xi_1^2\Big\},$$

and after optimization over r we arrive at (5.4.3).

In the case $\xi_k = \frac{1}{2}\,(X_k - Y_k)^2$ with i.i.d. X_k such that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$, $\mathbb{E}X_1^4 = \beta_4$, we have

$$\mathbb{E}\xi_1^2 = \frac{1}{2}\,\mathbb{E}\xi_1^4 + \frac{3}{2}(\mathbb{E}\xi_1^2)^2 \leq 2\beta_4,$$

and (5.4.3) yields

$$\mathbb{P}\{|X-Y|^2 \leq 2\lambda n\} \leq \exp\Big\{-\frac{(1-\lambda)^2}{4\beta_4}\,n\Big\}.$$

To obtain (5.4.2), it remains to put here $\lambda = 1/2$. $\square$

Let us now turn to the integrals

$$I_s(\theta) \,=\, 1_{\Omega_s} \int_{1/L_3}^{1/L_s} \frac{|f_\theta(t)|}{t}\,dt, \qquad \Omega_s = \{\theta \in S^{n-1} : L_s \leq L_3 \leq 1\}, \tag{5.4.5}$$

appearing in the Berry–Esseen-type bound (5.2.9) for the scheme of the weighted sums with $L_s = L_s(\theta)$. Since in general $L_s \geq \beta_s\, n^{-\frac{s-2}{2}}$, necessarily $I_s(\theta) = 0$, if $\beta_s > n^{\frac{s-2}{2}}$.

Lemma 5.4.2 *Given an integer $s \geq 4$, we have*

$$c_s\, \mathfrak{s}_{n-1}\big\{I_s(\theta) \geq \beta_s\, n^{-\frac{s-2}{2}}\big\} \,\leq\, \exp\big\{-n^{2/3}\big\} + \exp\Big\{-\frac{cn}{\beta_4}\Big\}. \tag{5.4.6}$$

In particular,

$$\mathbb{E}_\theta\, I_s(\theta) \leq C_s \beta_s\, n^{-\frac{s-2}{2}}. \tag{5.4.7}$$

Proof Introduce the sets on the unit sphere $\Omega_0 = \{L_3 < 2cb_n\}$, $\Omega_1 = \{L_3 \geq 2cb_n\}$, where $b_n = \beta_3/\sqrt{n}$ and $c = 33$. By Lemma 5.3.3 with $p = 3$,

$$\mathfrak{s}_{n-1}(\Omega_1) \,\leq\, \exp\big\{-(2n)^{2/3}\big\}.$$

Since $L_s \geq n^{-\frac{s-2}{2}}$, while $|f_\theta(t)| \leq 1$, we get, for all $\theta \in S^{n-1}$,

$$I_s(\theta) \,=\, 1_{\Omega_s} \int_{1/L_3}^{1/L_s} \frac{|f_\theta(t)|}{t}\,dt \,\leq\, \int_1^{n^{\frac{s-2}{2}}} \frac{1}{t}\,dt \,=\, \frac{s-2}{2}\,\log n,$$

and conclude that

$$\begin{aligned} \mathbb{E}_\theta\,\big[I_s(\theta)\,1_{\Omega_1}\big] &\leq \frac{s-2}{2}\,\log n\ \mathfrak{s}_{n-1}(\Omega_1) \,\leq\, \frac{s-2}{2}\,\log n\ \exp\big\{-(2n)^{2/3}\big\} \\ &\leq Cs\,\exp\big\{-n^{2/3}\big\}. \end{aligned} \tag{5.4.8}$$

Given $\theta \in \Omega_0 \cap \Omega_s$, let us extend the integration in (5.4.5) to the interval $[T_0, T]$ with endpoints $T_0 = \max\{1, (2cb_n)^{-1}\}$ and $T = \frac{1}{\beta_s}\, n^{\frac{s-2}{2}}$, and with the requirement that $T_0 \leq T$. Since, by Lemma 5.4.1,

$$\mathbb{E}_\theta\,|f_\theta(t)| \leq 3\,e^{-t^2/4} + 3\,e^{-n/(32\,\beta_4)},$$

we obtain that

$$\begin{aligned} \mathbb{E}_\theta\, I_s(\theta)\, 1_{\Omega_0} &\le 3\, \mathbb{E}_\theta \int_{T_0}^{T} \big(e^{-t^2/4} + e^{-n/(32\,\beta_4)}\big)\, \frac{dt}{t} \\ &\le 6\, e^{-T_0^2/4} + 3 \log \frac{T}{T_0}\, e^{-n/(32\,\beta_4)} \\ &\le 6\, \exp\Big\{ - \frac{1}{(4cb_n)^2} \Big\} + 3 \log \frac{n^{\frac{s-2}{2}}}{\beta_s}\, e^{-n/(32\,\beta_4)}. \end{aligned} \tag{5.4.9}$$

