EXPONENTIAL INEQUALITIES IN PROBABILITY SPACES REVISITED

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Abstract. We revisit several results on exponential integrability in probability spaces and derive some new ones. In particular, we give a quantitative form of recent results by Cianchi-Musil and Pick in the framework of Moser-Trudinger-type inequalities, and recover Ivanisvili-Russell's inequality for the Gaussian measure. One key ingredient is the use of a dual argument, which is new in this context, that we also implement in the discrete setting of the Poisson measure on integers.

1. Introduction

The aim of this paper is to develop a number of upper bounds on exponential moments of functions on the Euclidean and abstract probability spaces under certain smoothness-type conditions. For the first motivating example, suppose that we are given a Borel probability measure μ on $\mathbb{R}^n$ satisfying a logarithmic Sobolev inequality

$$(1) \qquad \int fe^f d\mu \le c \int |\nabla f|^2 e^f d\mu \quad \text{for all smooth } f \text{ on } \mathbb{R}^n \text{ such that } \int e^f d\mu = 1$$

with some constant $c > 0$ depending on the measure, only. We note that here and all along the paper, the integrals are understood over the whole region. Under this analytic hypothesis, it was shown in [10] that the exponential inequality

$$(2) \qquad \int e^f d\mu \le \left(\int e^{\alpha |\nabla f|^2} d\mu \right)^{\frac{c}{\alpha - c}},$$

holds true for all smooth f on $\mathbb{R}^n$ with $\int f d\mu = 0$ and for an arbitrary value $\alpha > c$. Here ∇f is the gradient of f and $|\cdot|$ stands for the Euclidean length.

A classical example of (1) is provided by the standard Gaussian measure $\mu = \gamma_n$ with density

$$\varphi_n(x) := (2\pi)^{-\frac{n}{2}} e^{-|x|^2/2},$$

for which $c = 1/2$ is optimal [39, 26]. More generally, according to the Bakry-Émery criterion [4], the relation (1) with constant $c = 1/(2\rho)$ holds for any μ with density $e^{-V(x)}$ satisfying $\mathrm{Hess}(V) \ge \rho > 0$ (as a matrix). We refer the interested reader to the textbooks and monographs [2, 1, 30, 27] for an introduction to the log-Sobolev inequality and its numerous connections with other fields of mathematics (convex geometry, information theory, statistical mechanics...).

In [10], the authors deal with a more general setting of a metric space $\mathcal{X}$ in place of $\mathbb{R}^n$ and some "derivation" operator in place of $|\nabla f|$. The inequality (2) also holds in the discrete setting, for example, on a graph for an appropriate Dirichlet form in the associated log-Sobolev inequality.

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Since $c = 1/2$ for the Gaussian measure, the admissible range of the parameter α is $(1/2, \infty)$. As for the critical value, it was observed by Talagrand that $\int e^f d\gamma_n < \infty$ whenever $\int f d\gamma_n = 0$ and $\int e^{\frac{1}{2}|\nabla f|^2} d\gamma_n < \infty$ (which is mentioned without proof in [10, after Corollary 2.2]). A quantitative form of this statement has been recently obtained by Ivanisvili and Russell [28], by proving that

$$\log \int e^f d\gamma_n \leq 10 \int \frac{e^{\frac{1}{2}|\nabla f|^2}}{1 + |\nabla f|} d\gamma_n \quad \text{for all smooth } f \text{ with} \int f d\gamma_n = 0. \tag{3}$$

This result nicely complements the family of inequalities (2) by dealing with the extremal value $\alpha = 1/2$.

A second closely related observation, which was another starting point of our investigation, is due to Cianchi, Musil, and Pick. In [15, Theorem 1.1], it was shown that, if f has γ_n-mean zero and satisfies $\int e^{\frac{1}{\kappa^\beta}|\nabla f|^\beta} d\gamma_n < M$ for some $\beta > 0$ and $M > 1$, then with some constant $C(M)$ depending on M only, we have

$$\int \exp\left\{|f|^{\frac{2\beta}{\beta+2}}\right\} d\gamma_n < C(M) \tag{4}$$

as soon as

$$\kappa \leq \kappa_\beta := \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\beta}. \tag{5}$$

Here, the particular value $\beta = 2$ also leads to a refined form of Talagrand's observation,

$$\int e^{|f|} d\gamma_n < C(M), \text{ whenever } \int e^{\frac{1}{2}|\nabla f|^2} d\gamma_n < M,$$

although (3) is more quantitative for such parameter β. In [16], Cianchi, Musil, and Pick show improved bounds in the sense that the integrand of (4) has a faster growth, i.e. they proved that if $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function that diverges to ∞ as $t \to \infty$ with a sufficiently mild growth, then $\int \exp\left\{(|f|)^{\frac{2\beta}{\beta+2}}\right\} \varphi(|f|) d\gamma_n < C(M)$. In a subsequent paper [17], the same authors derive similar bounds under the constraint $\int \exp\{\lambda |Lf|^\beta\} d\gamma_n < M$ in place of $\int \exp\{\frac{1}{\kappa^\beta}|\nabla f|^\beta\} d\gamma_n < M$ with a different exponent than $2\beta/(\beta + 2)$, and where $L = \Delta - x \cdot \nabla$ is the Ornstein-Uhlenbeck operator.

One of the main features of all these results is that they are dimension free, unlike the Moser-Trudinger inequality in the Euclidean space over the Lebesgue measure. We refer the reader to the introduction of [15, 17], or to the textbook [38] for more information on Moser-Trudinger inequality, a historical presentation and references.

It is therefore natural to consider on the Euclidean space equipped with a probability measure μ the following general exponential inequality

$$\int e^f d\mu \leq F\left(\int G(|\nabla f|)\, d\mu\right) \quad \text{for all smooth } f \text{ with} \int f d\mu = 0, \tag{6}$$

where $F, G \colon [0, \infty) \to [0, \infty)$ are given non-decreasing functions. More general spaces may also be involved in this family of Sobolev-type relations. Note that, applying (6) to constant functions shows that, necessarily $F \geq 1$.

In this paper our aim is to establish inequalities of the type (6) in different settings including the framework of the Γ_2 formalism. First, under the so-called Γ_2 condition (which is stronger than the log-Sobolev inequality), and using the semi-group technique, we sharpen (2), by proving a similar relation with a better exponent. Moreover, we obtain local inequalities for the semi-group in place of the measure μ. Then, we extend the approach of [10] to the class of sub-Gaussian measures under a weaker hypothesis in the form of a modified log-Sobolev inequality. This will lead us to new exponential inequalities for measures with sub-Gaussian tails like the ones with densities proportional to $\exp\{-|x|^p\}$, $p \in (1,2)$ (on each fibre).

On the other hand, we introduce a simple and direct argument, essentially based on convex duality, to recover the inequality (3) for the Gaussian measure, in a slightly modified form. One interesting point is that, though being totally different, our approach leads to the same conclusion as in [28]. We believe that one cannot improve such a relation (except for the numerical constant 10). In Section 2 we give more on this intuition.

Our dual argument reveals to be robust enough to be able to give a quantitative form of Cianchi, Musil, and Pick [15] for some range of the parameter β, namely $\beta \in (\sqrt{5}-1, 2)$, when $\kappa = \kappa_\beta$. Finally, we will deal in the final section with the discrete setting.

2. Perturbation, stability, optimality and Reduction

This section is preparatory. Here, we first prove that the inequality (6) satisfies some sort of stability under bounded perturbation of the density of the probability measure μ. Using the Caffarelli contraction theorem, we also show that (6), being valid for the Gaussian measure, is extended to the similar relation for any contraction of γ_n. Then we will show that the constraint $\int f d\mu = 0$ can be changed, under mild assumption, into $m_f = 0$, where m_f is the median of f.

Finally, using Ehrhard's rearrangement technique, we prove that the inequality (6) is by essence one dimensional and may be further reduced to the half-line for the class of non-decreasing functions satisfying $f(0) = 0$ (Hardy-type constraints), in some specific case.

2.1. Bounded perturbation.

Recall that a probability measure μ on $\mathbb{R}^n$ satisfies a Poincaré inequality with constant C_μ if

$$\int f^2 d\mu \leq C_\mu \int |\nabla f|^2 d\mu$$

holds for all f smooth enough and satisfying $\int f d\mu = 0$.

The main result of this section is the following:

Theorem 2.1. *Given $F, G\colon [0,\infty) \to [0,\infty)$, where F is non-decreasing, and let μ be a probability measure on $\mathbb{R}^n$ satisfying (6) and a Poincare inequality with constant C_μ.*

Let ν be a probability measure absolutely continuous with respect to μ with relative density $h = d\nu/d\mu$ such that $a \leq h(x) \leq b$ for all $x \in \mathbb{R}^n$ with some constants $0 < a < b < \infty$.

Then, for any smooth f such that $\int f d\nu = 0$,

$$\int e^f d\nu \leq 1 + \exp\left\{ \frac{\sqrt{bC_\mu}}{a} (\int |\nabla f|^2 d\nu)^{1/2} \right\} \widetilde{F}\left(\int G\left(|\nabla f|\right) d\nu \right)$$

with $\widetilde{F}(t) := b(F(t/a) - 1)$, $t \geq 0$.

Remark 2.2. *As is standard, applying (6) to $f = \varepsilon g$ with $\int g d\mu = 0$ leads, in the limit $\varepsilon \to 0$, to a Poincaré inequality. More precisely, if $F(G(0)) = 1$, then necessarily $G'(0) = 0$, and then (6) implies a Poincare inequality with constant $C_\mu = G''(0)F'(G(0))$. Therefore the Poincaré inequality assumption comes for free in some cases.*

The argument employs the following known lemma whose proof we give for completeness.

Lemma 2.3. *Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a C^1-convex function, and let μ a probability measure on $\mathbb{R}$. For any μ-integrable function $f\colon \mathbb{R} \to \mathbb{R}$,*

$$\int \Phi(f)\, d\mu - \Phi\left(\int f d\mu \right) = \inf_{t\in\mathbb{R}} \int \big(\Phi(f) - \Phi(t) - \Phi'(t)(f-t) \big)\, d\mu.$$

Proof. By the integrability of μ and the convexity of Φ, the integral $\int \Phi(f)\,d\mu$ is well-defined in the Lebesgue sense, taking values in $(-\infty,+\infty]$. Moreover, using the inequality $\Phi(a) \geq \Phi(t) + \Phi'(t)(a-t)$, $a, t \in \mathbb{R}$, we have

$$\int (\Phi(f) - \Phi(t) - \Phi'(t)(f-t))d\mu = \int \Phi(f)d\mu - \Phi(t) - \Phi'(t)\left(\int f d\mu - t\right)$$
$$\geq \int \Phi(f)d\mu - \Phi\left(\int f d\mu\right),$$

with equality if $t = \int f d\mu$. □

Proof of Theorem 2.1. Let f be a smooth function satisfying $\int f d\nu = 0$. Applying Lemma 2.3 (twice) to $\Phi(x) = e^x$, we have

$$\begin{aligned}\int e^f\,d\nu - 1 &= \int \Phi(f)\,d\nu - \Phi\left(\int f d\nu\right) \\ &= \inf_{t\in\mathbb{R}} \int \big(\Phi(f) - \Phi(t) - \Phi'(t)(f-t)\big)\,d\nu \\ &\leq b \inf_{t\in\mathbb{R}} \int \big(\Phi(f) - \Phi(t) - \Phi'(t)(f-t)\big)\,d\mu \;=\; b\left(\int e^f\,d\mu - \exp\left\{\int f d\mu\right\}\right).\end{aligned}$$

Here, we used the convexity of Φ which ensures that $\Phi(f) - \Phi(t) - \Phi'(t)(f-t) \geq 0$. It follows that

$$\begin{aligned}\int e^f\,d\nu &\leq 1 + b\exp\left\{\int f d\mu\right\}\left(F\left(\int G(|\nabla f|)\,d\mu\right) - 1\right) \\ &\leq 1 + b\exp\left\{\int f d\mu\right\}\left(F\left(\frac{1}{a}\int G(|\nabla f|)\,d\nu\right) - 1\right).\end{aligned}$$

Now, since μ satisfies by assumption the following Poincaré inequality $\mathrm{Var}_\mu(f) := \int f^2 d\mu - (\int f d\mu)^2 \leq C_\mu \int |\nabla f|^2 d\mu$, so does ν with constant bC_μ/a. Indeed, using the variational formula for the variance $\mathrm{Var}_P(f) = \inf_{m\in\mathbb{R}} \int (f-m)^2 dP$ twice (for $P = \nu$ and then for $P = \mu$), it holds

$$\mathrm{Var}_\nu(f) \leq b\mathrm{Var}_\mu(f) \leq bC_\mu \int |\nabla f|^2 d\mu \leq \frac{bC_\mu}{a}\int |\nabla f|^2 d\nu.$$

It follows by Cauchy-Schwarz' inequality (recall that $\int f d\nu = 0$)

$$\left(\int f d\mu\right)^2 \leq \int f^2 d\mu \leq \frac{1}{a}\int f^2 d\nu = \frac{1}{a}\mathrm{Var}_\nu(f) \leq \frac{bC_\mu}{a^2}\int |\nabla f|^2 d\nu.$$

This leads to the expected result and ends the proof of the theorem. □

2.2. **Stability by contraction.** A probability measure μ on $\mathbb{R}^n$ is often said to be strongly log-concave, if it has a log-concave density with respect to the standard Gaussian measure, that is, when $\mu(dx) = e^{-V(x)}\,d\gamma_n(x)$ with V convex.

