ON MILMAN'S ELLIPSOIDS AND M–POSITION OF CONVEX BODIES *†

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Abstract

Milman's ellipsoids and an M-position of convex bodies are described in terms of isotropic restricted Gaussian measures.

1 Introduction

For symmetric convex bodies A and B in $\mathbf{R}^n$, put

$$M(A,B) = \left(\frac{|A+B|}{|A\cap B|}\cdot\frac{|A^{\mathrm{o}}+B^{\mathrm{o}}|}{|A^{\mathrm{o}}\cap B^{\mathrm{o}}|}\right)^{1/n}.$$

Here and below we denote by $|A|$ the n-dimensional volume of a set A in $\mathbf{R}^n$, and by $A^{\mathrm{o}} = \{x\in\mathbf{R}^n : \langle x,y\rangle \leq 1, \forall y\in A\}$ its polar.

A main result about the quantity $M(A,B)$ is the following theorem due to V. D. Milman. Let us make the convention that all ellipsoids (in particular, all Euclidean balls) have the center at the origin.

Theorem 1.1 (V. D. Milman [M1]). *For any symmetric convex body K in $\mathbf{R}^n$, there exists an ellipsoid $\mathcal{E}$ such that*

$$M(K,\mathcal{E}) \leq C, \tag{1.1}$$

where C is a universal constant.

An ellipsoid $\mathcal{E}$ which appears in this statement is called Milman's ellipsoid or, for short, an M-ellipsoid (although the definition involves an implicit constant C). This deep result contains as corollaries a number of important facts in Convex Geometry, such as the reverse Santalo inequality due to J. Bourgain and V. D. Milman [B-M], Milman's reverse Brunn-Minkowski inequality [M1], the duality of entropy numbers [K-M]. There are some other equivalent definitions of M-ellipsoids, for example, in terms of the entropy numbers. For different proofs, see subsequent works of V. D. Milman [M2-4], and the book by G. Pisier [P], which contains an excellent exposition and historical remarks.

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Note that if $\mathcal{E}$ is an M-ellipsoid for K, then the polar ellipsoid $\mathcal{E}^{\circ}$ is an M-ellipsoid for the polar body K°. In general

$$M(T(A),T(B)) = M(A,B) = M(A^{\circ},B^{\circ}) \tag{1.2}$$

for any linear invertible map $T : \mathbf{R}^n \to \mathbf{R}^n$, so the M-functional represents an affine invariant of the couple (A,B), as well as of the couple (A°,B°). Hence, in Theorem 1.1 one may always choose T such that $T(\mathcal{E})$ is a Euclidean ball. In this case, one says that $T(K)$ is in M-position (i.e., in "main" position according to [M4], which also corresponds to the notion of a "regular" position in [P]). In other words, a symmetric convex body K is in M-position, when the inequality (1.1) holds true for some Euclidean ball $\mathcal{E}$.

Since Theorem 1.1 only states the existence of an M-ellipsoid, it is natural to ask how to find or constructively describe it. Equivalently, one may wonder how to find an M-position for K (that is, a map T). One way towards a solution to this question seems the notion of the isotropic position.

Let us recall that a symmetric log-concave probability measure μ on $\mathbf{R}^n$ with a (symmetric log-concave) density f is isotropic, if for any vector θ from the unit sphere S^{n-1},

$$f(0)^{2/n} \int \langle x,\theta\rangle^2 \, d\mu(x) \, = \, L_\mu^2, \tag{1.3}$$

for some positive L_μ, called an isotropic constant of μ. By simple algebra, any symmetric log-concave measure μ can be put in the isotropic position, and often the condition (1.3) has a matter of normalization, only. As a particular case, a symmetric convex body K with unit volume is called isotropic with an isotropic constant $L_K > 0$, if the restricted Lebesgue measure on K with the indicator density function $f = 1_K$ is isotropic, i.e., for any $\theta \in S^{n-1}$,

$$\int_K \langle x,\theta\rangle^2 \, dx \, = \, L_K^2.$$

There is a good reason to expect that any symmetric convex body, which is in the isotropic position, is in M-position. As was noticed in [B-M-K], if this was true, the isotropic constants would be bounded from above by an absolute constant (this assertion represents an equivalent formulation of the so-called hyperplane conjecture). In this note we show that, regardless of whether this is true or not, an M-position of convex bodies may indeed be related to the isotropy – but in a different class of log-concave probability distributions.