Due to the assumption $\mathbb{E}X_1^2 = 1$, the function $p \to \beta_p^{1/(p-2)}$ is non-decreasing in p, so, $\beta_4 \le \beta_s^{2/(s-2)}$ and

$$\log \frac{n^{\frac{s-2}{2}}}{\beta_s}\, e^{-n/(64\,\beta_4)} \le \log \frac{n^{\frac{s-2}{2}}}{\beta_s}\, \exp\Big\{ - \frac{1}{64}\, \beta_s^{-\frac{2}{s-2}}\, n \Big\} \le C_s.$$

This simplifies (5.4.9) to

$$c_s\, \mathbb{E}_\theta\, I_s(\theta)\, 1_{\Omega_0} \ \le\ \exp\Big\{ - \frac{n}{(4c\beta_3)^2} \Big\} + e^{-n/(64\,\beta_4)} \ \le\ 2e^{-c'n/\beta_4},$$

where we used $\beta_3^2 \le \beta_4$. Together with (5.4.8), we thus arrive at

$$c_s\, \mathbb{E}_\theta\, I_s(\theta) \ \le\ \exp\big\{ -n^{2/3} \big\} + \exp\Big\{ - \frac{cn}{\beta_4} \Big\},$$

which yields (5.4.6)–(5.4.7), by applying Markov's inequality and using $\beta_4 \le \beta_s^{2/(s-2)}$. $\square$

5.5 Proof of Theorem 5.1.1

We continue to keep our standard notations in the scheme of the weighted sums

$$S_\theta = \theta_1 X_1 + \dots + \theta_n X_n, \qquad \theta = (\theta_1, \dots, \theta_n) \in S^{n-1},$$

with i.i.d. random variables $X_1, \dots, X_n$ such that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$, $\beta_s = \mathbb{E}\,|X_1|^s < \infty$ for an integer $s \ge 3$. Let us write down the bound of Proposition 5.2.3 for this scheme:

$$c_s\, \rho(F_\theta, \Phi_{s-1,\theta}) \ \le\ L_s(\theta) + I_s(\theta).$$

Here $L_s = L_s(\theta) = \beta_s l_s(\theta) = \beta_s \sum_{k=1}^n |\theta_k|^s$ and

$$I_s(\theta) \ = \ 1_{\Omega_s} \int_{1/L_3}^{1/L_s} \frac{|f_\theta(t)|}{t}\,dt, \qquad \Omega_s = \{\theta \in S^{n-1} : L_s \le L_3 \le 1\}.$$

As we know from Lemma 5.3.1, $\mathbb{E}_\theta L_s(\theta) \le c_s \beta_s\, n^{-\frac{s-2}{2}}$, which is sharpened in Lemma 5.3.3 to

$$\mathfrak{s}_{n-1}\big\{L_s(\theta) \ge c_s \beta_s r\, n^{-\frac{s-2}{2}}\big\} \ \le \ \exp\big\{-(rn)^{2/s}\big\}, \qquad r \ge 1,$$

with $c_s \ = \ (\sqrt{s} + 2)^s$. Since Lemma 5.4.2 provides similar bounds for $I_s(\theta)$, we obtain:

Theorem 5.5.1 *Assuming that $\beta_s < \infty$, let $\Phi_{s-1,\theta}$ be the Edgeworth correction for F_θ of an integer order $s \ge 4$. Then*

$$\mathbb{E}_\theta\, \rho(F_\theta, \Phi_{s-1,\theta}) \ \le \ c_s \beta_s\, n^{-\frac{s-2}{2}}. \tag{5.5.1}$$

Moreover, for all $r \ge 1$,

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, \Phi_{s-1,\theta}) \ge c_s r\, \beta_s\, n^{-\frac{s-2}{2}}\Big\} \ \le \ C_s \varepsilon_s(n,r), \tag{5.5.2}$$

where

$$\varepsilon_s(n,r) \ = \ \exp\big\{-\min\big((rn)^{2/s}, n^{2/3}, cn/\beta_4\big)\big\}.$$

The upper bound on the right-hand side of (5.5.2) has almost an exponential decay with respect to n. For example, when $s = 4$ and with $r = 1$ in (5.5.2), we get

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, \Phi_{3,\theta}) \ge \frac{C\beta_4}{n}\Big\} \ \le \ C\, e^{-c\sqrt{n}}, \qquad n \ge \beta_4^2. \tag{5.5.3}$$