Theorem 2.4 (Caffarelli [13, 14])**.** *If μ is strongly log-concave on $\mathbb{R}^n$, then there exists a 1-Lipschitz map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\int h(T)\,d\gamma_n = \int h\,d\mu$ for all bounded measurable functions h on $\mathbb{R}^n$.*

In other words, T pushes forward γ_n to μ, or μ appears as image of γ_n under T. With the help of this theorem, we obtain the following elementary, but useful result.

Proposition 2.5. *Let $F, G\colon [0,\infty) \to [0,\infty)$ be non-decreasing and continuous, and let μ be a strongly log-concave probability measure on $\mathbb{R}^n$. Assume that for all smooth f with $\int f d\gamma_n = 0$, it holds that*

$$\int e^f d\gamma_n \le F\left(\int G\left(|\nabla f|\right) d\gamma_n\right),$$

then

$$\int e^g d\mu \le F\left(\int G\left(|\nabla g|\right) d\mu\right)$$

for all smooth g with $\int g d\mu = 0$.

Proof. The hypothesis about γ_n is extended to all locally Lipschitz functions f with γ_n-mean zero, in which case the modulus of the gradient may be defined by

$$|\nabla f(x)| = \limsup_{y\to x} \frac{|f(x)-f(y)|}{|x-y|}, \quad x \in \mathbb{R}^n.$$

Let T be the Lipschitz map from Theorem 2.4, and let g be a smooth function with μ-mean zero. Note that by a standard approximation argument, we can assume that g is bounded. Then $f = g(T)$ is locally Lipschitz and satisfies, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |\nabla f(x)| &= \limsup_{y\to x} \frac{|g(Tx)-g(Ty)|}{|x-y|} \\ &= \limsup_{y\to x} \left[\frac{|g(Tx)-g(Ty)|}{|Tx-Ty|}\,\frac{|Tx-Ty|}{|x-y|}\right] \le |\nabla g|(Tx). \end{aligned}$$

Applying the exponential inequality to f with respect to γ_n, we get

$$\begin{aligned} \int e^g d\mu &= \int e^f d\gamma_n \le F\left(\int G\left(|\nabla f|\right) d\gamma_n\right) \\ &\le F\left(\int G\left(|\nabla g|(T)\right) d\gamma_n\right) = F\left(\int G\left(|\nabla g|\right) d\mu\right). \end{aligned}$$

□

2.3. **Zero mean versus zero median.** While the inequality (6) appears under the condition $\int f\, d\mu = 0$, it is useful to know if one may replace with a similar condition such as $m_f = 0$, where

$$m_f := \inf\left\{t \in \mathbb{R} : \mu(\{x \in \mathbb{R}^n : f(x) > t\}) \le \frac{1}{2}\right\}$$

denotes the maximal median of f under the measure μ. Here we describe one specific example, in which the answer is affirmative. Namely, consider

$$F(t) = e^t \quad \text{and} \quad G(t) = \frac{1}{1+t}\, e^{t^2/2},$$

therefore dealing with the inequality (3) (but the computations to come can easily be adapted to many other examples, including F and G as in (2) for instance).

Recall that a probability measure μ on $\mathbb{R}^n$ is said to satisfy the Maz'ya-Cheeger inequality if there exists a constant $c \in (0,\infty)$ such that for all f smooth,

$$\int |f - m_f|\, d\mu \le c \int |\nabla f| d\mu. \tag{7}$$

As is well-known (see [33, 32]), this hypothesis may be equivalently stated in terms of the isoperimetric-type inequality

$$\min\{\mu(A), 1-\mu(A)\} \le c\mu^+(A),$$

relating the μ-perimeter $\mu^+(A) = \liminf_{\varepsilon\downarrow 0} (\mu(A_\varepsilon) - \mu(A))/\varepsilon$ to the μ-size of an arbitrary Borel set A in $\mathbb{R}^n$ (where A_ε denotes an ε-neighborhood of A). It is stronger than a Poincaré-type, although in general it is not comparable to the log-Sobolev inequality for the measure μ

Now assume that, for some $a \in (0,\infty)$,

$$\log \int e^f \, d\mu \le a \int G(|\nabla f|)\, d\mu \quad \text{for all } f \text{ with } \int f\, d\,\mu = 0.$$

Consider a smooth function g with $m_g = 0$. Applying the latter inequality to $f = g - \int g d\mu$, we get

$$\log\left(\int e^g d\mu\right) \le \int g\, d\mu + a \int G\left(|\nabla g|\right) d\mu.$$

By Maz'ya-Cheeger's inequality (7),

$$\int g d\mu \le \int |g - m_g|\, d\mu \le c \int |\nabla g|\, d\mu.$$

Therefore, since $t \le 2e^{t^2/2}/(1+t)$ for all $t \ge 0$, we end up with

$$\log\left(\int e^g d\mu\right) \le (a + 2c) \int G\left(|\nabla g|\right) d\mu.$$

The same argument in fact shows that the two conditions are equivalent in (3), at the expense of some loss in the constant.

2.4. **Optimality in (3).** Recall that the inequality (3) states that, for any f with $\int f d\gamma_n = 0$,

$$\log\left(\int e^f \, d\gamma_n\right) \le 10 \int \frac{e^{|\nabla f|^2/2}}{1 + |\nabla f|} d\gamma_n$$

We will develop an alternative approach, based on a dual convexity argument, leading to

$$\log\left(\int e^f \, d\gamma_n\right) \le 8 \int \frac{e^{|\nabla f|^2/2}}{\sqrt{1 + (|\nabla f|^2/2)}} d\gamma_n.$$

Let us show that such choice of the function $G(t) = e^{t^2/2}/\sqrt{1+t^2/2}$ appearing in the integrand on the right-hand side is essentially optimal. Restricting ourselves to the one dimensional case, suppose that, for all bounded, locally Lipshitz functions f on the real line,

$$\log\left(\int e^f \, d\gamma\right) \le \int f d\gamma + F\left(\int \frac{e^{f'^2/2}}{H\left(f'\right)} d\gamma\right) \tag{8}$$

for some non-decreasing function $H \ge 1$, where $\gamma = \gamma_1$, and F is non-decreasing. For the particular functions $f_N(x) = \frac{1}{2}\left(x \wedge N\right)^2$, the left-hand side satisfies

$$\log\left(\int e^{f_N} d\gamma\right) \ge \log\left(\int_{-N}^N \frac{dx}{\sqrt{2\pi}}\right) \ge \log(2N) - 1,$$

while the right-hand side can be bounded above by

$$\frac{1}{2} + F\left(2\int_0^N \frac{1}{H(t)}\, dt + 1\right),$$

since $\int f_N d\gamma \le \frac{1}{2}\int x^2 d\gamma = \frac{1}{2}$. Therefore, the inequality (8) cannot hold if $1/H$ is integrable at infinity. This excludes, for example, H behaving like $t(\log t)^{1+\varepsilon}$ for large t. Therefore, heuristically the biggest admissible function H lies "between" t and $t\log t$ at infinity. Observe that the above example does not exclude an inequality of the type

$$\log\log\left(\int e^f\,d\gamma\right) \le F\left(\int \frac{e^{f'^2/2}}{(1+|f'|)\log(1+|f'|)}d\gamma\right)$$

that we don't know if it could hold, or not.

On the other hand, for $H(t) = 1+|t|$, the above example shows that $F(t) \ge t$ in the large, and therefore the logarithm is needed in the left hand side of (3), in contrast with the inequality (2).

To conclude this section, we observe that for the special choice $H(x) = 1$, linear perturbation of the quadratic example used above shows that an inequality of the form

$$\int e^f\,d\gamma \le \left(\int e^{f'^2/2}d\gamma\right)^2$$

could hold for all f with mean zero with respect to the Gaussian measure.

2.5. **Reduction to dimension 1.** In this section we show that, under mild assumption on F and G, the inequality (6) for the standard Gaussian measure γ_n in dimension n holds if and only if it holds in dimension 1 for γ_1. Therefore, by essence this inequality for the standard Gaussian is one dimensional. To achieve this, one could use the localization technique of Lovász and Simonovits [31] (see also [29, 22]). Here we will instead use the Gaussian Rearrangement technique of Ehrhard [19, 20].

Proposition 2.6. *Let $F, G\colon [0,\infty) \to [0,\infty)$ be non-decreasing. Assume that G is convex and that f is smooth. Then, the exponential inequality*

$$\int e^f\,d\gamma_n \le F\left(\int G\left(|\nabla f|\right)d\gamma_n\right), \qquad \int f\,d\gamma_n = 0,$$

holds in any dimension if and only if it holds in dimension $n = 1$ for the class of all non-decreasing smooth functions f.

Remark 2.7. *The same result holds if one replaces the left integral $\int e^f d\gamma$ by $\int H(f)\,d\gamma_n$, where H is non-negative.*

Proof. The Gaussian rearrangement of a Borel set A of $\mathbb{R}^n$ is the half space

$$A^* = \{x_1 > \lambda\} \subset \mathbb{R}^n,$$

where the number λ is chosen so that A and A^* have the same Gaussian measure $\gamma_n(A) = \gamma_n(A^*) = \gamma_1([x_1,\infty))$. Put $\Phi(x) = \int_{-\infty}^x \varphi(t)\,dt$, where φ denotes the standard normal density. More generally, we define the Gaussian rearrangement of a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$ as a non-decreasing function

$$f^*(x) \coloneqq \inf\{a \in \mathbb{R} : \Phi(x_1) \le \gamma_n(\{f \le a\})\}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

It is clear that f^* depends only on the first coordinate. With a slight abuse of notation we will write f^* also for the function defined on the real line, i.e. $f^*(x_1) = f^*(x_1, x_2, \dots, x_n)$ for an arbitrary choice of $x_2, \dots, x_n$. The main feature of the Gaussian rearrangement is that f and f^* are equimeasurable, therefore

$$\int_{\mathbb{R}^n} e^f\,d\gamma_n = \int_{\mathbb{R}} e^t\gamma_n(\{f>t\})dt = \int_{\mathbb{R}} e^t\gamma_n(\{f^*>t\})dt = \int_{\mathbb{R}^n} e^{f^*}d\gamma_n = \int_{\mathbb{R}} e^{f^*}d\gamma_1.$$

Moreover, for any non-decreasing convex function $G\colon [0,\infty) \to \mathbb{R}$, the following Pólya-Szego type inequality

$$\int_{\mathbb{R}^n} G(|\nabla f|)\,d\gamma_n \ge \int_{\mathbb{R}^n} G(|\nabla f^*|)\,d\gamma_n = \int_{\mathbb{R}} G(|f^{*\prime}|)\,d\gamma_1$$

holds true for any smooth f. Since $\int f d\gamma_n = \int f^* d\gamma_1 = 0$, assuming that the exponential integral inequality holds in dimension $n = 1$, we get

$$\int_{\mathbb{R}^n} e^f \, d\gamma_n \, = \, \int_{\mathbb{R}} e^{f^*} d\gamma_1 \, \leq \, F\left(\int_{\mathbb{R}} G(|f^{*\prime}|) \, d\gamma_1\right) \, \leq \, F\left(\int_{\mathbb{R}^n} G(|\nabla f|) \, d\gamma_n\right),$$

by the monotonicity of F in the last step. □

2.6. **Reduction to the half line, monotone functions, and a Hardy-type inequality.** The aim of this section is to show that one can relate the one-dimensional version of the inequality (6) to an inequality of Hardy type on the half-line. Furthermore, at the expense of some loss in the constants, it is proved that in the one dimensional version of (6) one can restrict oneself to monotone functions. For simplicity, we will only deal with symmetric probability measures. We will use the following notation: given a function F on $[0, \infty)$,

$$F_2(x) := \sup\left\{F(x_1) + F(x_2) : x_1, x_2 \geq 0, \; x_1 + x_2 = x\right\}, \quad x \geq 0. \tag{9}$$

If F is non-decreasing, then $F_2(x) \leq 2F(x)$, and if in addition F is convex, then necessarily $F_2(x) = F(x) + F(0)$ for all $x \geq 0$.

We say that μ (on the line) satisfies Hardy's inequality if

$$\int_0^\infty |f - f(0)| \, d\mu \leq A \int_0^\infty |f'| \, d\mu \tag{10}$$

for C^1 functions f on the half-line with some constant $A \in (0, \infty)$.

Lemma 2.8. *Let $F, G \colon [0, \infty) \to [0, \infty)$ be non-decreasing and let μ be a symmetric probability measure on the line. Assume that for all non-decreasing smooth enough functions $f \colon [0, \infty) \to \mathbb{R}$,*

$$\int_0^\infty e^f \, d\mu - \frac{1}{2} e^{f(0)} \leq e^{f(0)} F\left(\int_0^\infty G(|f'|) \, d\mu\right). \tag{11}$$

Then, for all smooth $g \colon \mathbb{R} \to \mathbb{R}$ with $\int G(|g'|) d\mu < \infty$,

$$\int_{-\infty}^\infty e^g \, d\mu - \exp\left\{\int_{-\infty}^\infty g \, d\mu\right\} \leq e^{g(0)} \, F_2\left(\int_{-\infty}^\infty G(|g'|) \, d\mu\right).$$

Moreover, if μ satisfies the Hardy-type inequality (10), then, for all g with $\int_{-\infty}^\infty g \, d\mu = 0$,

$$\int_{-\infty}^\infty e^g d\mu \leq 1 + \exp\left\{A \int_{-\infty}^\infty |g'| \, d\mu\right\} F_2\left(\int_{-\infty}^\infty G(|g'|) d\mu\right). \tag{12}$$

Remark 2.9. By symmetry, $\mu(-\infty, 0) = \mu(0, \infty) = 1/2$. This explains the factor $1/2$ appearing in (11). Observe also that this inequality is invariant by changing f into $f + c$ for any constant c. The inequality (11) can be seen as an exponential Hardy-type inequality.