Denote by γ the standard Gaussian measure on $\mathbf{R}^n$ with density $\varphi(x) = (2\pi)^{-n/2}\, e^{-|x|^2/2}$, $x \in \mathbf{R}^n$. For symmetric convex bodies K in $\mathbf{R}^n$, we consider the normalized restrictions of this measure to K, defined by

$$\gamma_K(A) = \gamma(A \cap K)/\gamma(K)$$

on Borel subsets A of the space. Theorem 1.1 may be complemented with the following:

Theorem 1.2. *Given a symmetric convex body K in $\mathbf{R}^n$ with volume $|K| = 1$, assume the normalized restricted Gaussian measure γ_K is isotropic. Then K is in M-position. Moreover,*

$$L_{\gamma_K} \leq C, \tag{1.4}$$

for some positive numerical constant C.

Thus, the isotropic constants are universally bounded for the class of isotropic restricted Gaussian measures.

The argument leading to (1.4) essentially uses Theorem 1.1. To make its application convenient, first we discuss different equivalent representations for the functional $M(A,B)$, which are easily obtained by virtue of the reverse Santalo inequality and an extension of Roger-Shephard's inequality to the case of two bodies (Section 2). In Section 3, the isotropic positions for the normalized restricted Gaussian measures are described as solutions to the variational problem, where $\gamma(T(K))$ is to be maximized among all volume-preserving linear maps T. On this step, we involve a generalized form of the so-called B-conjecture, considered and solved in [CE-F-M]. A final step of the proof, based on the concentration property of restricted Gaussian measures, is made in Section 4.

2 Representations for $M(A,B)$

We need one generalization of the well-known Roger-Shephard's difference body inequality.

Proposition 2.1 (C. A. Roger and G. C. Shephard [R-S2]). *For all convex bodies A and B in $\mathbf{R}^n$,*

$$|A-B|\,|A\cap B| \,\leq\, \frac{(2n)!}{n!^{\,2}}\,|A|\,|B|.$$

In case of one convex body, that is, when $A=B$, the above inequality is reduced to

$$|A-A| \,\leq\, \frac{(2n)!}{n!^{\,2}}\,|A|.$$

It was first proved in [R-S1], and later Roger and Shephard obtained a more general form, involving two convex bodies; cf. [R-S2], Theorem 1 on p.273. It can also be derived from Berwald's Khinchin-type inequality for the class of concave functions, cf. [Ber], [Bor].

If convex bodies A and B in $\mathbf{R}^n$ are symmetric (which is always assumed in the sequel), Proposition 2.1 implies

$$|A|^{1/n}\,|B|^{1/n} \,\leq\, |A+B|^{1/n}\,|A\cap B|^{1/n} \,\leq\, 4\,|A|^{1/n}\,|B|^{1/n}, \tag{2.1}$$

where the (trivial) left inequality is added to compare with the right inequality.

Definition. For two expressions Q and Q', depending on the dimension n, we write $Q\sim Q'$, if for any $n\geq 1$,

$$cQ\leq Q'\leq c'Q,$$

with some numerical positive constants c, c'.