However, F_θ is still approximated by a function depending on θ. According to (5.2.7),

$$\begin{aligned}
\Phi_{3,\theta}(x) &= \Phi(x) - \frac{\gamma_3(\theta)}{3!}\,(x^2-1)\varphi(x) \\
&= \Phi(x) - \frac{\alpha_3}{6}\,(x^2-1)\varphi(x)\,\alpha_3(\theta), \qquad \alpha_3 = \mathbb{E}X_1^3, \ \ \alpha_3(\theta) = \sum_{k=1}^n \theta_k^3.
\end{aligned}$$

To eliminate the correction term in the case $\alpha_3 \neq 0$, note that $|x^2 - 1|\,\varphi(x) \leq 1$, leading to

$$\rho(\Phi_{3,\theta}, \Phi) \leq \beta_3\, |\alpha_3(\theta)|.$$

But, $\alpha_3(\theta)$ is of order $1/n$, as indicated in Lemma 5.3.2. Using (5.5.1)–(5.5.2), this gives

$$\mathbb{E}_\theta\, \rho(F_\theta, \Phi) \leq \frac{C}{n}\, \beta_4$$

and

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, \Phi) \geq \frac{Cr}{n}\, \beta_4\Big\} \;\leq\; C \exp\big\{-c \min\big((rn)^{1/2}, n^{2/3}, n/\beta_4, r^{2/3}\big)\big\}$$

with arbitrary $r \geq 1$, which may be assumed to satisfy $r \leq n/(C\beta_4)$. But, in this case, within a universal factor the quantities $(rn)^{1/2}$, $n^{2/3}$ and n/β_4 dominate $r^{2/3}$. We thus arrive at the Klartag-Sodin theorem for the i.i.d. situation with a slight improvement of the power of r (which was actually mentioned in [7]). In addition, one may emphasize a concentration threshold phenomenon as in (5.5.3) for the case where $\alpha_3 = 0$.

Corollary 5.5.2 *If β_4 is finite, then for all $r \geq 1$,*

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, \Phi) \geq \frac{Cr}{n}\, \beta_4\Big\} \;\leq\; C\, \exp\big\{-cr^{2/3}\big\}.$$

Moreover, if $\alpha_3 = 0$ and $n \geq \beta_4^2$, then

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, \Phi) \geq \frac{C\beta_4}{n}\Big\} \;\leq\; C\, \exp\big\{-c\sqrt{n}\big\}.$$

On the other hand, if $\alpha_3 = 0$ and β_5 is finite, one may turn to the next Edgeworth correction which is given according to (5.2.8) by

$$\begin{aligned}\Phi_{4,\theta}(x) &= \Phi(x) - \frac{\gamma_4(\theta)}{4!}\, H_3(x)\, \varphi(x) \\ &= \Phi(x) - \frac{\beta_4 - 3}{24}\, H_3(x)\, \varphi(x)\, l_4(\theta), \qquad l_4(\theta) = \sum_{k=1}^n \theta_k^4, \end{aligned} \tag{5.5.4}$$

with $H_3(x) = x^3 - 3x$. This approximation also depends on θ, but the correction term does not have mean zero.

Proof of Theorem 5.1.1. Using $\mathbb{E}_\theta\, l_4(\theta) = \frac{3}{n+2}$, let us rewrite the above as

$$\Phi_{4,\theta}(x) \ = \ G(x) + \frac{\beta_4 - 3}{4n(n+2)}\, H_3(x)\, \varphi(x) - \frac{\beta_4 - 3}{24}\, H_3(x)\, \varphi(x)\, \Big(l_4(\theta) - \frac{3}{n+2}\Big),$$

where

$$G(x) = \Phi(x) - \frac{\beta_4 - 3}{8n}\, H_3(x)\, \varphi(x)$$

does not contain θ anymore. Since $|\beta_4 - 3| \leq 2\beta_4$ and $|H_3(x)|\, \varphi(x) \leq 1$, it follows that

$$\rho(\Phi_{4,\theta}, G) \ \leq \ \frac{\beta_4}{2n^2} + \frac{\beta_4}{12}\, \Big|l_4(\theta) - \frac{3}{n+2}\Big|, \tag{5.5.5}$$

which in turn, recalling the bound $\mathrm{Var}_\theta(l_4) < 24\, n^{-3}$, yields

$$\mathbb{E}_\theta\, \rho(\Phi_{4,\theta}, G) \leq 2\beta_4\, n^{-3/2}.$$

But, according to Theorem 5.5.1 with $s = 5$,

$$\mathbb{E}_\theta\, \rho(F_\theta, \Phi_{4,\theta}) \ \leq \ C\beta_5\, n^{-3/2}.$$

These two bounds yield the inequality (5.1.4) of Theorem 5.1.1, by applying the triangle inequality for the distance ρ.