Proof. Fix $g \colon \mathbb{R} \to \mathbb{R}$ and assume without loss of generality that $g(0) = 0$. Applying Lemma 2.3 with $\Phi(x) = e^x$ and $t = g(0) = 0$, we have

$$\int_{-\infty}^\infty e^g d\mu - \exp\left\{\int_{-\infty}^\infty g \, d\mu\right\} \leq \left(\int_0^\infty e^g d\mu - \frac{1}{2} - \int_0^\infty g \, d\mu\right) + \left(\int_{-\infty}^0 e^g d\mu - \frac{1}{2} - \int_{-\infty}^0 g \, d\mu\right)$$

Define

$$f_+(x) := \int_0^x g'(t) \, \mathbb{1}_{g'(t)>0} \, dt, \quad f_-(x) := -\int_{-x}^0 g'(t) \, \mathbb{1}_{g'(t)<0} \, dt, \quad x \geq 0.$$

By construction, both f_+ and f_- are non-negative, non-decreasing, and satisfy $g \leq f_+$ on the positive axis, while $g(-x) \leq f_-(x)$ for all $x \geq 0$. Furthermore, $f_+'(x) \leq |g'(x)|$ and

$f'_-(x) \leq |g'(-x)|$, $x \geq 0$, and for any measurable function H on the line, by symmetry of μ, $\int_{-\infty}^0 H d\mu = \int_0^\infty H(-x)d\mu$. Therefore, since $x \mapsto e^x - x$ non-decreasing on $(0, \infty)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^g d\mu - e^{\int_{-\infty}^{\infty} g d\mu} &\leq \left(\int_0^\infty e^{f_+} d\mu - \frac{1}{2} - \int_0^\infty f_+ d\mu\right) + \left(\int_0^\infty e^{f_-} d\mu - \frac{1}{2} - \int_0^\infty f_- d\mu\right) \\ &\leq \left(\int_0^\infty e^{f_+} d\mu - \frac{1}{2}\right) + \left(\int_0^\infty e^{f_-} d\mu - \frac{1}{2}\right)\end{aligned}$$

where for the last inequality we used the fact that $f_\pm \geq 0$. Thanks to our assumption, we finally get

$$\begin{aligned}\int_{-\infty}^{\infty} e^g d\mu - e^{\int_{-\infty}^{\infty} g d\mu} &\leq F\left(\int_0^\infty G(f'_+) d\mu\right) + F\left(\int_0^\infty G(f'_-) d\mu\right) \\ &\leq F_2\left(\int_0^\infty G(f'_+) d\mu + \int_0^\infty G(f'_-) d\mu\right) \\ &\leq F_2\left(\int_0^\infty G(|g'|) d\mu + \int_0^\infty G(|g'(-x)|) d\mu\right) = F_2\left(\int G(|g'|) d\mu\right)\end{aligned}$$

where we used that F_2 is non-decreasing (which is a consequence of the fact that F is non-decreasing). This proves the first part of the lemma.

For $g\colon \mathbb{R} \to \mathbb{R}$ with $\int_{-\infty}^{\infty} g d\mu = 0$ we have just proved that

$$\int_{-\infty}^{\infty} e^g d\mu \leq 1 + e^{g(0)} F_2\left(\int_{-\infty}^{\infty} G(|g'|) d\mu\right) = 1 + e^{-\int_{-\infty}^{\infty}(g-g(0))d\mu} F_2\left(\int_{-\infty}^{\infty} G(|g'|) d\mu\right).$$

Hardy-type inequality (10) and the symmetry of μ guarantee that

$$\int_{-\infty}^{\infty} |g - g(0)| d\mu = \int_0^\infty |g(x) - g(0)| d\mu + \int_0^\infty |g(-x) - g(0)| d\mu \leq A \int_{-\infty}^{\infty} |g'| d\mu.$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} e^g d\mu &\leq 1 + e^{\int_{-\infty}^{\infty} |g-g(0)| d\mu} F_2\left(\int_{-\infty}^{\infty} G(|g'|) d\mu\right) \\ &\leq 1 + e^{A\int_{-\infty}^{\infty} |g'| d\mu} F_2\left(\int_{-\infty}^{\infty} G(|g'|) d\mu\right)\end{aligned}$$

leading to the second part of the lemma. □

We end this section by focusing on the Gaussian measure γ_1 and the special case $F(x) = ae^{bx} - c$ for some constants $a, b, c > 0$. The lemma will be applied later on to $G(x) = e^{x^2/2}/\sqrt{1 + (x^2/2)}$, which amounts to considering the inequality (3) since G compares with $e^{x^2/2}/(1 + x)$. The choice of G is governed by the fact that it is convex non-decreasing, a property that is not shared by the map $x \mapsto e^{x^2/2}/(1 + x)$ considered in [28].

Lemma 2.10. *Let $F(x) = ae^{bx} - c$ for $a, b, c > 0$ with $a \geq c$, and let $G\colon [0, \infty) \to [0, \infty)$ be non-decreasing convex function. Assume that the Gaussian measure γ_1 satisfies*

$$\int_0^\infty e^f d\gamma_1 - \frac{1}{2} e^{f(0)} \leq e^{f(0)} F\left(\int_0^\infty G(|f'|) d\gamma_1\right)$$

for all smooth non-decreasing $f\colon [0, \infty) \to [0, \infty)$. Then, for any dimension n and any smooth $g\colon \mathbb{R}^n \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} g d\gamma_n = 0$, the following inequality holds

$$(13)\quad \int_{-\infty}^{\infty} e^g d\gamma_n \leq 1 + a\exp\left\{(d+b)\int_{-\infty}^{\infty} G(|\nabla g|) d\gamma_n\right\} + (a - 2c)\exp\left\{d\int_{-\infty}^{\infty} G(|\nabla g|) d\gamma_n\right\}$$

with $d := \sqrt{\frac{\pi}{2}} \max_{x \geq 0} \frac{x}{G(x)}$.

Remark 2.11. *Observe that if $d = \infty$, the right-hand side of (13) is infinite, and the conclusion of the lemma is useless.*

Proof. First we claim that the Hardy-type inequality (10) holds for γ_1 with constant $A = \sqrt{\frac{\pi}{2}}$. To prove that this constant is best possible, we may use a result by Muckenhoupt [35]. Indeed, this author proved that the best constant in (10) is

$$A = \sup_{r>0} e^{r^2/2} \int_r^\infty e^{-x^2/2} dx.$$

Now observe that

$$\int_r^\infty e^{-x^2/2} dx \le \frac{1}{r} \int_r^\infty x e^{-x^2/2} dx = \frac{e^{-r^2/2}}{r}.$$

Therefore, if we denote $H(r) = e^{r^2/2} \int_r^\infty e^{-x^2/2} dx$, $r > 0$, we have

$$H'(r) = r e^{r^2/2} \int_r^\infty e^{-x^2/2} dx - 1 \le 0.$$

In turn, the best constant is $A = H(0) = \sqrt{\frac{\pi}{2}}$ as announced.

Now Lemma 2.8 implies that

$$\int_{\mathbb{R}} e^g d\gamma_1 \le 1 + e^{A \int |g'| d\gamma_1} F_2 \left(\int_{\mathbb{R}} G(|g'|) d\gamma_1 \right)$$

holds for all $g \colon \mathbb{R} \to \mathbb{R}$ smooth with $\int g d\gamma_1 = 0$. Observe that, for $F(x) = ae^{bx} - c$, $F_2(x) = F(x) + F(0) = ae^{bx} + a - 2c$. Hence, the latter inequality reads

$$\begin{aligned}\int_{-\infty}^\infty e^g d\gamma_1 &\le 1 + e^{A \int_{-\infty}^\infty |g'| d\gamma_1} \left(a e^{b \int_{-\infty}^\infty G(|g'|) d\gamma_1} + a - 2c \right) \\ &\le 1 + a \exp\left\{ (d+b) \int_{-\infty}^\infty G(|g'|) d\gamma_1 \right\} + (a - 2c) \exp\left\{ d \int_{-\infty}^\infty G(|g'|) d\gamma_1 \right\}.\end{aligned}$$

The expected result follows from Proposition 2.6. □

3. Exponential inequality via Γ_2 condition and semi-group

In this section we obtain some exponential inequality using the so-called Γ_2-condition and via semi-group techniques. Let us start by collecting some useful and well known facts on the Γ_2 calculus. We refer to [2], [1, Chapter 5] and to the excellent [5, Section 2] for more details and comments.

3.1. **Γ_2 formalism.** The general setting is given by an abstract Markov generator L, on some probability space $(\mathcal{X}, \mathcal{B}, \mu)$, associated to the semi-group $(P_t)_{t \ge 0}$. We assume that L is self-adjoint in $\mathbb{L}^2(\mu)$ and has domain $\mathcal{D}(L)$. We further assume the existence of a set $\mathcal{A}$ of a good family of functions (see [1, Definition 2.4.2]). The set $\mathcal{A}$ is assumed to be contained and to be dense in all $\mathbb{L}^p(\mu)$, $1 < p < \infty$, it is stable under the action of P_t, $t > 0$, and L and also by composition by any $\mathcal{C}^\infty$-smooth functions. Finally we assume that $\mathcal{A}$ contains all constant functions and $\lim_{t \to \infty} P_t f = \int f d\mu$ for all $f \in \mathcal{A}$.

For the Ornstein-Uhlenbeck process $\mathcal{A}$ may consist of all $\mathcal{C}^\infty$ smooth functions with successive derivatives slowly growing at infinity (*i.e.* such that $|f| \le P$ at infinity, for some polynomial P and the same for the derivatives). For a compact connected complete Riemannian manifold and $L = \Delta + X$, with Δ the Laplace-Beltrami operator, and X a smooth vector field (with no constant term), see below, $\mathcal{A}$ may consist of all $\mathcal{C}^\infty$ smooth functions.

For non-compact Riemanian manifolds the situation is more complicated since compactly supported $\mathcal{C}^\infty$-smooth functions are not stable under P_t in general. We refer to [21] for more on this issue.

Now, following P.-A. Meyer and Bakry-Émery, we introduce the "carré du champs" operator

$$\Gamma(f,g) \coloneqq \frac{1}{2}\left(L(fg) - fLg - gLf\right)$$

and its iterated

$$\Gamma_2(f,g) \coloneqq \frac{1}{2}\left(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\right), \qquad f,g \in \mathcal{A}.$$

For simplicity we write $\Gamma(f) := \Gamma(f,f)$ and $\Gamma_2(f) = \Gamma_2(f,f)$.

On a connected complete Riemannian manifold $\mathcal{X} = M$, denote by dx and the Riemannian volume element. For $L = \Delta + \nabla V$, with V smooth enough and $\int e^V dx = 1$, $\Gamma(f) = |\nabla f|^2$ is the Riemannian length of the gradient. The Bochner's Formula indicates that $\Gamma_2(f) = Ric(\nabla f, \nabla f) + \|Hess(f)\|_2^2$ where Ric is the Ricci tensor of M and $\|Hess(f)\|_2^2$ the Hilbert-Schmidt norm of the tensor of the second order derivatives of f. The potential $V(x) = -|x|^2/2$ in $\mathbb{R}^n$ corresponds to the Ornstein-Uhlenbeck semi-group.

Next we say that L has curvature $\rho \in \mathbb{R}$ (Γ_2-condition) if

$$\Gamma_2(f) \geq \rho\Gamma(f), \qquad \forall f \in \mathcal{A}. \tag{14}$$

As is well known the Ornstein-Uhlenbeck operator in $\mathbb{R}^n$ has curvature $\rho = 1$, the operator $L = \Delta + \nabla V$, on a manifold with Ricci curvature bounded below by R, has curvature bounded below by ρ if $R + \nabla\nabla V \geq \rho g$ where g is the Riemannian metric.

One of the main feature of the above general framework is that it allows to prove similar results in different settings ($\mathbb{R}^n$, manifold etc.). One important property that illustrate this fact is the commutation of the semi-group and the carré du champs operator. Indeed, it is well-known that the curvature condition (14) is equivalent to saying that, for all $f \in \mathcal{A}$, it holds

$$\Gamma(P_t f) \leq e^{-2\rho t} P_t \Gamma(f). \tag{15}$$

Also, the operator L satisfies in our general framework, the following integration by part formula, for $f, g \in \mathcal{A}$,

$$\int fLg\,d\mu = -\int \Gamma(f,g)\,d\mu. \tag{16}$$

In most places, we may need that Γ represents a derivation. This is the case when L is a diffusion, which means that, for any $\Phi\colon \mathbb{R}^n \to \mathbb{R}$, $\mathcal{C}^\infty$ and any $f = (f_1, \ldots, f_n) \in \mathcal{A}^n$, it holds that

$$L(\Phi(f)) = \sum_{i=1}^n \partial_i \Phi(f) L f_i + \sum_{i,j=1}^n \partial_{ij}^2 \Phi(f) \Gamma(f_i, f_j).$$

Here $\partial_i, \partial_{ij}^2$ are shorthand notation for the first and second order derivatives with respect to the i-th, i,j-th variables respectively. One crucial consequence of the diffusion property is that

$$\Gamma(\Phi(f), g) = \sum_{i=1}^n \partial_i \Phi(f) \Gamma(f_i, g), \qquad f, g \in \mathcal{A}. \tag{17}$$

The above diffusion property is satisfied by all examples given above.