For example, (2.1) gives the equivalence

$$|A+B|^{1/n}\,|A\cap B|^{1/n}\sim |A|^{1/n}\,|B|^{1/n}$$

within the factors 1 and 4. Applying this twice in the definition of the M-functional, we obtain

$$M(A,B) \sim \frac{|A|^{1/n}\,|B|^{1/n}}{|A\cap B|^{2/n}} \cdot \frac{|A^{\mathrm{o}}|^{1/n}\,|B^{\mathrm{o}}|^{1/n}}{|A^{\mathrm{o}}\cap B^{\mathrm{o}}|^{2/n}}.$$

By the Santalo and reverse Santalo inequalities, written as the equivalence $|K|^{1/n}\,|K^{\mathrm{o}}|^{1/n} \sim \frac{1}{n}$ (cf. [B-M]), we then get that

$$M(A,B)^{1/2} \sim \frac{1}{n} \cdot \frac{1}{|A\cap B|^{1/n} \cdot |A^{\mathrm{o}}\cap B^{\mathrm{o}}|^{1/n}}. \tag{2.2}$$

On the other hand, since $\frac{1}{2}\,(A^{\mathrm{o}}\cap B^{\mathrm{o}}) \subset (A+B)^{\mathrm{o}} \subset A^{\mathrm{o}}\cap B^{\mathrm{o}}$, we always have

$$|(A+B)^{\mathrm{o}}|^{1/n} \sim |A^{\mathrm{o}}\cap B^{\mathrm{o}}|^{1/n}.$$

Hence, by the Santalo and the reverse Santalo inequalities, applied to $K = A+B$,

$$|A+B|^{1/n} \sim \frac{1}{n\,|A^{\mathrm{o}}\cap B^{\mathrm{o}}|^{1/n}},$$

so (2.1) implies

$$\frac{|A\cap B|^{1/n}}{|A^{\mathrm{o}}\cap B^{\mathrm{o}}|^{1/n}} \ \sim\ n\,|A|^{1/n}\,|B|^{1/n}.$$

Plugging this in (2.2), we get an equivalent expression for the M-functional, which does not involve polar bodies.

Corollary 2.2. *For all symmetric convex bodies A and B in $\mathbf{R}^n$,*

$$M(A,B)^{1/2} \ \sim\ \frac{|A|^{1/n}\,|B|^{1/n}}{|A\cap B|^{2/n}}, \tag{2.3}$$

as well as

$$M(A,B)^{1/2} \ \sim\ \frac{|A+B|^{2/n}}{|A|^{1/n}\,|B|^{1/n}}.$$

Multiplying the two relations, we also have

$$M(A,B)^{1/2} \ \sim\ \frac{|A+B|^{1/n}}{|A\cap B|^{1/n}}.$$

All these representations remain to hold for the polar bodies by the polar invariance of M (property (1.2)) and seem to be more-less known, although we could not find a direct reference.

Now, for a symmetric convex body K in $\mathbf{R}^n$ with volume $|K| = 1$, introduce the functional

$$M(K) \ =\ \inf_{\mathcal{E}} M(K,\mathcal{E}), \tag{2.4}$$

where the infimum runs over all ellipsoids $\mathcal{E}$ in $\mathbf{R}^n$. Then, Milman's Theorem 1.1 is telling us that $M(K)$ is bounded from above by a universal constant. Using Corollary 2.2, this quantity may be related to a simpler functional

$$m(K) = \sup_{|\mathcal{E}|=1} |K \cap \mathcal{E}|^{1/n}.$$

Corollary 2.3. *For any symmetric convex body K in $\mathbf{R}^n$ with volume $|K| = 1$, up to some positive absolute constants, we have*

$$\frac{C_0}{m(K)} \leq M(K) \leq \frac{C_1}{m(K)^4}. \tag{2.5}$$

Indeed, by (2.3) with $A = K$,

$$M(K,B)^{-1/2} \sim \frac{|K \cap B|^{2/n}}{|B|^{1/n}}, \tag{2.6}$$

which implies

$$\frac{1}{C}\, M(K,B)^{-1/2} \leq |B|^{1/n} \leq C\, M(K,B)^{1/2}$$

with some absolute $C \geq 1$. Hence, for the optimal ellipsoid $\mathcal{E}$ in (2.4), we have $\frac{1}{\lambda} \leq |\mathcal{E}|^{1/n} \leq \lambda$, where $\lambda = C\, M(K)^{1/2}$. By (2.6), this gives

$$M(K)^{-1/2} \sim \sup \Big\{ \frac{|K \cap \mathcal{E}|^{2/n}}{|\mathcal{E}|^{1/n}} : \frac{1}{\lambda} \leq |\mathcal{E}|^{1/n} \leq \lambda \Big\}. \tag{2.7}$$

Restricting the sup on the right-hand side to the ellipsoids with unit volume, we get immediately that $M(K)^{-1/2} \geq c\, m(K)^2$, which is the bound on the right-hand side of (2.5).