Moreover, applying Lemma 5.3.2, from (5.5.5) it also follows that

$$\mathfrak{s}_{n-1}\big\{\rho(\Phi_{4,\theta}, G) \geq Cr\beta_4\, n^{-3/2}\big\} \ \leq \ C \exp\big\{-cr^{1/2}\big\}, \qquad r \geq 1. \tag{5.5.6}$$

Combining this with the inequality (5.5.2) and recalling that $\beta_4 \leq \beta_5^{2/3} \leq \beta_5$, we get

$$\mathfrak{s}_{n-1}\big\{\rho(F_\theta, G) \geq Cr\beta_5\, n^{-3/2}\big\} \ \leq \ C \exp\big\{-c \min\big((rn)^{2/5}, n^{2/3}, n\beta_5^{-2/3}, r^{1/2}\big)\big\}$$

with an arbitrary value $r \geq 1$. Note that $|G(x)| \leq C\beta_4 \leq C\beta_5$, so that we may restrict ourselves to the region

$$1 \leq r \leq n^{3/2}/(C\beta_5)$$

(since otherwise the left probability is zero). But in this case, necessarily $\beta_5 \leq n^{3/2}/C$, and both $(rn)^{1/2}$ and $n\beta_5^{-2/3}$ dominate $r^{1/2}$. Hence, the above bound is simplified to

$$\mathfrak{s}_{n-1}\big\{\rho(F_\theta, G) \geq Cr\beta_5\, n^{-3/2}\big\} \ \leq \ C \exp\big\{-c \min\big(n^{2/3}, r^{1/2}\big)\big\}.$$

Here, $r^{1/2}$ is dominated by $n^{2/3}$ in the region $r \le n^{4/3}$, in which case we arrive at the desired inequality

$$\mathfrak{s}_{n-1}\big\{\rho(F_\theta, G) \ge Cr\beta_5\, n^{-3/2}\big\} \,\le\, C \exp\big\{-cr^{1/2}\big\}. \tag{5.5.7}$$

As for larger values of r, the usual Berry–Esseen inequality (5.1.1) with a purely Gaussian approximation is more accurate. Indeed, together with Lemma 5.3.3 for $p = 3$, the bound (5.1.1), which is known to hold with $c = 1$, gives, for all $r \ge n$,

$$\begin{aligned}\mathfrak{s}_{n-1}\big\{\rho(F_\theta, \Phi) \ge Cr\beta_3\, n^{-3/2}\big\} &\le \mathfrak{s}_{n-1}\big\{l_3(\theta) \ge Crn^{-3/2}\big\} \\ &\le \exp\big\{-r^{2/3}\big\},\end{aligned} \tag{5.5.8}$$

which sharpens (5.5.7). At the expense of a worse rate, one may replace here Φ with $\Phi_{4,\theta}$. From (5.5.4), $\rho(\Phi_{4,\theta}, \Phi) \le \beta_4\, l_4(\theta)$. Hence, applying Lemma 5.3.3 with $p = 4$ and with $r/\sqrt{n}$ in place of r (which is justified as long as $Cr \ge 121\sqrt{n}$), we get

$$\begin{aligned}\mathfrak{s}_{n-1}\big\{\rho(\Phi_{4,\theta}, \Phi) \ge Cr\beta_4\, n^{-3/2}\big\} &\le \mathfrak{s}_{n-1}\big\{n\, l_4(\theta) \ge Crn^{-1/2}\big\} \\ &\le \exp\big\{-r^{1/2}\, n^{1/4}\big\} \,\le\, \exp\big\{-r^{1/2}\big\}.\end{aligned}$$

Combining this with (5.5.8), we get

$$\mathfrak{s}_{n-1}\big\{\rho(F_\theta, \Phi_{4,\theta}) \ge Cr\beta_4\, n^{-3/2}\big\} \,\le\, 2\, \exp\big\{-r^{1/2}\big\}, \qquad r \ge n.$$

Finally, by (5.5.6),

$$\mathfrak{s}_{n-1}\Big\{\rho(F_\theta, G) \ge Cr\beta_4\, n^{-3/2}\Big\} \,\le\, (C+2)\, \exp\big\{-cr^{1/2}\big\}.$$

This means that we have obtained the required bound (5.5.7) for all values $r \ge 1$. It remains to rescale the parameter r to arrive at the inequality (5.1.5) of Theorem 5.1.1. □

5.6 General Lower Bounds

Let U be a function of bounded variation on the real line with $U(-\infty) = U(\infty) = 0$. By analogue with Berry–Esseen-type theorems, a standard approach to the problem of lower bounds for the L^∞-norm $\|U\| = \sup_x |U(x)|$ may be based on the study of the associated Fourier–Stieltjes transform

$$u(t) = \int_{-\infty}^{\infty} e^{itx}\, dU(x), \qquad t \in \mathbb{R}.$$

For example, we have the following estimate derived in [2], cf. Theorem 19.2.