Finally, we observe that, for diffusions, (15) is also equivalent to the stronger commutation

$$\sqrt{\Gamma(P_t f)} \leq e^{-\rho t} P_t(\sqrt{\Gamma(f)}), \qquad f \in \mathcal{A}. \tag{18}$$

3.2. **Inequality (2) revisited.** In this section we provide a semi-group approach, under the Γ_2 condition, of Inequality (2) that improves the exponent $\frac{c}{\alpha-c}$. The result will however still exclude the extremal value $\alpha=c$. On the other hand, since it is classical that the curvature condition $\Gamma_2 \geq \rho\Gamma$, $\rho>0$, implies the log-Sobolev inequality, our result deal with less general situations than in [10] (there exist probability measures satisfying the log-Sobolev inequality and not the Γ_2 condition). Also, our result represents a local version (in the sense that it involves the semi-group P_t) of (2), which, to the best of our knowledge, is new. In other words, the approach in [10] is more robust while our approach by semi-group gives a stronger statement.

Theorem 3.1. *Let L be some Markov diffusion operator satisfying the curvature condition $\Gamma_2(f) \geq \rho\Gamma(f)$, $\rho>0$, for all $f \in \mathcal{A}$. Then, for all f and all $t>0$ the following inequality holds*

$$\log P_t(e^f) - P_t f \leq c_\alpha(t)\left(\log P_t\left(e^{\alpha\Gamma(f)}\right)\right)$$

for all $\alpha > \frac{e^{2\rho t}+e^{-2\rho t}-2}{2\rho(e^{2\rho t}-1)}$ with

$$c_\alpha(t) = \frac{\sqrt{1-e^{-2\rho t}}}{2\sqrt{2\rho\alpha}}\log\left(\frac{(e^{\rho t}a_t-1)(e^{\rho t}a_t+1+a_t^2-e^{-2\rho t})}{(e^{\rho t}a_t+1)(-e^{\rho t}a_t+1+a_t^2-e^{-2\rho t})}\right)$$

and

$$a_t^2 := 2\rho\alpha(e^{2\rho t}-1).$$

In particular, for any $\alpha > \frac{1}{2\rho}$ and any f satisfying $\int f d\mu = 0$, it holds

$$\int e^f d\mu \leq \left(\int e^{\alpha\Gamma(f)}d\mu\right)^{\frac{1}{2\sqrt{2\rho\alpha}}\log\frac{\sqrt{2\rho\alpha}+1}{\sqrt{2\rho\alpha}-1}}.$$

Remark 3.2. *Since $\frac{1}{2x}\log\left(\frac{x+1}{x-1}\right) \leq \log\left(\frac{x^2}{x^2-1}\right)$, we immediately get that*

$$\frac{1}{2\sqrt{2\rho\alpha}}\log\frac{\sqrt{2\rho\alpha}+1}{\sqrt{2\rho\alpha}-1} \leq \log\frac{2\rho\alpha}{2\rho\alpha-1} \leq \frac{1}{2\rho\alpha-1}.$$

The result above is therefore stronger than (2) as announced (recall that for the Γ_2-condition implies the log-Sobolev inequality (1) with constant $c=1/(2\rho)$, hence in our setting the exponent in the right hand side of (2) is precisely $\frac{1}{2\rho\alpha-1}$).

Proof. The second part of the theorem is a direct consequence of the first, in the limit $t\to\infty$ since $\lim_{t\to\infty} P_t e^f = \int e^f d\mu$ and $\lim_{t\to\infty} P_t f = \int f d\mu = 0$.

For the first part, observe that

$$\log P_t(e^f) - P_t f = -\int_0^t \frac{d}{ds}\log P_{t-s}\left(e^{P_s f}\right)ds.$$

By the diffusion property, we have

$$-\frac{d}{ds}\log P_{t-s}\left(e^{P_s f}\right) = \frac{P_{t-s}\left(Le^{P_s f} - e^{P_s f}LP_s f\right)}{P_{t-s}\left(e^{P_s f}\right)} = \frac{P_{t-s}\left(e^{P_s f}\Gamma(P_s f)\right)}{P_{t-s}\left(e^{P_s f}\right)}.$$

Therefore,

$$\log P_t(e^f) - P_t f = \int_0^t \frac{P_{t-s}\left(e^{P_s f}\Gamma(P_s f)\right)}{P_{t-s}\left(e^{P_s f}\right)}ds = \int_0^t P_{t-s}\left(h\Gamma(f_s)\right)ds$$

where we set for simplicity $h = e^{P_s f}/P_{t-s}\left(e^{P_s f}\right)$, which is a density with respect to P_{t-s}, and $f_s = P_s f$. Applying the entropic inequality (see *e.g.* [1, Chapter 1]), for any $\theta>0$, it holds that

$$P_{t-s}(h\Gamma(P_s f)) \leq \frac{1}{\theta}\mathrm{Ent}_{P_{t-s}}(h) + \frac{1}{\theta}\log P_{t-s}\left(e^{\theta\Gamma(f_s)}\right)$$

where $\mathrm{Ent}_{P_t}(f^2) := P_t(f^2 \log f^2) - P_t(f^2) \log P_t(f^2)$ denotes the entropy of f^2 with respect to P_t. Now the curvature condition implies (and in fact is equivalent to) the following local log-Sobolev inequality (see *e.g.* [1, Theorem 5.4.7])

$$\mathrm{Ent}_{P_t}(f^2) \leq \frac{2}{\rho}(1 - e^{-2\rho t}) P_t(\Gamma(f)). \tag{19}$$

At time $t - s$ with $f = \sqrt{h}$, (19) reads

$$\mathrm{Ent}_{P_{t-s}}(h) \leq \frac{1 - e^{-2\rho(t-s)}}{2\rho} P_{t-s}(h\Gamma(f_s)).$$

Therefore, for any $\theta > \frac{1-e^{-2\rho(t-s)}}{2\rho}$ it holds

$$\begin{aligned}
P_{t-s}(h\Gamma(f_s)) &\leq \frac{2\rho}{2\rho\theta - 1 + e^{-2\rho(t-s)}} \log P_{t-s}\left(e^{\theta\Gamma(f_s)}\right) \\
&\leq \frac{2\rho}{2\rho\theta - 1 + e^{-2\rho(t-s)}} \log P_{t-s}\left(e^{\theta e^{-2\rho s} P_s(\Gamma(f))}\right) \\
&= \frac{2\rho q(s)}{2\rho\theta - 1 + e^{-2\rho(t-s)}} \log \left(P_{t-s}\left(e^{q(s) P_s(\Gamma(f))}\right)\right)^{1/q(s)}
\end{aligned}$$

where for the second inequality we used the commutation property (15) and in the last equality we choose θ so that

$$\theta e^{-2\rho s} = q(s) := \frac{\alpha(e^{2\rho t} - 1)}{e^{2\rho(t-s)} - 1}.$$

This choice is licit as soon as $\alpha > \frac{e^{2\rho t} + e^{-2\rho t} - 2}{2\rho(e^{2\rho t} - 1)}$ (which guaranties that $\theta > \frac{1-e^{-2\rho(t-s))}}{2\rho}$). Now by Lemma 3.4, applied at time 0 and s, with $p = \alpha$, we get

$$\begin{aligned}
P_{t-s}(h\Gamma(f_s)) &\leq \frac{2\rho q(s)}{2\rho\theta - 1 + e^{-2\rho(t-s)}} \log \left(P_t\left(e^{q(0)(\Gamma(f))}\right)\right)^{1/q(0)} \\
&= \frac{2\rho q(s)/q(0)}{2\rho\theta - 1 + e^{-2\rho(t-s)}} \log P_t\left(e^{\alpha(\Gamma(f))}\right).
\end{aligned}$$

Using the explicit expressions of θ and $q(s)$ in terms of s, t, ρ, we conclude after some algebra that

$$\log P_t\left(e^f\right) - P_t f \leq \log P_t\left(e^{\alpha\Gamma(f)}\right) \int_0^t \frac{2\rho(e^{2\rho t} - 1)}{2\rho\alpha e^{2\rho s}(e^{2\rho t} - 1) + (e^{2\rho(t-s)} - 1)(e^{-2\rho(t-s)} - 1)} ds.$$

The expected result follows from Lemma 3.6. The proof is complete. □

Remark 3.3. *In the course of the proof, we could have used a cruder but shorter argument. Indeed, by Jensen's inequality*

$$P_{t-s}\left(e^{\theta e^{-2\rho s} P_s(\Gamma(f))}\right) \leq P_{t-s} P_s\left(e^{\theta e^{-2\rho s}\Gamma(f)}\right) = P_t\left(e^{\theta e^{-2\rho s}\Gamma(f)}\right).$$

Therefore, for $\theta = \theta(s) = \alpha e^{2\rho s}$, we obtain (for $\alpha > (1 - e^{-2t})/2\rho$)

$$\begin{aligned}
\log P_t(e^f) - P_t f &\leq \log P_t\left(e^{\alpha\Gamma(f)}\right) \int_0^t \frac{2\rho}{2\rho\theta(s) - 1 + e^{-2\rho(t-s)}} ds \\
&= \log P_t\left(e^{\alpha\Gamma(f)}\right) \log\left(\frac{2\rho\alpha e^{2\rho t}}{2\rho\alpha e^{2\rho t} + 1} \times \frac{2\rho\alpha + e^{-2\rho t}}{2\rho\alpha - 1 + e^{-2\rho t}}\right)
\end{aligned}$$

(where the equality follows by computing explicitly the integral (changing variables $u = e^{2\rho s}(2\rho\alpha + e^{-2\rho t}) - 1$ *(the denominator)). In the limit* $t \to \infty$ *the latter implies*

$$\int e^f \, d\mu \leq \left(\int e^{\alpha \Gamma(f)} d\mu \right)^{\log\left(\frac{2\rho\alpha}{2\rho\alpha - 1}\right)}$$

for any $\alpha > \frac{1}{2\rho}$ *which already constitutes an improvement with respect to* (2)*, under the curvature condition.*

In the proof of Theorem 3.1 we used the following lemmas.

Lemma 3.4. *Let* L *be a Markov diffusion operator satisfying the curvature condition* $\Gamma_2(f) \geq \rho\Gamma(f)$ *for some* $\rho > 0$ *and all* $f \in \mathcal{A}$*. Let* $0 \leq s \leq s' < t$*. Then any* $f \in \mathcal{A}$ *non-negative satisfies*

$$(P_{t-s'}(e^{q(s')P_{s'}f}))^{\frac{1}{q(s')}} \leq (P_{t-s}(e^{q(s)P_s f}))^{\frac{1}{q(s)}}$$

where

$$q(s) = \frac{p(e^{2\rho t} - 1)}{e^{2\rho(t-s)} - 1}$$

with $p > 0$.

Remark 3.5. *Lemma 3.4 corresponds to some local hypercontractivity property that is usually stated for the invariant measure* μ *in place of* P_t*, see* [2, 1]*. Some similar local version of the hypercontractivity property already exist in* [3, Theorem 3.1]*. It is plain and somehow classical that the statement of Lemma 3.4 is equivalent to what appears in* [3, Theorem 3.1][1] *but we need the specific form of the statement above that we could not find or directly derive from* [3]*.*

Proof. Fix t and set $\psi(s) = \frac{1}{q(s)} \log\left(P_{t-s}\left(e^{q(s)P_s f}\right)\right)$. Taking the derivative, we get, after some algebra (using the diffusion property)

$$\begin{aligned}
\psi'(s) &= -\frac{q'(s)}{q(s)^2} \log\left(P_{t-s}\left(e^{q(s)P_s f}\right)\right) + \frac{P_{t-s}\left(-Le^{qP_s f} + e^{qP_s f}[q'P_s f + qLP_s f]\right)}{q(s)P_{t-s}\left(e^{q(s)P_s f}\right)} \\
&= \frac{q'(s)}{q(s)^2} \frac{1}{P_{t-s}\left(e^{q(s)P_s f}\right)} \left(\mathrm{Ent}_{P_{t-s}}\left(e^{qP_s f}\right) - \frac{4q}{q'} P_{t-s}\left(\Gamma(e^{qP_s f/2})\right)\right).
\end{aligned}$$

Our choice of q insures that

$$\frac{4q(s)}{q'(s)} = \frac{2}{\rho}(1 - e^{-2\rho(t-s)})$$

so that, applying the local log-Sobolev inequality (19) at time $t-s$ to $f^2 = e^{qP_s f}$, we conclude that $\psi'(s) \leq 0$. The expected result immediately follows. □

Lemma 3.6. *For any* $t > 0$ *it holds*

$$\begin{aligned}
\int_0^t & \frac{2\rho(e^{2\rho t} - 1)}{2\rho\alpha e^{2\rho s}(e^{2\rho t} - 1) + (e^{2\rho(t-s)} - 1)(e^{-2\rho(t-s)} - 1)} ds \\
&= \frac{\sqrt{1 - e^{-2\rho t}}}{2\sqrt{2\rho\alpha}} \log\left(\frac{(e^{\rho t}a_t - 1)(e^{\rho t}a_t + 1 + a_t^2 - e^{-2\rho t})}{(e^{\rho t}a_t + 1)(-e^{\rho t}a_t + 1 + a_t^2 - e^{-2\rho t})}\right)
\end{aligned}$$

with

$$a_t^2 := 2\rho\alpha(e^{2\rho t} - 1).$$

[1] and in fact the conclusion of the Lemma implies the curvature condition so that both properties are equivalent

Proof. Changing variables $u = e^{2\rho s}$, we observe that

$$\begin{aligned}
I_t &:= \int_0^t \frac{2\rho(e^{2\rho t}-1)}{2\rho\alpha e^{2\rho s}(e^{2\rho t}-1)+(e^{2\rho(t-s)}-1)(e^{-2\rho(t-s)}-1)}ds \\
&= \int_1^{e^{2\rho t}} \frac{e^{2\rho t}-1}{u\left[2\rho\alpha u(e^{2\rho t}-1)+2-\frac{e^{2\rho t}}{u}-e^{-2\rho t}u\right]}du \\
&= \int_1^{e^{2\rho t}} \frac{e^{2\rho t}-1}{u^2c_t+2u-e^{2\rho t}}du
\end{aligned}$$

where we set for simplicity $c_t := 2\rho\alpha(e^{2\rho t}-1)-e^{-2\rho t}$. Denote $d_t^2 := \frac{1}{c_t^2}+\frac{e^{2\rho t}}{c_t}$ so that

$$u^2c_t+2u-e^{2\rho t} = c_t\left(\left(u+\frac{1}{c_t}\right)^2-d_t^2\right).$$

Changing again variables $v = u+\frac{1}{c_t}$, it follows that

$$I_t = \frac{e^{2\rho t}-1}{c_t}\int_{1+\frac{1}{c_t}}^{e^{2\rho t}+\frac{1}{c_t}} \frac{dv}{v^2-d_t^2} = \frac{e^{2\rho t}-1}{c_t}\left[\frac{1}{2d_t}\log\left(\frac{v-d_t}{v+d_t}\right)\right]_{1+\frac{1}{c_t}}^{e^{2\rho t}+\frac{1}{c_t}}.$$

This leads to the expected result after some algebra. □

3.3. **Extension to Sub-Gaussian measures.** As already mentioned, the main assumption in the original approach of the inequality (2) in [10] is the log-Sobolev inequality. For the one parameter family of measures μ_p with density $Z_p^{-1}e^{-|x|^p}$ on the line, say, with $Z = \int_{\mathbb{R}} e^{-|x|^p}dx$, it is known that a log-Sobolev inequality holds if and only if $p \geq 2$.