On the other hand, put $\mathcal{E}' = \frac{1}{|\mathcal{E}|^{1/n}}\, \mathcal{E}$, so that $|\mathcal{E}'| = 1$. Assuming $|\mathcal{E}|^{1/n} \geq \frac{1}{\lambda}$ and using also that $|K| = 1$, we have

$$\begin{aligned} \frac{|K \cap \mathcal{E}|^{2/n}}{|\mathcal{E}|^{1/n}} \ &\leq \ \frac{|K \cap \mathcal{E}|^{1/n}}{|\mathcal{E}|^{1/n}} \ = \ \Big| \Big(\frac{1}{|\mathcal{E}|^{1/n}}\, K \Big) \cap \mathcal{E}' \Big|^{1/n} \\ &\leq \ |(\lambda K) \cap \mathcal{E}'|^{1/n} \ \leq \ \lambda |K \cap \mathcal{E}'|^{1/n} \ \leq \ \lambda m(K). \end{aligned}$$

Taking the sup over all $\mathcal{E}$ and applying (2.7), we arrive at

$$M(K)^{-1/2} \leq C \lambda m(K) = C' M(K)^{1/2}\, m(K),$$

which is equivalent to the bound on the left-hand side of (2.5). Corollary 2.3 follows.

Remark. According to (2.6), if K is a symmetric convex body in $\mathbf{R}^n$ with volume $|K| = 1$, we have $\widetilde{M}(K) \sim m(K)^{-4}$ for a slightly modified functional

$$\widetilde{M}(K) \ = \ \inf_{|\mathcal{E}|=1} M(K, \mathcal{E}).$$

3 Restricted Gaussian measures in isotropic position

Recall that the standard (n-dimensional) Gaussian measure γ is defined on Borel subsets of $\mathbf{R}^n$ by

$$\gamma(A) = (2\pi)^{-n/2} \int_A e^{-|x|^2/2} \, dx.$$

Proposition 3.1. *Given a symmetric convex body K in $\mathbf{R}^n$, the normalized restricted Gaussian measure γ_K is isotropic, if and only if in the class of all volume preserving linear maps $T : \mathbf{R}^n \to \mathbf{R}^n$ the maximum to*

$$\gamma(T(K)) = (2\pi)^{-n/2} \int_{T(K)} e^{-|x|^2/2} \, dx$$

is attained for the identity map $T(x) = x$.

Proof. For $Q = T'T$ put

$$u(Q) = (2\pi)^{n/2} \, \gamma(T(K)) = \int_K e^{-\langle Qx,x\rangle^2/2} \, dx.$$

So, maximum to $\gamma(T(K))$ over all linear maps T with $|\det T| = 1$ is attained at the identity map, if and only if in the class $\mathcal{M}$ of all symmetric positive definite matrices Q with $\det Q = 1$ the functional $u(Q)$ attains a maximum for the unit matrix I_n.

Note that u does attain a maximum at some Q in $\mathcal{M}$, since $u(Q) \to 0$ when the maximal eigenvalue of Q grows to infinity. To find a necessary condition, assume that Q provides a local maximum to u. Given an arbitrary symmetric $n \times n$ matrix E and numbers ε small enough, define

$$Q_\varepsilon = \frac{I_n + \varepsilon E}{\det(I_n + \varepsilon E)} = I_n + \varepsilon \, F + o(\varepsilon),$$

where $F = E - (\mathrm{Tr} E) \, I_n$. The latter may be any symmetric $n \times n$ matrix with trace $\mathrm{Tr} F = 0$. Hence $\langle Q_\varepsilon x, x\rangle = \langle x, x\rangle + \varepsilon \, \langle Fx, x\rangle + o(\varepsilon)$, as $\varepsilon \to 0$ uniformly over all $x \in K$, and by Taylor's expansion,

$$u(Q_\varepsilon) = u(I_n) - \frac{\varepsilon}{2} \int_K \langle Fx, x\rangle \, e^{-|x|^2/2} \, dx + o(\varepsilon).$$