Lemma 5.6.1 *For any $T > 0$,*

$$\|U\| \ \geq \ \frac{1}{3T}\bigg|\int_0^T u(t)\,\Big(1-\frac{t}{T}\Big)\,dt\bigg|. \tag{5.6.1}$$

In the scheme of the weighted sums, introduce the characteristic function $f(t) = \mathbb{E}_\theta\, f_\theta(t)$ of the average distribution function $F(x) = \mathbb{E}_\theta\, F_\theta(x) = \mathbb{E}_\theta\, \mathbb{P}\{S_\theta \leq x\}$. Lemma 5.6.1 may be used to derive:

Lemma 5.6.2 *Given a function G of bounded variation such that $G(-\infty) = 0$ and $G(\infty) = 1$, for any $T > 0$,*

$$\mathbb{E}_\theta\, \rho(F_\theta, G) \ \geq \ \frac{1}{6\sqrt{2}\,T}\ \mathbb{E}_\theta\,\bigg|\int_0^T (f_\theta(t) - f(t))\,\Big(1-\frac{t}{T}\Big)dt\bigg|. \tag{5.6.2}$$

Proof Given a complex-valued random variable ξ with finite first absolute moment, for any complex number b,

$$\mathbb{E}\,|\xi - b| \geq \frac{1}{2\sqrt{2}}\,\mathbb{E}\,|\xi - \mathbb{E}\xi|. \tag{5.6.3}$$

For the proof of this claim, first note that, by the triangle inequality,

$$\mathbb{E}\sqrt{\eta_0^2 + \eta_1^2} = \mathbb{E}\,|\eta| \geq |\mathbb{E}\eta| = \sqrt{(\mathbb{E}\eta_0)^2 + (\mathbb{E}\eta_1)^2}$$

for any complex-valued random variable η with $\eta_0 = \mathrm{Re}(\eta)$, $\eta_1 = \mathrm{Im}(\eta)$. Replacing η_0 with $|\eta_0|$ and η_1 with $|\eta_1|$, the above can be formally sharpened to

$$\mathbb{E}\sqrt{\eta_0^2 + \eta_1^2} \ \geq \sqrt{(\mathbb{E}\,|\eta_0|)^2 + (\mathbb{E}\,|\eta_1|)^2}. \tag{5.6.4}$$

Now, write $\xi = \xi_0 + i\xi_1$. Since the inequality (5.6.3) is shift invariant, we may assume that both ξ_0 and ξ_1 have median at zero. In that case, for any $b = b_0 + ib_1$, $b_0, b_1 \in \mathbb{R}$,

$$\mathbb{E}\,|\xi| \leq \mathbb{E}\,|\xi_0| + \mathbb{E}\,|\xi_1| \leq \mathbb{E}\,|\xi - b_0| + \mathbb{E}\,|\xi - b_1|.$$

so,

$$\mathbb{E}\,|\xi - \mathbb{E}\xi| \leq 2\,\mathbb{E}\,|\xi| \leq 2\,\mathbb{E}\,|\xi - b_0| + 2\,\mathbb{E}\,|\xi - b_1|.$$

Using (5.6.4) with $\eta_0 = \xi - b_0$ and $\eta_1 = \xi - b_1$, this gives

$$\begin{aligned}\mathbb{E}\,|\xi - b| &\geq \sqrt{(\mathbb{E}\,|\xi - b_0|)^2 + (\mathbb{E}\,|\xi - b_1|)^2} \\ &\geq \frac{1}{\sqrt{2}}\,\mathbb{E}\,|\xi - b_0| + \frac{1}{\sqrt{2}}\,\mathbb{E}\,|\xi - b_1| \ \geq\ \frac{1}{2\sqrt{2}}\,\mathbb{E}\,|\xi - \mathbb{E}\xi|,\end{aligned}$$

and we arrive at (5.6.3).

Finally, denote by g the Fourier–Stieltjes transform of G. We apply (5.6.3) on the probability space $(S^{n-1}, \mathfrak{s}_{n-1})$ with

$$\xi = \int_0^T f_\theta(t)\,\Big(1 - \frac{t}{T}\Big)dt, \qquad b = \int_0^T g(t)\,\Big(1 - \frac{t}{T}\Big)dt.$$

In view of (5.6.1) applied to $U = F_\theta - G$, we then get

$$\begin{aligned}\mathbb{E}_\theta\,\rho(F_\theta, G) \ =\ \mathbb{E}_\theta\,\|F_\theta - G\| &\geq \frac{1}{3T}\,\mathbb{E}_\theta\,\Big|\int_0^T (f_\theta(t) - g(t))\,\Big(1 - \frac{t}{T}\Big)dt\,\Big| \\ &\geq \frac{1}{6\sqrt{2}\,T}\,\mathbb{E}_\theta\,\Big|\int_0^T (f_\theta(t) - f(t))\,\Big(1 - \frac{t}{T}\Big)dt\,\Big|.\end{aligned}$$

□

5.7 Approximation by Mean Characteristic Functions

To apply the lower bound (5.6.2), we need to look once more at the asymptotic behaviour of characteristic functions $f_\theta(t)$, at least near zero. To this aim, Proposition 5.2.2 may still be used. In the scheme of the weighted sums, it gives the next two assertions for Edgeworth approximations of orders 4 and 5. Put $\alpha_3 = \mathbb{E}X_1^3$ and $f(t) = \mathbb{E}_\theta\, f_\theta(t)$.