In this section we make use of a different functional inequality in order to derive some exponential integrability bounds in the flavour of (2) but for the measure μ_p, $1 < p \leq 2$. Our approach is an extension of the proof of [10]. In particular the semi-group approach of Theorem 3.1 cannot be applied as it is since we would need a commutation property similar to (15) which does not hold for the diffusion semi-group associated to the measure μ_p, $p \in (1,2)$ (the curvature condition only reads $\Gamma_2 \geq 0$).

We work on $\mathbb{R}^n$ but our result should extend to general diffusion as in Section 3.1. The choice of $\mathbb{R}^n$ is motivated by the fact that, to the best of our knowledge, the modified log-Sobolev inequality we are about to introduce has been studied and proved only in $\mathbb{R}^n$ (see [23, 24, 8]).

A probability measure μ on $\mathbb{R}^n$ satisfies the modified log-Sobolev inequality with function $H\colon \mathbb{R} \to [0,\infty)$ if there exists a constant $c \in (0,\infty)$ such that for all locally Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ the following holds

$$\mathrm{Ent}_\mu(f^2) \leq c\int \sum_{i=1}^n H\left(\frac{\partial_i f}{f}\right) f^2 d\mu$$

where $\partial_i f = \frac{\partial}{\partial x_i}f$ and

$$\mathrm{Ent}_\mu(f) := \int f\log f d\mu - \int f d\mu \log\int f d\mu = \sup\left\{\int fg d\mu : \int e^g d\mu \leq 1\right\}.$$

Similar to the log-Sobolev inequality, this is equivalent to

$$\mathrm{Ent}_\mu(e^f) \leq c\int \sum_{i=1}^n H\left(\frac{\partial_i f}{2}\right) e^f d\mu, \tag{20}$$

for all locally Lipschitz functions f. The terminology goes back to [9] in their study of the concentration phenomenon for product of exponential measures, thus recovering a celebrated result by Talagrand [40]. The second author and Ledoux introduced and considered (20) for functions such that $\partial_i f/f \le \kappa < 1$ which amounts to taking $H(x) = 2x^2/(1-\kappa)$ if $|x| \le \kappa$ and $H(x) = \infty$ otherwise.

The same authors proved (20) for the family of measures μ_p, with $p > 2$ (they consider also measures of the form e^{-V} with V strictly uniformly convex), and $H(x) = c_p|x|^q$ with $q = \frac{p-1}{p} \in [1,2]$ the conjugate of p. Such inequalities are equivalent to q-log-Sobolev inequalities that are studied in depth in [11].

The case $p \in (1,2)$ was considered by Gentil, Guillin and Miclo [23] who established a modified log-Sobolev inequality for μ_p when $p \in (1,2)$, with H that compares to $\max(x^2, |x|^q)$, $q = \frac{p-1}{p} > 2$. The same authors extended their result to a wider class of log-concave measures on the line, with tail behavior between exponential and Gaussian [24]. Another approach, based on Hardy-type inequalities, was proposed in [8]. For more results related to the measures μ_p, $p \in (1,2)$, we refer the reader to [18, 37].

We are now in position to state our theorem.

Theorem 3.7. *Assume that the probability measure μ on $\mathbb{R}^n$ satisfies the modified log-Sobolev inequality (20) with a non-decreasing convex function $H\colon \mathbb{R} \to [0,\infty)$ vanishing at the origin (and constant c). Assume furthermore that $x \mapsto H(\sqrt{x})$ is also convex. Then, for all locally Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $\int f d\mu = 0$, the following inequality holds*

$$\int e^f d\mu \le \left(\int \exp\left\{ \alpha \sum_{i=1}^n H\left(\frac{\partial_i f}{2}\right) \right\} d\mu \right)^{\frac{c}{\alpha - c}}$$

for all $\alpha > c$.

Remark 3.8. The classical log-Sobolev inequality corresponds to $H(x) = x^2$. We therefore recover the result of [10] (for $\mathbb{R}^n$).

For the measure μ_p, $p \in (1,2)$ mentioned above, one can prove (20) [23, 8] with H that compares to $\max(|x|^q, x^2)$, with $q = \frac{p-1}{p} > 2$ the conjugate of p. At the price of changing the constant c, it is easy to construct an other H that satisfies the assumption of the theorem for μ_p for which the conclusion applies.

By mean of (20) for more general log-concave measures with tail between exponential and Gaussian [24, 8] the conclusion of the theorem can also be extended to such a class of probability measures.

It should be noticed that, except for the Gaussian measure, the optimal constant in Inequality (20) is not known. Therefore the constraint $\alpha > c$ might be sub-optimal.

Proof. We follow [10]. Fix $\lambda, \alpha \ge 0$. Using the variational characterization of the entropy $\mathrm{Ent}_\mu(e^{\lambda f}) = \sup\{\int e^{\lambda f} g d\mu : \int e^g d\mu = 1\}$, we get, for

$$g = \alpha \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right) - \log \int \exp\left\{ \alpha \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right) \right\} d\mu,$$

$$\mathrm{Ent}_\mu(e^{\lambda f}) \ge \alpha \int e^{\lambda f} \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right) d\mu - \int e^{\lambda f} d\mu \log \int \exp\left\{ \alpha \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right) \right\} d\mu.$$

On the other hand, the modified log-Sobolev inequality (20) applied to λf ensures that

$$\alpha \int e^{\lambda f} \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right) d\mu \ge \frac{\alpha}{c} \mathrm{Ent}_\mu(e^{\lambda f}).$$

Therefore, for $\alpha > c$,

$$\mathrm{Ent}_\mu(e^{\lambda f}) \leq \frac{c}{\alpha - c} \int e^{\lambda f} d\mu \log \int \exp\left\{\alpha \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right)\right\} d\mu$$

The rest of the argument is Herbst's argument. Set $G(\lambda) := \int e^{\lambda f} d\mu$ and observe that

$$\mathrm{Ent}_\mu(e^{\lambda f}) = \lambda G'(\lambda) - G(\lambda)\log G(\lambda)$$

so that the inequality above can be recast as

$$\frac{G'(\lambda)}{\lambda G(\lambda)} - \frac{\log(G(\lambda))}{\lambda^2} = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\log G(\lambda)\right) \leq \frac{c}{\alpha - c}\frac{1}{\lambda^2}\log \int \exp\left\{\alpha \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i f\right)\right\} d\mu.$$

Now introduce, for $x \geq 0$, $\omega(x) = \sup_{t>0} \frac{H(tx)}{H(t)}$ hence it follows that:

$$\frac{d}{d\lambda}\left(\frac{1}{\lambda}\log G(\lambda)\right) \leq \frac{c}{\alpha - c}\frac{\beta(\lambda^2)}{\lambda^2}$$

with

$$\beta(\lambda) := \log \int \exp\left\{\alpha\omega(\sqrt{\lambda})\sum_{i=1}^n H\left(\frac{\partial_i f}{2}\right)\right\} d\mu.$$

Since $H(\sqrt{x})$ is convex, it is easy to check that $\omega(\sqrt{x})$ is convex and therefore that β is also convex. Since $\omega(0) = 0$ (H vanishes at 0), the map $t \to \beta(t)/t$ is non-decreasing and therefore

$$\frac{\beta(\lambda^2)}{\lambda^2} \leq \beta(1) = \log \int \exp\left\{\alpha\omega(1)\sum_{i=1}^n H\left(\frac{\partial_i f}{2}\right)\right\} d\mu = \log \int \exp\left\{\alpha \sum_{i=1}^n H\left(\frac{\partial_i f}{2}\right)\right\} d\mu$$

since $\omega(1) = 1$. Integrating the differential inequality, we obtain

$$\log G(1) - \lim_{\lambda \to 0}\frac{1}{\lambda}\log G(\lambda) \leq \frac{c}{\alpha - c}\beta(1)$$

which leads to the desired conclusion since $\lim_{\lambda\to 0}\frac{\log G(\lambda)}{\lambda} = \int f d\mu = 0$. $\square$

3.4. **Exponential integrability of second order.** In [17] the authors are interested in the following second order type exponential integrability. They prove that, given $\beta \geq 1$ and $M > 0$, all f smooth enough, with $\int f d\gamma_n = 0$, and such that $\int e^{(\frac{\beta}{2}|Lf|)^\beta} d\gamma_n \leq M$ satisfy

$$\int e^{|f|^\beta} d\gamma_n \leq C(M)$$

for some constant $C(M)$ (independent of f), where $L = \Delta - x \cdot \nabla$ is the Ornstein-Uhlenbeck operator. In this section, we give a quantitative version of this statement in the special case $\beta = 1$. Our result extends [17] (for $\beta = 1$) to the class of measures on $\mathbb{R}^n$ that satisfy the log-Sobolev Inequality (this class is larger that the set of all strongly log-concave probability measures considered in [17]) and to the general framework allowed by Γ_2 formalism. Our result reads as follows.

Theorem 3.9. *Let L be some Markov diffusion operator reversible with respect to some probability measure μ. Assume that μ satisfies the following log-Sobolev inequality*

$$\mathrm{Ent}_\mu(e^f) \leq c\int e^f \Gamma(f) d\mu \tag{21}$$

for some $c \in (0,\infty)$ *and any* f *smooth enough. Then, for any* $\alpha > c$ *and any* f *with* $\int f d\mu = 0$, *the following inequality holds*

$$\log \int e^f d\mu \leq \frac{c}{\alpha - c} \int e^{\alpha |Lf|} d\mu.$$

Moreover we have:

$$\log \int e^{|f|} d\mu \leq \left(\frac{c}{e\alpha} + \log 2 + \frac{2c}{\alpha - c} \right) \int e^{\alpha |Lf|} d\mu.$$

Remark 3.10. *For the Gaussian measure* $c = 1/2$. *The constraint* $\alpha > 1/2$ *agrees with the optimal threshold* $\beta/2 = 1/2$ *of* [17].

Proof of Theorem 3.9. By the diffusion property, $\Gamma(e^f, f) = e^f \Gamma(f)$ so that the integration by parts formula ensures that $\int e^f \Gamma(f) d\mu = \int \Gamma(e^f, f) d\mu = -\int e^f Lf d\mu$. In particular, the log-Sobolev inequality can be recast as

$$\mathrm{Ent}_\mu(e^f) \leq c \int e^f (-Lf) d\mu.$$

Next, if f is such that $\int e^f d\mu = 1$, the entropic inequality ensures that, for any $t > 0$,

$$\int e^f (-Lf) d\mu \leq \frac{1}{t} \mathrm{Ent}_\mu(e^f) + \frac{1}{t} \log \left(\int e^{-tLf} d\mu \right).$$

It follows that for any $t > c$

$$\mathrm{Ent}_\mu(e^f) \leq \frac{c}{t - c} \log \left(\int e^{-tLf} d\mu \right).$$

In turn, applying this to λf, the following inequality holds by homogeneity

$$\mathrm{Ent}_\mu(e^{\lambda f}) \leq \frac{c}{t - c} \int e^{\lambda f} d\mu \log \left(\int e^{-t\lambda Lf} d\mu \right), \qquad \lambda \in [0,1].$$

Set as usual $H(\lambda) = \int e^{\lambda f} d\mu$. Then the latter reads

$$\lambda H'(\lambda) - H(\lambda) \log H(\lambda) \leq \frac{c}{t - c} H(\lambda) \log \left(\int e^{-t\lambda Lf} d\mu \right).$$

Dividing both sides by $\lambda^2 H(\lambda)$ as in the usual Herbst's argument, we obtain

$$\frac{d}{d\lambda} \left(\frac{\log H(\lambda)}{\lambda} \right) = \frac{H'(\lambda)}{\lambda H(\lambda)} - \frac{\log H(\lambda)}{\lambda^2} \leq \frac{c}{t - c} \frac{1}{\lambda^2} \log \left(\int e^{-t\lambda Lf} d\mu \right), \qquad t > c.$$

Jensen's inequality $\log \left(\int e^{-t\lambda Lf} d\mu \right) \leq \lambda \log \left(\int e^{-tLf} d\mu \right)$ would not be enough to make the right hand side integrable at 0 (with respect to λ). To overcome this lack of integrability, we will expand the exponential, taking advantage of the fact that $\int Lf d\mu = 0$ by reversibility (thus making the right hand side of order $\int (Lf)^2 d\mu$ when $\lambda \to 0$). More precisely, we have for any $\lambda \in (0,1]$

$$\begin{aligned}
\frac{1}{\lambda^2} \log \left(\int e^{-t\lambda Lf} d\mu \right) &= \frac{1}{\lambda^2} \log \left(1 + \sum_{k=2}^{\infty} \frac{(-t\lambda)^k}{k!} \int (Lf)^k d\mu \right) && \left(\text{since } \int Lf d\mu = 0\right) \\
&\leq \sum_{k=2}^{\infty} \frac{t^k \lambda^{k-2}}{k!} \int |Lf|^k d\mu && (\text{since } \log(1+x) \leq x) \\
&\leq \int e^{t|Lf|} d\mu.
\end{aligned}$$

Finally, we get

$$\frac{d}{d\lambda}\left(\frac{\log H(\lambda)}{\lambda}\right) \leq \frac{c}{t-c}\int e^{t|Lf|}d\mu$$

which leads to the first desired conclusion after integration from $\lambda = 0$ to $\lambda = 1$ since $\lim_{\lambda\to 0}\frac{\log H(\lambda)}{\lambda} = \int f d\mu$.