Since $u(Q_\varepsilon) \leq u(I_n)$ with arbitrary ε in some neighbourhood of zero, we conclude that

$$\int_K \langle Fx, x\rangle \, e^{-|x|^2/2} \, dx = 0$$

for any symmetric F such that $\mathrm{Tr} F = 0$. But this is equivalent to saying that there is a constant C such that, for all $i, j = 1, \dots, n$,

$$\int_K x_i x_j \, d\gamma(x) = C \delta_{ij}, \tag{3.1}$$

where δ_{ij} denotes Kronecker's symbol. Thus, $\int_K \langle x, \theta\rangle^2 \, d\gamma(x) = C$, for any unit vector θ, that is, γ_K is isotropic.

The converse statement is more delicate. Assume γ_K is isotropic. We need to show that

$$u(Q) \leq u(I_n), \tag{3.2}$$

for any symmetric positive definite matrix Q with $\det Q = 1$.

Let us represent $Q = UDU^{-1}$, where U is an orthogonal matrix, and $D = D(\lambda_1, \dots, \lambda_n)$ is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n > 0$ on the main diagonal, such that $\lambda_1 \dots \lambda_n = 1$. Then

$$u(Q) = \int_{U(K)} e^{-\langle Dx, x\rangle^2/2}\, dx.$$

But, as follows from the very definition, the restricted Gaussian measures $\gamma_{U(K)}$ will be isotropic for any orthogonal U, as long as γ_K is isotropic. Replacing $U(K)$ with K, the inequality (3.2) is therefore reduced to

$$u(D(\lambda)) \leq u(I_n), \tag{3.3}$$

for any collection $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_i > 0$ and $\lambda_1 \dots \lambda_n = 1$.

At this step we involve the following observation made by D. Cordero-Erausquin, M. Fradelizi and B. Maurey in their study and proof of the so-called B-conjecture, cf. [CE-F-M], Theorem 1. It is stated below as a lemma, where $D(\lambda)$ is treated as a linear map.

Lemma 3.2 (D. Cordero-Erausquin, M. Fradelizi and B. Maurey [CE-F-M]). *For any symmetric convex body K in $\mathbf{R}^n$, the function*

$$(t_1, \dots, t_n) \longrightarrow \gamma(D(e^{t_1}, \dots, e^{t_n})(K))$$

is log-concave on $\mathbf{R}^n$.

To continue the proof of Proposition 3.1, introduce the function on $\mathbf{R}^{n-1}$

$$v(t_1, \dots, t_{n-1}) = \log u(D(e^{t_1}, \dots, e^{t_n})),$$

where $t_n = -(t_1 + \dots + t_{n-1})$. The required property (3.3), where one can take $\lambda_i = e^{t_i}$, is equivalent to the statement that v attains a maximum at the origin. But by Lemma 3.2, v is concave, so it is enough to check that $\nabla v(0) = 0$. To this aim, write

$$v(t_1, \dots, t_{n-1}) = \log \int_K \exp\Big[-\frac{1}{2}\sum_{i=1}^n e^{2t_i} x_i^2\Big]\, dx.$$

The direct differentiation gives, for any $i = 1, \dots, n-1$,

$$\begin{aligned} \frac{\partial v(0)}{\partial t_i} &= \frac{1}{u(I_n)}\Big[\int_K x_n^2\, e^{-|x|^2/2}\, dx - \int_K x_i^2\, e^{-|x|^2/2}\, dx\Big] \\ &= \int x_n^2\, d\gamma_K(x) - \int x_i^2\, d\gamma_K(x) \ = \ 0, \end{aligned}$$

according to the isotropy assumption (3.1). Hence, $\nabla v(0) = 0$.