Lemma 5.7.1 *If β_4 is finite, then for all $\theta \in S^{n-1}$ except for a set on the sphere of measure at most $C\beta_3\, e^{-\sqrt{n}}$, in the interval $|t| \leq T_n = \sqrt{n}/(33\,\beta_3)$, we have*

$$f_\theta(t) - f(t) \ =\ \alpha_3\,\alpha_3(\theta)\,\frac{(it)^3}{3!}\,e^{-t^2/2} + \varepsilon \tag{5.7.1}$$

with

$$|\varepsilon| \ \leq\ C\beta_4\,n^{-1}\,t^4\,e^{-t^2/8} + C\beta_3\,\exp\big\{-\sqrt{n}\big\}.$$

Lemma 5.7.2 *If β_5 is finite and $\alpha_3 = 0$, then for all $\theta \in S^{n-1}$ except for a set of measure at most $C\beta_4\, \exp\{-n^{2/5}\}$, in the interval $|t| \le T_n$, we have*

$$f_\theta(t) - f(t) \,=\, (\beta_4 - 3)\Big(l_4(\theta) - \frac{3}{n+2}\Big)\,\frac{t^4}{4!}\,e^{-t^2/2} + \varepsilon, \tag{5.7.2}$$

where

$$|\varepsilon| \,\le\, C\beta_5\, n^{-3/2}\, |t|^5\, e^{-t^2/8} + C\beta_4\, \exp\big\{-n^{2/5}\big\}.$$

Proof According to Definition 5.2.1 with $s = 4$ and $s = 5$, $f_\theta(t)$ is approximated by the functions of the form (5.2.2)–(5.2.3), that is, by

$$g_{3,\theta}(t) = e^{-t^2/2}\Big(1 + \gamma_3(\theta)\frac{(it)^3}{3!}\Big), \qquad \gamma_3(\theta) = \alpha_3\, \alpha_3(\theta),$$
$$g_{4,\theta}(t) = e^{-t^2/2}\Big(1 + \gamma_4(\theta)\frac{t^4}{4!}\Big), \qquad \gamma_4(\theta) = (\beta_4 - 3)\, l_4(\theta),$$

where we assume that $\alpha_3 = 0$ in the second case. More precisely, if $\beta_4 < \infty$, then for any $\theta \in S^{n-1}$, in the interval $|t| \le 1/L_3(\theta)$, we have that

$$|f_\theta(t) - g_{3,\theta}(t)| \,\le\, C\beta_4\, l_4(\theta)\, t^4\, e^{-t^2/8}. \tag{5.7.3}$$

Moreover, if $\beta_5 < \infty$ and $\alpha_3 = 0$, then in the same interval,

$$|f_\theta(t) - g_{4,\theta}(t)| \,\le\, C\beta_5\, l_5(\theta)\, |t|^5\, e^{-t^2/8}. \tag{5.7.4}$$

Using the results from Sect. 5.3, one can simplify these relations for a majority of the coefficients. As was already stressed (as a consequence of Lemma 5.3.3 with $p = 3$),

$$L_3(\theta) \le 33\,\frac{\beta_3}{\sqrt{n}} = \frac{1}{T_n}$$

for all θ from a set Ω on the sphere of measure at least $1 - \exp\{-n^{2/3}\}$. Therefore, the bounds (5.7.3)–(5.7.4) are fulfilled for all $|t| \le T_n$ and for all $\theta \in \Omega$.

Moreover, by Lemma 5.3.3,

$$\mathfrak{s}_{n-1}\big\{l_4(\theta) \ge Cn^{-1}\big\} \le \exp\big\{-\sqrt{n}\big\}.$$

Therefore, (5.7.3) leads to a simpler version

$$\big|f_\theta(t) - g_{\theta,3}(t)\big| \,\le\, C\beta_4\, n^{-1}\, t^4\, e^{-t^2/8}, \qquad |t| \le T_n, \tag{5.7.5}$$

which holds for all θ except for a set $\mathcal{F}$ on the sphere of measure at most $2\exp\{-\sqrt{n}\}$.