The second part of the theorem is a consequence of the first. Given f with $\int f d\mu = 0$, set $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Then, since $\log(a + b) \leq \log a + \log b + \log 2$ for all $a, b \geq 1$, the first part of the theorem applied twice by scaling argument implies that

$$\begin{aligned}
\log\int e^{|f|}d\mu &\leq \log\left(\int e^{f_+}d\mu + \int e^{f_-}d\mu\right)\\
&\leq \log\left(\int e^{f_+}d\mu\right) + \log\left(\int e^{f_-}d\mu\right) + \log 2\\
&\leq \int (f_+ + f_-)d\mu + \log 2 + \frac{c}{\alpha - c}\left(\int e^{\alpha|Lf_+|}d\mu + \int e^{\alpha|Lf_-|}d\mu\right)\\
&\leq \int |f|d\mu + \log 2 + \frac{2c}{\alpha - c}\int e^{\alpha|Lf|}d\mu.
\end{aligned}$$

Since μ satisfies the log-Sobolev inequality (21), it satisfies a Poincaré inequality with constant $c/2$ (see *e.g.* [1, Chapter 1]). In particular (recall that $\int f d\mu = 0$)

$$\int f^2 d\mu \leq \frac{c}{2}\int \Gamma(f)d\mu.$$

Since $\int \Gamma(f)d\mu = -\int fLf d\mu$, the Cauchy-Schwartz Inequality implies

$$\int f^2 d\mu \leq \frac{c}{2}\sqrt{\int f^2 d\mu \int (Lf)^2 d\mu}$$

from which we deduce that

$$\int f^2 d\mu \leq \frac{c^2}{4}\int (Lf)^2 d\mu.$$

Using again Cauchy-Schwartz' inequality, we obtain

$$\int |f|d\mu \leq \sqrt{\int f^2 d\mu} \leq \frac{c}{2}\sqrt{\int (Lf)^2 d\mu}.$$

Now we observe that $x^2 \leq \frac{4}{e^2}e^x$ for $x \geq 0$, hence $\int \alpha^2 (Lf)^2 d\mu \leq \frac{4}{e^2}\int e^{\alpha|Lf|}d\mu$ and therefore

$$\int |f|d\mu \leq \frac{c}{2\alpha}\sqrt{\frac{4}{e^2}\int e^{\alpha|Lf|}d\mu} \leq \frac{c}{e\alpha}\int e^{\alpha|Lf|}d\mu, \qquad \alpha > 0.$$

We get for any $\alpha > c$

$$\begin{aligned}
\log\int e^{|f|}d\mu &\leq \frac{c}{e\alpha}\int e^{\alpha|Lf|}d\mu + \log 2 + \frac{2c}{\alpha - c}\int e^{\alpha|Lf|}d\mu\\
&\leq \left(\frac{c}{e\alpha} + \log 2 + \frac{2c}{\alpha - c}\right)\int e^{\alpha|Lf|}d\mu.
\end{aligned}$$

The desired conclusion follows. □

4. A convexity argument

In this section we will give an alternative proof of Inequality (3) (in a slight modified form). Our proof is based on an argument of duality that applies to part of the more general situation considered in [15].

4.1. Inequality (3) revisited.

Set for simplicity $\gamma = \gamma_1$ for the standard Gaussian measure on the line, with density $e^{-x^2/2}/\sqrt{2\pi}$. Our next result is one of our main results and is, as already mentioned in Section 2, some sort of exponential Hardy-type inequality.

Theorem 4.1. *For all $f\colon [0,\infty) \to \mathbb{R}$ with $f(0) = 0$, it holds that*

$$\log \int_0^\infty e^{|f(x)|} e^{-\frac{x^2}{2}} dx \le \int_0^\infty \frac{e^{|f'(x)|^2/2}}{\sqrt{1+\frac{|f'(x)|^2}{2}}} e^{-\frac{x^2}{2}} dx + 5.14.$$

Before moving to the proof of Theorem 4.1, let us show how to recover Inequality (3) from it.

Corollary 4.2. *For any dimension n and any f with $\int f d\gamma_n = 0$, the following inequality holds*

$$\log \int e^f d\gamma_n \le 8 \int_{\mathbb{R}^n} \frac{e^{\frac{|\nabla f|^2}{2}}}{\sqrt{1+\frac{|\nabla f|^2}{2}}} d\gamma_n \le 14 \int_{\mathbb{R}^n} \frac{e^{\frac{|\nabla f|^2}{2}}}{1+|\nabla f|} d\gamma_n.$$

Proof of Corollary 4.2. The exponential Hardy-type inequality of Theorem 4.1 implies that

$$\int_0^\infty e^f d\gamma - \frac{1}{2} \le F\left(\int G(|f'|) d\gamma \right), \qquad f(0) = 0$$

with $F(x) = ae^{bx} - c$, $a = \frac{e^{5.14}}{\sqrt{2\pi}}$, $b = \sqrt{2\pi}$, $c = \frac{1}{2}$ and $G(x) = e^{x^2/2}/\sqrt{1+(x^2/2)}$. Therefore the assumptions of Lemma 2.10 are satisfied. Since $\max_{x \ge 0} \frac{x}{G(x)} = \sqrt{1+\sqrt{2}}e^{-\frac{1}{\sqrt{2}}}$ (the maximum is reached at $x = 2^{1/4}$),

$$d := \sqrt{\frac{\pi}{2}} \max_{x\ge 0} \frac{x}{G(x)} \simeq 0.9602 \le 1$$

and $d + b \simeq 3.467 \le 3.5$. Lemma 2.10 implies that, for any dimension n and any f with $\int f d\gamma_n = 0$, it holds

$$\log \int e^g d\gamma_n \le \log \left(1 + a \exp\left\{ 3.5 \int G(|\nabla f|) d\gamma_n \right\} + (a-1) \exp\left\{ d \int G(|\nabla f|) d\gamma_n \right\} \right)$$

It is not difficult to check that $x \mapsto \Psi(x) := \frac{1}{\log(x)} \log\left(1 + ax^{3.5} + (a-1)x\right)$ is non-increasing on $[e,\infty)$ so that, for $x = \exp\left\{\int G(|\nabla f|) d\gamma_n\right\}$,

$$\log \int e^f d\gamma_n \le \Psi(e) \int_{\mathbb{R}^n} \frac{e^{\frac{|\nabla f|^2}{2}}}{\sqrt{1+\frac{|\nabla f|^2}{2}}} d\gamma_n \le 8 \int_{\mathbb{R}^n} \frac{e^{\frac{|\nabla f|^2}{2}}}{\sqrt{1+\frac{|\nabla f|^2}{2}}} d\gamma_n$$

where we used that $\Psi(e) \le 8$ (by numerical computation). To conclude, it is enough to observe that $\frac{1}{\sqrt{1+\frac{x^2}{2}}} \le \frac{\sqrt{3}}{1+x}$ (the maximum is reached at $x = 2$), and that $8\sqrt{3} \le 14$. $\square$

Proof of Theorem 4.1. Since $f(0)=0$, for $x>0$, we have by Young's inequality,

$$\begin{aligned}|f(x)| &= |\int_0^x f'(t)e^{t^2/2}e^{-t^2/2}dt| \\ &\le \int_0^x G(|f'(t)|)e^{-t^2/2}dt + \int_0^x G^*(e^{t^2/2})e^{-t^2/2}dt\end{aligned}$$

for any $G\colon [0,\infty)\to\mathbb{R}$ and $G^*(y)=\sup_{x>0}\{xy-G(x)\}$, $y\ge 1$. It follows that

$$\log\int_0^\infty e^{|f(x)|}e^{-\frac{x^2}{2}}dx \le \int_0^\infty G(|f'(x)|)e^{-\frac{x^2}{2}}dx + \log\int_0^\infty e^{\int_0^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx.$$

To end the proof, it remains to show that the second term in the right hand side of the latter is bounded above. Having a non-explicit constant would just necessitate the asymptotic of G^* for the choice $G(x)=e^{V(x)}$ with $V(x):=\frac{x^2}{2}-\frac{1}{2}\log(1+\frac{x^2}{2})$, $x\ge 0$. This could be achieved in few lines. We choose however to make some effort to keep track of the constants, making the presentation much lengthier. In other words the proof of Theorem 4.1 is essentially finished, what remains to be done is fixing some heavy technicalities to get an explicit constant in the end.

Observe that $V'(x)=\frac{x(1+x^2)}{2+x^2}$ and set

$$W(x)=V(x)+\log V'(x)=\frac{x^2}{2}+\log H(x), \qquad H(x)=\frac{\sqrt{2}x(1+x^2)}{(2+x^2)^{3/2}}, \quad x>0.$$

The maps W,H are non-decreasing from $(0,\infty)$ to $(-\infty,\infty)$ and from $(0,\infty)$ to $(0,\sqrt{2})$ respectively, so that we can safely define their inverse W^{-1},H^{-1} (see Lemma 4.3). Set $A_o=H^{-1}(1)$ and $x_o=W^{-1}(0)$. We observe that

$$G^*(y)=y\left(W^{-1}(\log y)-\frac{1}{V'(W^{-1}(\log y))}\right), \qquad y\ge 1$$

and (for $t\ge 0$)

$$\begin{aligned}G^*(e^{t^2/2})e^{-t^2/2}-t &= W^{-1}(t^2/2)-\frac{1}{V'(W^{-1}(t^2/2))}-t \\ &= x-\frac{1}{V'(x)}-\sqrt{2W(x)}\end{aligned}$$

where we have set $x=W^{-1}(t^2/2)$. Denote $t_o=\sqrt{2W(4)}\simeq 4.057$ and observe that, by monotonicity of W, there is a correspondence between the notations $[0,t_o]$ and $[x_o,4]$ through the change of variables $x=W^{-1}(t^2/2))$. Therefore, thanks to Item (iii) of Lemma 4.3 that guarantees that the above quantity is non-positive, the following holds

$$\begin{aligned}\int_0^\infty e^{\int_0^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx &= \int_0^{t_o} e^{\int_0^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx+\int_{t_o}^\infty e^{\int_0^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx \\ &\le \int_0^{t_o}dx+\int_{t_o}^\infty e^{\int_{t_o}^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx. \\ &= t_o+\int_{t_o}^\infty e^{\int_{t_o}^x\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx.\end{aligned}$$

Observe that, for any $x\ge A_o$ $W(x)\ge x^2/2$ so that (composing by W^{-1} that is non-decreasing) for all $t\ge A_o$ (and therefore for all $t\ge t_o$), $W^{-1}(t^2/2)\le t$. It follows from

Item (iv) of Lemma 4.3 that

$$\int_{t_o}^{x}\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt \le -\int_{t_o}^{x}\frac{1.228}{W^{-1}(t^2/2)}dt \le -\int_{t_o}^{x}\frac{1.228}{t}dt = -1.228\log\left(\frac{x}{t_o}\right).$$

In turn,

$$\int_{t_o}^{\infty} e^{\int_{t_o}^{x}\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx \le \int_{t_o}^{\infty}\left(\frac{x}{t_o}\right)^{-1.228}dx = \frac{1}{0.228t_o}.$$

We conclude that

$$\int_{0}^{\infty} e^{\int_{0}^{x}\left[G^*(e^{t^2/2})e^{-t^2/2}-t\right]dt}dx \le t_o+\frac{1}{0.228t_o} \simeq 5.138 \le 5.14$$

as expected. □

The next lemma collects some technical facts about the function V and W appearing in the proof of the previous theorem.