Proposition 3.1 is proved.

Recall that

$$m(K) = \sup_{|\mathcal{E}|=1} |K \cap \mathcal{E}|^{1/n}, \tag{3.4}$$

where the supremum is taken over all ellipsoids with unit volume. Note that this quantity does not depend on the "position" of K.

Corollary 3.3. *Let K be a symmetric convex body in $\mathbf{R}^n$ with volume $|K| = 1$. If the normalized restricted Gaussian measure γ_K is isotropic, then*

$$\gamma(K)^{1/n} \sim \gamma(K \cap D)^{1/n} \sim m(K),$$

where D is the Euclidean ball in $\mathbf{R}^n$ of unit volume with center at the origin.

Proof. Since the density of γ does not exceed $(2\pi)^{-n/2}$, we have

$$\gamma(K \cap D) \leq (2\pi)^{-n/2}\, |K \cap D| \leq (2\pi)^{-n/2}\, m(K)^n. \tag{3.5}$$

Now, consider a volume preserving linear map $T : \mathbf{R}^n \to \mathbf{R}^n$, such that for $K' = T(K)$ the supremum in (3.4) is attained at $\mathcal{E} = D$. Since D has radius of order $\sqrt{n}$,

$$\begin{aligned} \gamma(K') \ \geq \ \gamma(K' \cap D) = (2\pi)^{-n/2} \int_{K' \cap D} e^{-|x|^2/2}\, dx \ &\geq \ c^n\, |K' \cap D| \\ &= \ c^n\, m(K')^n \ = \ c^n\, m(K)^n, \end{aligned}$$

for some absolute constant $c > 0$. Using the isotropy assumption for γ_K (Proposition 3.1), we arrive at $\gamma(K) \geq \gamma(K') \geq c^n\, m(K)^n$. Thus,

$$\gamma(K)^{1/n} \geq c\, m(K).$$

Moreover, since $\gamma(D)^{1/n} \geq c'$ with some absolute constant $c' > 0$,

$$\gamma(K \cap D)^{1/n} \geq \gamma(K)^{1/n} \gamma(D)^{1/n} \geq c''\, m(K), \tag{3.6}$$

where the first inequality is a simple part of the Gaussian correlation inequality. More generally, one has $\mu(K \cap D) \geq \mu(K)\mu(D)$, for any spherically invariant probability measure on $\mathbf{R}^n$, cf. [S-S-Z]. Thus, combining (3.5) with (3.6), we get

$$\gamma(K \cap D)^{1/n} \sim m(K).$$

Using once more $\gamma(D)^{1/n} \geq c'$ and the first inequality in (3.6), we also have that $\gamma(K)^{1/n} \sim \gamma(K \cap D)^{1/n}$. This finishes the proof.

Remark 3.4. Without the assumption that γ_K is isotropic, we only have a lower bound

$$m(K) \geq c\, \gamma(K)^{1/n},$$

where K is an arbitrary symmetric convex body in $\mathbf{R}^n$ with $|K| = 1$, and $c > 0$ is an absolute constant. This is seen by combining (3.5) with the first inequality in (3.6).

Remark 3.5. One may also relate $m(K)$ to the isotropic constant L_K and other related quantities. Let K be a symmetric convex body in $\mathbf{R}^n$ with $|K| = 1$. The isotropic constant does not depend on the "position" and is defined by

$$L_K^2 \,=\, \inf_T \int_K \frac{|Tx|^2}{n}\,dx,$$

where the infimum is taken over all volume preserving linear maps $T : \mathbf{R}^n \to \mathbf{R}^n$. If K is isotropic (so that the above infimim is attained for the identity map), by Jensen's inequality,

$$\gamma(K) \,\geq\, (2\pi)^{-n/2}\, e^{-\frac{1}{2}\int_K |x|^2\,dx} \,=\, (2\pi)^{-n/2}\, e^{-nL_K^2/2}.$$

Hence, $\gamma(K)^{1/n} \,\geq\, \frac{1}{\sqrt{2\pi}}\, e^{-L_K^2}$ and, by Remark 3.4, $m(K) \,\geq\, c\, e^{-L_K^2/2}$ with some absolute constant $c > 0$.