Clearly, one may replace f_θ and $g_{3,\theta}$ in (5.7.5) by their mean values $f(t) = \mathbb{E}_\theta\, f_\theta(t)$ and $g(t) = \mathbb{E}_\theta\, g_{3,\theta}(t) = e^{-t^2/2}$ at the expense of an error not exceeding

$$\mathfrak{s}_{n-1}(\mathcal{F})\, \sup_t\, |f_\theta(t) - g_{3,\theta}(t)| \,\le\, C\beta_3\, \exp\big\{-\sqrt{n}\big\}.$$

Averaging over θ in (5.7.5), it thus yields in the same interval

$$\big|f(t) - g(t)\big| \,\le\, C\beta_4\, n^{-1}\, t^4\, e^{-t^2/8} + C\beta_3\, \exp\big\{-\sqrt{n}\big\}. \tag{5.7.6}$$

Finally, combining the latter with (5.7.5), one may bound the expression

$$(f_\theta(t) - g_{3,\theta}(t)) - (f(t) - g(t)) \,=\, (f_\theta(t) - f(t)) - \alpha_3\, \alpha_3(\theta)\, \frac{(it)^3}{3!}\, e^{-t^2/2}$$

by the same quantity as on the right-hand side of (5.7.6). This proves Lemma 5.7.1.

Now, turning to (5.7.4), we apply Lemma 5.3.3 with $p = 5$, when it gives

$$\mathfrak{s}_{n-1}\big\{l_5(\theta) \ge Cn^{-3/2}\big\} \le \exp\big\{-n^{2/5}\big\}.$$

Hence, we get a simpler version

$$\big|f_\theta(t) - g_{4,\theta}(t)\big| \,\le\, C\beta_5\, n^{-3/2}\, |t|^5\, e^{-t^2/8}, \qquad |t| \le T_n, \tag{5.7.7}$$

which holds for all θ except for a set $\mathcal{F}$ on the sphere of measure at most $2\exp\{-n^{2/5}\}$. Again, at the expense of an error not exceeding

$$\mathfrak{s}_{n-1}(\mathcal{F})\, \sup_t\, |f_\theta(t) - g_{4,\theta}(t)| \,\le\, C\beta_4\, \exp\big\{-n^{2/5}\big\},$$

one may replace f_θ and $g_{4,\theta}$ in (5.7.7) by their mean values $f(t)$ and $g(t)$, where now

$$g(t) = \mathbb{E}_\theta\, g_\theta(t) = e^{-t^2/2}\Big(1 + \frac{\alpha}{n+2}\, t^4\Big), \qquad \alpha = \frac{\beta_4 - 3}{8}.$$

Averaging over θ in (5.7.7), it thus yields

$$\big|f(t) - g(t)\big| \,\le\, C\beta_5\, n^{-3/2}\, |t|^5\, e^{-t^2/8} + C\beta_4\, \exp\big\{-n^{2/5}\big\}. \tag{5.7.8}$$

Finally, combining the latter with (5.7.5), one may bound the expression

$$(f_\theta(t) - g_{4,\theta}(t)) - (f(t) - g(t)) = (f_\theta(t) - f(t)) - (\beta_4 - 3)\Big(l_4(\theta) - \frac{3}{n+2}\Big)\, \frac{t^4}{4!}\, e^{-t^2/2}$$

by the same quantity as on the right-hand side of (5.7.8). This proves Lemma 5.7.2. □

5.8 Proof of Theorem 5.1.2

First, let us apply Lemma 5.6.2 by virtue of the representation (5.7.1) from Lemma 5.7.1, which holds for all θ in $\mathcal{F} \subset S^{n-1}$ of measure at least $1 - C\beta_3 \exp\{-n^{1/2}\}$ in the interval $|t| \leq \sqrt{n}/(33\,\beta_3)$. Given $0 < T \leq 1$, we have

$$\int_0^T t^3\, e^{-t^2/2} \Big(1 - \frac{t}{T}\Big)\, dt \ \geq\ \frac{1}{4} \int_0^{T/2} t^3\, dt \ =\ \frac{1}{256}\, T^4.$$

On the other hand,

$$\int_0^T t^4\, e^{-t^2/8} \Big(1 - \frac{t}{T}\Big)\, dt \ \leq\ \frac{1}{5}\, T^5.$$

Therefore, for all $\theta \in \mathcal{F}$ and $n \geq (33\,\beta_3)^2$,

$$\bigg|\int_0^T (f_\theta(t) - f(t)) \Big(1 - \frac{t}{T}\Big)\, dt\bigg| \geq c\, |\alpha_3|\, |\alpha_3(\theta)|\, T^4 - \frac{C}{n}\, \beta_4 T^5 - C\beta_3\, \exp\big\{ - n^{1/2}\big\}\, T.$$