Lemma 4.3. *Set*

$$V(x)=\frac{x^2}{2}-\frac{1}{2}\log(1+\frac{x^2}{2}), \quad H(x)=\frac{\sqrt{2}x(1+x^2)}{(2+x^2)^{3/2}}, \quad x\ge 0,$$

(observe that $V'(x)=\frac{x(1+x^2)}{2+x^2}$*) and*

$$W(x)=V(x)+\log V'(x)=\frac{x^2}{2}+\log H(x).$$

Then
(i) the map H *is non-decreasing (one to one from* $(0,\infty)$ *to* $(0,\sqrt{2})$*) and* $A_o := H^{-1}(1) \in (2.13, 2.14)$*;*
(ii) the map W *is non-decreasing (one to one from* $(0,\infty)$ *to* $(-\infty,\infty)$*) and* $x_o := W^{-1}(0) \in (1.05, 1.06)$*;*
(iii) for all $x \ge x_o$*,*

$$x-\frac{1}{V'(x)}-\sqrt{2W(x)} \le 0;$$

(iv) for all $x\ge 4$*,*

$$x-\frac{1}{V'(x)}-\sqrt{2W(x)} \le -\frac{1.228}{x}$$

Proof. Point (i) and (ii) are obvious and left to the reader. By numerical computations $A_o\in(2.13,2.14)$ and $x_o\in(1.05,1.06)$. For (iii) we observe that, for all $x\ge x_o$,

$$\begin{aligned} x-\frac{1}{V'(x)}-\sqrt{2W(x)} &= -\frac{1}{V'(x)}+\frac{x^2-2W(x)}{x+\sqrt{2W(x)}} \\ &= -\frac{1}{V'(x)}-\frac{2\log H(x)}{x+\sqrt{2W(x)}}. \end{aligned} \tag{22}$$

For $x\ge A_o$, $H(x)\ge 1$ and therefore the above quantity is negative. Now assume that $x\in(x_o,A_o)$. We first deal with $x\in(x_o,2^{1/4})$. In that regime we write

$$x-\frac{1}{V'(x)}-\sqrt{2W(x)} \le x-\frac{1}{V'(x)} = \frac{x^4-2}{x(1+x^2)} \le 0.$$

For $x \in (2^{1/4}, A_o)$, since $H(x) \le 1$, $V'(x) \le x$ and $\sqrt{2W(x)} \ge 0$, the following holds

$$\begin{aligned} x - \frac{1}{V'(x)} - \sqrt{2W(x)} &= -\frac{1}{V'(x)} - \frac{2\log H(x)}{x + \sqrt{2W(x)}} \\ &\le -\frac{1}{x} - \frac{2\log H(x)}{x} \\ &= -\frac{2}{x}\log(\sqrt{e}H(x)). \end{aligned}$$

By monotonicity of H on $(2^{1/4}, A_o)$ we have $-\log(\sqrt{e}H(x)) \le -\log(\sqrt{e}H(2^{1/4})) \le 0$ since $\sqrt{e}H(2^{1/4}) = \sqrt{2e}2^{1/4}(1+\sqrt{2})/(2+\sqrt{2})^{3/2} \ge 1$. This proves Point (iii).

To prove the statement of Point (iv), we come back to (22) and use the fact that $V'(x) \le x$ to write

$$\begin{aligned} x - \frac{1}{V'(x)} - \sqrt{2W(x)} &\le -\frac{1}{x} - \frac{2\log H(x)}{x + \sqrt{2W(x)}} \\ &= -\frac{1 + I(x)}{x} \end{aligned}$$

with

$$I(x) := \frac{2x\log H(x)}{x + \sqrt{2W(x)}} = \frac{2\log H(x)}{1 + \sqrt{1 + \frac{2\log H(x)}{x^2}}}.$$

Since Since $y \mapsto \frac{y}{1+\sqrt{1+y}}$ and H are non-decreasing, we get, for any $x \ge 4$ (by numerical computations)

$$I(x) \ge \frac{2\log H(4)}{1 + \sqrt{1 + \frac{2\log H(4)}{x^2}}} \ge \frac{2\log H(4)}{1 + \sqrt{1 + \frac{2\log H(4)}{4^2}}} \ge 0.228.$$

The expected result of Point (iv) follows. □

4.2. **Extension.** In the next theorem, we extend the previous approach to more general exponential inequalities. To keep the length of the paper reasonable we will not, in this section, keep track of the explicit constants.

Our starting point is the following (exponential Hardy-type inequality).

Theorem 4.4. *For all $\beta \in (\sqrt{5}-1, 2)$, there exists a constant $c_\beta \in (0,\infty)$ such that for all $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ the following inequality holds*

$$\log \int_0^\infty e^{|f(x)|^{\frac{2\beta}{\beta+2}}} e^{-\frac{x^2}{2}}\,dx \le \left(\int_0^\infty e^{|\kappa f'(x)|^\beta} e^{-\frac{x^2}{2}}\,dx\right)^{\frac{2\beta}{\beta+2}} + c_\beta$$

with $\kappa = \frac{\sqrt{2}\beta}{\beta+2}$. One can take

$$c_\beta = \frac{5}{(2-\beta)^2} - \log(\beta - \sqrt{5} + 1).$$

Remark 4.5. The constant $c_\beta = \frac{5}{(2-\beta)^2} - \log(\beta - \sqrt{5} + 1)$ explodes both at $\beta = \sqrt{5} - 1$ and $\beta = 2$. Indeed, for some technical reason the above result does not contain the case $\beta = 2$ that was independently considered in Theorem 4.1. It appears that many simplification occur when $\beta = 2$ that cannot be transposed to $\beta \ne 2$.

Let us mention that we were not able to extend the semi-group approach developed in [28] or in Section 3 to prove the Theorem above.

Finally we observe that, with much more effort, it is possible to improve the right-hand side of the exponential Hardy-type inequality appearing in the theorem, by adding an extra factor in the denominator. Namely, one can replace $e^{|\kappa f'|^\beta}$ by $e^{|\kappa f'|^\beta}/H(f')$ for some non-decreasing function H tending to infinity at infinity.

Proof. The proof follows the line of Theorem 4.1. By Young's inequality,

$$\begin{aligned}|f(x)| &= |\int_0^x f'(t)e^{t^2/2}e^{-t^2/2}dt| \\ &\le \int_0^x G(|f'(t)|)e^{-t^2/2}dt + \int_0^x G^*(e^{t^2/2})e^{-t^2/2}dt\end{aligned}$$

for any $G\colon [0,\infty) \to \mathbb{R}$ and $G^*(y) = \sup_{x>0}\{xy - G(x)\}$, $y \ge 1$. It follows that

$$\begin{aligned}&\int_0^\infty e^{|f(x)|^{\frac{2\beta}{\beta+2}}} e^{-\frac{x^2}{2}}\, dx \le \\ &\int_0^x \exp\left\{\left(\int_0^\infty G(|f'(t)|)e^{-\frac{t^2}{2}}\, dt + \int_0^x \left[G^*(e^{t^2/2})e^{-t^2/2}\right] dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\} dx.\end{aligned}$$

Our aim is to analyse the second term in the right hand side of the latter. Assume that $G(x) = e^{V(x)}$ with $V(x) = (\kappa x)^\beta$, $x \ge 0$ and $\kappa := \frac{\sqrt{2}\beta}{\beta+2}$. Set

$$W(x) := V(x) + \log V'(x) = (\kappa x)^\beta + \log(\beta\kappa^\beta x^{\beta-1}), \quad x > 0.$$

The map W is one to one non-decreasing from $(0,\infty)$ to $(-\infty,\infty)$ so that we can define its inverse W^{-1}. In particular, a direct computation shows that

$$G^*(y) = y\left(W^{-1}(\log y) - \frac{1}{V'(W^{-1}(\log y))}\right), \quad y \ge 1,$$

so that (for $t \ge 0$)

$$G^*(e^{t^2/2})e^{-t^2/2} = W^{-1}(t^2/2) - \frac{1}{V'(W^{-1}(t^2/2))}.$$

Set

$$M := \int_0^\infty G(|f'(x)|)e^{-\frac{x^2}{2}}\, dx = \int_0^\infty e^{|\kappa f'(x)|^\beta} e^{-\frac{x^2}{2}}\, dx \quad \text{and} \quad x_o := \sqrt{2W\left(1/\left(\kappa\beta^{\frac{1}{\beta}}\right)\right)}.$$

Then, thanks to Lemma 4.6 (that guarantees that $G^*(e^{t^2/2})e^{-t^2/2} \le 0$ for all $t \in (0, x_o)$) the following inequality holds

$$\begin{aligned}&\int_0^\infty \exp\left\{\left(\int_0^x G(|f'(x)|)e^{-\frac{x^2}{2}}\, dx + \int_0^x \left[G^*(e^{t^2/2})e^{-t^2/2}\right] dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\} dx \\ &\le e^{M^{\frac{2\beta}{\beta+2}}} \int_0^{x_o} e^{-\frac{x^2}{2}}\, dx + \int_{x_o}^\infty \exp\left\{\left(M + \int_{x_o}^x \left[G^*(e^{t^2/2})e^{-t^2/2}\right] dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\} dx.\end{aligned}$$

Observe that by the last part of Lemma 4.6, $x_o \le 1$ so that $\int_0^{x_o} e^{-x^2/2}dx \le \int_0^1 e^{-x^2/2}dx \le 1$. Combining these observations and the fact that $G^*(e^{t^2/2})e^{-t^2/2} \le W^{-1}(t^2/2)$ (note that $W^{-1} \ge 0$), one has

$$\int_0^\infty e^{|f(x)|^{\frac{2\beta}{\beta+2}}} e^{-\frac{x^2}{2}}\, dx \le e^{M^{\frac{2\beta}{\beta+2}}} + \int_{x_o}^\infty \exp\left\{\left(M + \int_{x_o}^x W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\} dx.$$

Since $0 \le \frac{2\beta}{\beta+2} \le 1$,

$$\left(M + \int_{x_o}^{x} W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}} \le M^{\frac{2\beta}{\beta+2}} + \left(\int_{x_o}^{x} W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}}.$$

Therefore, as an intermediate result, it holds that

$$\int_0^\infty e^{|f(x)|^{\frac{2\beta}{\beta+2}}} e^{-\frac{x^2}{2}} dx \le e^{M^{\frac{2\beta}{\beta+2}}} \left(1 + \int_{x_o}^{\infty} \exp\left\{\left(\int_{x_o}^{x} W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\} dx\right).$$

Our next aim is to analyse the term $\int_{x_o}^{x} W^{-1}(t^2/2)dt$ for $x \ge x_o$ in order to prove that the integral factor is finite. By Lemma 4.6 and a change of variables, we have

$$\begin{aligned}\int_{x_o}^{x} W^{-1}(t^2/2)dt &\le \frac{1}{\kappa}\int_{\sqrt{2/3}}^{x}\left(\frac{t^2}{2} - \frac{\beta-1}{\beta}\log\left(\frac{t^2}{2}\right) + 1\right)^{\frac{1}{\beta}} dt \\ &= \frac{1}{\kappa}\int_{\frac{1}{3}}^{\frac{x^2}{2}}\left(u - \frac{\beta-1}{\beta}\log u + 1\right)^{\frac{1}{\beta}} \frac{du}{\sqrt{2u}}.\end{aligned}$$

Using the concavity property

$$(1+y)^\gamma \le 1 + \gamma y, \quad y > -1, \quad \gamma \in [0,1] \tag{23}$$

with $\gamma = \frac{1}{\beta}$, it follows that (for $u \ge 1/3$)

$$\begin{aligned}\left(u - \frac{\beta-1}{\beta}\log u + 1\right)^{\frac{1}{\beta}} &= u^{\frac{1}{\beta}}\left(1 - \frac{\beta-1}{\beta}\frac{\log u}{u} + \frac{1}{u}\right)^{\frac{1}{\beta}} \\ &\le u^{\frac{1}{\beta}}\left(1 + \frac{1}{\beta}\left(-\frac{\beta-1}{\beta}\frac{\log u}{u} + \frac{1}{u}\right)\right) \\ &= u^{\frac{1}{\beta}} - \frac{\beta-1}{\beta^2}u^{-\frac{\beta-1}{\beta}}\log u + \frac{1}{\beta}u^{-\frac{\beta-1}{\beta}}.\end{aligned}$$

As a consequence, by direct computation and some algebra, the following holds

$$\begin{aligned}&\int_{\frac{1}{3}}^{\frac{x^2}{2}}\left[u - \frac{\beta-1}{\beta}\log u + 1\right]^{\frac{1}{\beta}}\frac{du}{\sqrt{2u}} \\ &\le \frac{1}{\sqrt{2}}\int_{\frac{1}{3}}^{\frac{x^2}{2}}\left[u^{\frac{2-\beta}{2\beta}} - \frac{\beta-1}{\beta^2}u^{-\frac{3\beta-2}{2\beta}}\log u + \frac{1}{\beta}u^{-\frac{3\beta-2}{2\beta}}\right]du \\ &= \frac{1}{\sqrt{2}}\left[\frac{2\beta}{\beta+2}u^{\frac{2+\beta}{2\beta}} - \frac{\beta-1}{\beta^2}\left(\frac{2\beta}{2-\beta}u^{\frac{2-\beta}{2\beta}}\log u - \frac{4\beta^2}{(2-\beta)^2}u^{\frac{2-\beta}{2\beta}}\right) + \frac{2}{2-\beta}u^{\frac{2-\beta}{2\beta}}\right]_{\frac{1}{3}}^{\frac{x^2}{2}} \\ &= \kappa\left(\frac{x^2}{2}\right)^{\frac{2+\beta}{2\beta}} - \frac{\sqrt{2}(\beta-1)}{\beta(2-\beta)}\left(\frac{x^2}{2}\right)^{\frac{2-\beta}{2\beta}}\log\left(\frac{x^2}{2}\right) + \frac{\sqrt{2}\beta}{(2-\beta)^2}\left(\frac{x^2}{2}\right)^{\frac{2-\beta}{2\beta}} - c_\beta\end{aligned}$$

where in the last line we used the fact that $\kappa = \frac{\sqrt{2}\beta}{\beta+2}$, and we set

$$c_\beta := \frac{\kappa}{3^{\frac{2+\beta}{2\beta}}} + \frac{\sqrt{2}(\beta-1)}{\beta(2-\beta)}\frac{\log 3}{3^{\frac{2-\beta}{2\beta}}} + \frac{\sqrt{2}\beta}{(2-\beta)^2 3^{\frac{2-\beta}{2\beta}}} > 0.$$