However, the exponential dependence on L_K^2 is not optimal and can be improved by involving other than Gaussian probability measures (e.g. with heavy-tailed Cauchy densities) to get

$$m(K) \geq \frac{c}{L_K}. \tag{3.7}$$

The latter can also be derived from the reverse Brunn-Minkowski-type inequality in the form of K. Ball [Bal], who showed that with some numerical constant C, for all convex symmetric bodies K and K' in $\mathbf{R}^n$,

$$|K + K'|^{2/n} \leq C\bigg[\frac{1}{n\,|K|}\int_K |x|^2\,dx + \frac{1}{n\,|K'|}\int_{K'} |x|^2\,dx\bigg].$$

In particular, taking $K' = D$ the Euclidean ball in $\mathbf{R}^n$ of unit volume, and if K is isotropic and has volume one, then

$$|K + D|^{1/n} \leq C\, L_K,$$

where C is a different numerical constant and where we have used the fact L_K is separated from zero. Hence, using the left inequality in (2.1), we have $|K \cap D|^{1/n} \geq 1/(CL_K)$, which implies (3.7).

Although being a tautology, the relation (3.7) shows that Milman's Theorem 1.1 in the form $m(K) \geq c > 0$ would follow from the assertion of the slicing conjecture, telling that L_K is bounded from above by a universal constant. In fact, with similar arguments (3.7) may be sharpened as

$$m(K) \geq \frac{c}{\widetilde{L}_K}$$

in terms of $\widetilde{L}_K \,=\, \inf_{K'} L_{K'}$, where the infimum is taken over all convex bodies K' in $\mathbf{R}^n$ with baricenter at the origin, such that $|K'| = 1$ and $\frac{1}{2}\,K' \subset K \subset 2K'$. On the other hand, a remarkable theorem due to B. Klartag [K] concerning the isomorphic variant of the slicing problem asserts that $\widetilde{L}_K$ is indeed bounded from above by a universal constant. Hence, Klartag's theorem implies that $m(K)$ is separated from zero (and therefore implies Theorem 1.1, provided that one can use the reverse Santalo and the extended Roger-Shephard's inequalities).

4 Isotropic constants of restricted Gaussian measures

Let K be a symmetric convex body in $\mathbf{R}^n$ with volume $|K| = 1$. As it was already discussed, the quantity

$$M(K) = \inf_{\mathcal{E}} M(K, \mathcal{E}) \tag{4.1}$$

may be bounded both from above and below by negative powers of

$$m(K) = \sup_{|\mathcal{E}|=1} |K \cap \mathcal{E}|^{1/n}.$$

Thanks to Theorem 1.1, the latter quantity is separated from zero.

Now, assume the normalized restricted Gaussian measure γ_K is isotropic. As we know from Corollary 3.3, $\gamma(K \cap D)^{1/n} \sim m(K)$, where D is the Euclidean ball in $\mathbf{R}^n$ of unit volume. Together with (3.5) this gives, up to an absolute constant C,

$$m(K) \leq C\, |K \cap D|^{1/n}.$$

Hence, by Corollary 2.2, cf (2.6), and Corollary 2.3,

$$M(K, D) \sim |K \cap D|^{-4/n} \leq C^4\, m(K)^{-4} \leq C' M(K)^4.$$

Irrespectively of whether or not $\mathcal{E} = D$ realizes minimum to (4.1), if $M(K)$ is bounded by a universal constant, then so is $M(K, D)$. It is in this sense K is in an M-position (which is the first assertion in Theorem 1.2).