Integrating this inequality over the set $\mathcal{F}$ and using

$$\mathbb{E}_\theta\, |\alpha_3(\theta)|\, 1_{\mathcal{F}} \geq \frac{c}{n},$$

we arrive at

$$\frac{1}{T}\, \mathbb{E}_\theta \bigg|\int_0^T (f_\theta(t) - f(t)) \Big(1 - \frac{t}{T}\Big) dt\bigg| \ \geq\ \frac{T^3}{n}\, \big(c\, |\alpha_3| - C\beta_4 T\big) - C\beta_4\, \exp\big\{-n^{1/2}\big\}.$$

Choosing an appropriate value of $T \sim |\alpha_3|/\beta_4$ and applying Lemma 5.6.2, we get

$$\mathbb{E}_\theta\, \rho(F_\theta, G) \ \geq\ c\, \frac{|\alpha_3|^4}{\beta_4^3\, n} - C\beta_4\, \exp\big\{ - n^{1/2}\big\}$$

with an arbitrary function G of bounded variation such that $G(-\infty) = 0$, $G(\infty) = 1$. The latter immediately yields the required relation (5.1.6) for the range $n \geq n_0$ with constant $c = c_0\, |\alpha_3|^4/\beta_4^3$ and for a sufficiently large n_0 depending α_3 and β_4.

To involve the remaining values $2 \le n < n_0$, let us to note that the infimum in (5.1.6) is positive. Indeed, assuming the opposite for a fixed n, there would exist $G \in \mathcal{G}$ such that $\mathbb{E}_\theta\, \rho(F_\theta, G) = 0$ and hence $F_\theta(x) = G(x)$ for all $\theta \in S^{n-1}$ and all points x. In particular, all the weighted sums S_θ would be equidistributed. But this is only possible when all the random variables X_k have a standard normal distribution, according to the Pólya characterization theorem [12], cf. also [6]. And this contradicts to the assumption $\alpha_3 \neq 0$.

The second assertion, where $\alpha_3 = 0$, but $\beta_4 \neq 0$, is similar. We now apply Lemma 5.6.2 by virtue of the representation (5.7.2) of Lemma 5.7.2, which holds for all θ in a set $\mathcal{F} \subset S^{n-1}$ of measure at least $1 - C\beta_4 \exp\{-n^{2/5}\}$ in the same interval $|t| \le \sqrt{n}/(33\,\beta_3)$. Given $0 < T \le 1$, we have

$$\int_0^T t^4\, e^{-t^2/2} \Big(1 - \frac{t}{T}\Big) dt \ \ge\ \frac{1}{4} \int_0^{T/2} t^4\, dt \ =\ \frac{1}{640}\, T^5.$$

On the other hand,

$$\int_0^T t^5\, e^{-t^2/8} \Big(1 - \frac{t}{T}\Big) dt \ \le\ \frac{1}{6}\, T^6.$$

Therefore, for all $\theta \in \mathcal{F}$ and $n \ge (33\,\beta_3)^2$,

$$\bigg|\int_0^T (f_\theta(t) - f(t)) \Big(1 - \frac{t}{T}\Big) dt\bigg| \ge c\, \big|\beta_4 - 3\big|\, \Big|l_4(\theta) - \frac{3}{n+2}\Big|\, T^5 - \frac{C}{n^{3/2}}\, \beta_5\, T^6 - C\beta_4\, \exp\big\{-n^{2/5}\big\}.$$

Integrating this inequality over the set $\mathcal{F}$ and using

$$\mathbb{E}_\theta\, \Big|l_4(\theta) - \frac{3}{n+2}\Big|\, 1_{\mathcal{F}} \ge \frac{c}{n^{3/2}},$$

we arrive at

$$\frac{1}{T}\, \mathbb{E}_\theta \,\bigg|\int_0^T (f_\theta(t) - f(t)) \Big(1 - \frac{t}{T}\Big)\, dt\bigg| \ge \frac{T^4}{n^{3/2}} \big(c\, \big|\beta_4 - 3\big| - C\beta_5\, T\big) - C\beta_5\, \exp\big\{-n^{2/5}\big\}.$$

Choosing an appropriate value of $T \sim |\beta_4 - 3|/\beta_5$ and applying Lemma 5.6.2, we get

$$\mathbb{E}_\theta\, \rho(F_\theta, G)\ \ge\ c\, \frac{|\beta_4 - 3|^5}{\beta_5^4\, n} - C\beta_5\, \exp\big\{-n^{2/5}\big\}.$$

The latter yields the required relation (5.1.7) for the range $n \geq n_0$ with constant

$$c = c_0 \, |\beta_4 - 3|^5 / \beta_5^4$$

and with a sufficiently large n_0 depending β_4 and β_5. A similar argument as before allows us to involve the remaining values $2 \leq n < n_0$ as well. □

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