Now, let us look at the isotropic constant of γ_K. It is defined like in the general symmetric isotropic log-concave case (1.3) by

$$L_{\gamma_K}^2 \; = \; \frac{1}{\gamma(K)^{2/n}} \int \frac{|x|^2}{n}\, d\gamma_K(x). \tag{4.2}$$

Lemma 4.1. *Given a symmetric convex body K in $\mathbf{R}^n$ with volume $|K| = 1$, if γ_K is isotropic, then*

$$c \leq \int \frac{|x|^2}{n}\, d\gamma_K(x) \leq 1 \tag{4.3}$$

with some absolute constant $c > 0$. In particular,

$$L_{\gamma_K} \sim \gamma(K)^{-1/n}.$$

The right inequality in (4.3) remains to hold regardless of the volume of K and of whether γ_K is isotropic or not. However, the assumptions are important for the left inequality. This can be seen on the example of the parallepipeds

$$K = \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]^{n-1} \times \left[-\frac{1}{2\varepsilon^{n-1}}, \frac{1}{2\varepsilon^{n-1}}\right], \qquad \varepsilon > 0.$$

Indeed, in this particular case rewrite the left inequality in (4.3) equivalently as

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \cdots \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_{-\frac{1}{2\varepsilon^{n-1}}}^{-\frac{1}{2\varepsilon^{n-1}}} \left(\frac{x_1^2 + \ldots + x_n^2}{n} - c\right) e^{-\frac{1}{2}\,(x_1^2 + \ldots + x_n^2)}\, dx_1 \ldots dx_n \;\geq\; 0.$$

Dividing by ε^{n-1} and letting $\varepsilon \to 0$, in the limit we obtain that $\int_{-\infty}^{+\infty} (\frac{x_n^2}{n} - c)\, e^{-x_n^2/2}\, dx_n \geq 0$, so, $c \leq \frac{1}{n}$.

Proof of Lemma 4.1. Since γ_K has a log-concave density with respect to γ, it inherits many properties of the standard Gaussian measure. As an example, it satisfies an isoperimetric inequality similarly to the Gaussian case (cf. [B-L], [Bob], [C]). In addition, for any function u on $\mathbf{R}^n$ with Lipschitz semi-norm $\|u\|_{\mathrm{Lip}} \leq 1$,

$$\mathrm{Var}_{\gamma_K}(u) \leq \mathrm{Var}_{\gamma}(u) \leq 1.$$

One may take $u(x) = x_i$, so if K is symmetric, we get $\int x_i^2\, d\gamma_K \leq 1$. Hence, $\int |x|^2\, d\gamma_K(x) \leq n$, which is the right inequality in (4.3).

For the left inequality of the lemma (which is not needed for Theorem 1.2), one may use the well-known fact that the isotropic constants are separated from zero. Hence, from (4.2) and using Corollary 3.3 and Theorem 1.1, we have with some absolute constants

$$\int \frac{|x|^2}{n}\, d\gamma_K(x) \ \geq\ c_1 \gamma(K)^{2/n} \ \geq\ c_2\, m(K)^2 \ \geq\ c_3 \ >\ 0.$$

Proof of (1.4). Now, it is easy to complete the proof of Theorem 1.2. According to Lemma 4.1, if $|K| = 1$ and γ_K is isotropic,

$$L_{\gamma_K} \sim \frac{1}{\gamma(K)^{1/n}} \sim \frac{1}{\gamma(K \cap D)^{1/n}} \sim \frac{1}{m(K)} \leq C\, M(K).$$

It remains to apply Theorem 1.1.

Remark. If K is a symmetric convex body in $\mathbf{R}^n$ with $|K| = 1$, and if γ_K is not necessarily isotropic, then we only have an inequality

$$L_{\gamma_K} \leq \gamma(K)^{-1/n}.$$

Arguing as before, we have

$$\gamma(K)^{-1/n} \sim \gamma(K \cap D)^{-1/n} \leq C\, |K \cap D|^{-1/n}.$$

Hence, by Theorem 1.1, L_{γ_K} is still universally bounded, as long as K is in M-position.

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