Therefore, using (23) with $\gamma = \frac{2\beta}{\beta+2} \leq 1$, we obtain

$$\begin{aligned}
&\left(\int_{x_o}^{x} W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}} \\
&\leq \left(\left(\frac{x^2}{2}\right)^{\frac{2+\beta}{2\beta}} - \frac{\sqrt{2}(\beta-1)}{\kappa\beta(2-\beta)}\left(\frac{x^2}{2}\right)^{\frac{2-\beta}{2\beta}}\log\left(\frac{x^2}{2}\right) + \frac{\sqrt{2}\beta}{\kappa(2-\beta)^2}\left(\frac{x^2}{2}\right)^{\frac{2-\beta}{2\beta}}\right)^{\frac{2\beta}{\beta+2}} \\
&= \frac{x^2}{2}\left(1 - \frac{\sqrt{2}(\beta-1)}{\kappa\beta(2-\beta)}\frac{\log\left(x^2/2\right)}{x^2/2} + \frac{\sqrt{2}\beta}{\kappa(2-\beta)^2}\frac{1}{x^2/2}\right)^{\frac{2\beta}{\beta+2}} \\
&\leq \frac{x^2}{2}\left(1 + \frac{2\beta}{\beta+2}\left(-\frac{\sqrt{2}(\beta-1)}{\kappa\beta(2-\beta)}\frac{\log\left(x^2/2\right)}{x^2/2} + \frac{\sqrt{2}\beta}{\kappa(2-\beta)^2}\frac{1}{x^2/2}\right)\right) \\
&= \frac{x^2}{2} - \frac{2\sqrt{2}(\beta-1)}{\kappa(\beta+2)(2-\beta)}\log\left(\frac{x^2}{2}\right) + \frac{2\sqrt{2}\beta^2}{\kappa(\beta+2)(2-\beta)^2}.
\end{aligned}$$

At this step, using the expression of $\kappa = \frac{\sqrt{2}\beta}{\beta+2}$ and $x_o \geq \sqrt{2/3}$ (from Lemma 4.6), we can conclude that

$$\begin{aligned}
&\int_{x_o}^{\infty} \exp\left\{\left(\int_{x_o}^{x} W^{-1}(t^2/2)dt\right)^{\frac{2\beta}{\beta+2}} - \frac{x^2}{2}\right\}dx \\
&\leq e^{\frac{2\beta}{(2-\beta)^2}} \int_{\sqrt{2/3}}^{\infty} \left(\frac{2}{x^2}\right)^{\frac{2(\beta-1)}{\beta(2-\beta)}} dx \\
&= e^{\frac{2\beta}{(2-\beta)^2}} 2^{\frac{2(\beta-1)}{\beta(2-\beta)}} \int_{\sqrt{2/3}}^{\infty} \left(\frac{1}{x}\right)^{\frac{4(\beta-1)}{\beta(2-\beta)}} dx \\
&= e^{\frac{2\beta}{(2-\beta)^2}} 2^{\frac{2(\beta-1)}{\beta(2-\beta)}} \frac{\beta(2-\beta)}{\beta^2+2\beta-4}\left(\frac{3}{2}\right)^{\frac{\beta^2+2\beta-4}{2\beta(2-\beta)}} \\
&= \frac{\sqrt{2}\beta(2-\beta)}{\beta^2+2\beta-4}\exp\left\{\frac{2\beta}{(2-\beta)^2} + \frac{\beta^2+2\beta-4}{2\beta(2-\beta)}\log 3\right\}
\end{aligned}$$

where we used the fact that, for $\beta > \sqrt{5}-1$, $\frac{4(\beta-1)}{\beta(2-\beta)} > 1$ to ensure the convergence of the integral. Now it is not difficult to see that

$$\frac{\sqrt{2}\beta(2-\beta)}{\beta^2+2\beta-4} \leq \frac{8\sqrt{2}-16\sqrt{2/5}}{4(\beta-\sqrt{5}+1)} \leq \frac{1}{2(\beta-\sqrt{5}+1)} \qquad \beta \in (\sqrt{5}-1, 2)$$

(the maximum is reached at $\beta = \sqrt{5}-1$), and

$$\frac{\beta^2+2\beta-4}{2\beta(2-\beta)}\log 3 \leq \frac{1}{(2-\beta)^2}.$$

All together, we obtained for any $\beta \in (\sqrt{5}-1, 2)$

$$\begin{aligned}
\int_0^{\infty} e^{|f(x)|^{\frac{2\beta}{\beta+2}}} e^{-\frac{x^2}{2}}dx &\leq e^{M^{\frac{2\beta}{\beta+2}}}\left(1 + \frac{1}{2(\beta-\sqrt{5}+1)}\exp\left\{\frac{2\beta+1}{(2-\beta)^2}\right\}\right) \\
&\leq \frac{1}{\beta-\sqrt{5}+1}\exp\left\{\frac{5}{(2-\beta)^2}\right\}e^{M^{\frac{2\beta}{\beta+2}}}
\end{aligned}$$

This leads to the desired conclusion. □

Lemma 4.6. *Let $V(x) := (\kappa x)^\beta$, $x \geq 0$, $\beta \in (1,2)$, with $\kappa := \frac{\sqrt{2}\beta}{\beta+2}$ and $W(x) := (\kappa x)^\beta + \log(\beta\kappa^\beta x^{\beta-1})$, $x > 0$. Then, for all $x > 0$ the following inequality holds*

$$W^{-1}(x) \leq \frac{1}{\kappa}\left(x - \frac{\beta-1}{\beta}\log x + 1\right)^{\frac{1}{\beta}}.$$

Furthermore,

$$x \leq W\left(\frac{1}{\kappa\beta^{\frac{1}{\beta}}}\right) \quad \Longleftrightarrow \quad W^{-1}(x) - \frac{1}{V'(W^{-1}(x))} \leq 0.$$

Finally, for all $\beta \in (\sqrt{5}-1, 2)$, it holds $\frac{1}{2} \geq W\left(\frac{1}{\kappa\beta^{\frac{1}{\beta}}}\right) \geq 1/3$.

Proof. By studying the map $x \mapsto x - \frac{\beta-1}{\beta}\log x + (2-\beta)^2$ on $(0,\infty)$ we observe that it is non-increasing and then non-decreasing, its minimum is achieved at $x_o = \frac{\beta-1}{\beta}$ so that $x - \frac{\beta-1}{\beta}\log x + (2-\beta)^2 > 0$ for all $x > 0$. Therefore, we can compute

$$\begin{aligned}
&W\left(\frac{1}{\kappa}\left(x - \frac{\beta-1}{\beta}\log x + 1\right)^{\frac{1}{\beta}}\right) \\
&= x - \frac{\beta-1}{\beta}\log x + 1 + \log(\beta\kappa) + \frac{\beta-1}{\beta}\log\left(x - \frac{\beta-1}{\beta}\log x + 1\right) \\
&= x + \frac{\beta-1}{\beta}\log\left(1 - \frac{\beta-1}{\beta}\frac{\log x}{x} + \frac{1}{x}\right) + \log(e\beta k).
\end{aligned}$$

Now the map $x \mapsto -\frac{\beta-1}{\beta}\frac{\log x}{x} + \frac{1}{x}$ is non-increasing and then non-decreasing, with a minimum achieved at $e^{\frac{2\beta-1}{\beta-1}}$ so that, for any $x > 0$

$$\begin{aligned}
W\left(\frac{1}{\kappa}\left(x - \frac{\beta-1}{\beta}\log x + 1\right)^{\frac{1}{\beta}}\right) &\geq x + \frac{\beta-1}{\beta}\log\left(1 - \frac{\beta-1}{\beta}e^{-\frac{2\beta-1}{\beta-1}}\right) + \log(e\beta k) \\
&\geq x + \frac{\beta-1}{\beta}\log\left(1 - \frac{\beta-1}{\beta e^2}\right) + \log(e\beta k) \\
&\geq x + \frac{1}{2}\log\left(1 - \frac{1}{2e^2}\right) + \log(e\beta k)
\end{aligned}$$

where we used that $\frac{2\beta-1}{\beta-1} \geq 2$ and, in the last line, that $y \mapsto y\log(1-y)$ is non-increasing on $(0,1)$. Now it is clear that

$$\frac{1}{2}\log\left(1 - \frac{1}{2e^2}\right) + \log(e\beta k) = \log\left(\frac{\sqrt{2e^2-1}\beta^2}{\beta+2}\right) > 0$$

for any $\beta \in (1,2)$ so that $W\left(\frac{1}{\kappa}\left(x - \frac{\beta-1}{\beta}\log x + 1\right)^{\frac{1}{\beta}}\right) \geq x$, $x > 0$. Since W is non-decreasing, the first expected result follows.

For the second part of the lemma, set $y = W^{-1}(x)$ and observe that $y - \frac{1}{V'(y)} \leq 0$ if and only if $yV'(y) = \beta\kappa^\beta y^\beta \leq 1$, proving the desired result by monotonicity of W.

For the last conclusion of the lemma, note that $W\left(\frac{1}{\kappa\beta^{\frac{1}{\beta}}}\right) = \frac{\log(e\beta)}{\beta} + \log(\kappa)$. In particular, the map $\beta \in (1,2) \mapsto W\left(\frac{1}{\kappa\beta^{\frac{1}{\beta}}}\right)$ is non-decreasing so that, for $\beta \in (\sqrt{5}-1, 2)$, it holds

$$\frac{1}{2} = \frac{\log(2e)}{2} + \log\left(\frac{\sqrt{2}}{2}\right) \geq W\left(\frac{1}{\kappa\beta^{\frac{1}{\beta}}}\right) \geq \frac{\log(e(\sqrt{5}-1))}{\sqrt{5}-1} + \log\left(\frac{\sqrt{2}(\sqrt{5}-1)}{\sqrt{5}+1}\right) \simeq 0.36 \geq \frac{1}{3}.$$

$\square$

5. Discrete setting

Consider a graph (G, V) with vertex set V. Given a probability measure μ on G and an operator $L = (L(x,y))_{x,y\in V}$ symmetric in $\mathbb{L}^2(\mu)$ and satisfying $L(x,y) \geq 0$ for all $x \neq y$ and $\sum_y L(x,y) = 0$ for all x, we are interested in the following modified log-Sobolev inequality

$$\mathrm{Ent}_\mu(e^f) \leq c \sum_{x,y\in V} \mu(x)L(x,y)\left(e^{f(y)} - e^{f(x)}\right)(f(y) - f(x)). \tag{24}$$

The discrete gradient $\nabla e^f \nabla f$ reduces to $e^f|\nabla f|^2$ in the continuous setting. Therefore Inequality (24) is equivalent to the usual log-Sobolev inequality in the continuous. However, in the discrete setting, the behavior of the classical log-Sobolev inequality, that reads

$$\mathrm{Ent}_\mu(e^f) \leq \frac{c}{2} \sum_{x,y\in V} \mu(x)L(x,y)\left(e^{f(y)} - e^{f(x)}\right)^2$$

might be very different from the modified one. For instance, the Poisson measure on the integers, associated to a proper birth and death process, called $M/M/\infty$ in the literature, is known to satisfy the modified log-Sobolev inequality (24) with optimal constant the parameter of the Poisson measure, while the log-Sobolev inequality above does not hold.

There is a large activity on this topic, in relation with models coming from statistical mechanics, transport theory and some notion of Ricci curvature on graphs. To give a complete list of the literature is out of reach. Let us mention [12, 25] for two papers involving some of the authors, in this direction.

We may use the following classical notations. Given f, $Lf(x) = \sum_y L(x,y)f(y)$. Also, by reversibility, it is easy to see that

$$\int f(-Lg)d\mu = \frac{1}{2}\sum_{x,y} \mu(x,y)L(x,y)(g(y) - g(x))(f(y) - f(x)).$$

Therefore the modified log-Sobolev inequality can equivalently be written as

$$\mathrm{Ent}_\mu(e^f) \leq c\int e^f(-Lf)d\mu.$$

Theorem 5.1. *Assume that, on a graph (G, V), the probability measure and operator L as above are satisfying the modified log-Sobolev Inequality (24) with constant $c \in (0, \infty)$. Then, for any $\alpha > c$, and any f with $\int f d\mu = 0$ the following inequality holds*

$$\log \int e^f d\mu \leq \frac{c}{\alpha - c}\int e^{\alpha|Lf|}d\mu.$$

Proof. Fix $\lambda \in (0, 1]$ and a function f with mean zero with respect to μ. By the variational formula of the entropy $\mathrm{Ent}_\mu(h) = \sup\{\int hg d\mu : \log\int e^g d\mu \leq 1\}$, applied with $g = -\alpha\lambda Lf - \log\int e^{-\alpha\lambda Lf}d\mu$, together with the modified log-Sobolev inequality we have

$$\alpha\lambda\int e^{\lambda f}(-Lf)d\mu - \int e^{\lambda f}d\mu \log\int e^{-\alpha\lambda Lf}d\mu \leq \mathrm{Ent}_\mu(f^{\lambda f}) \leq c\int e^{\lambda f}(-L(\lambda f))d\mu$$

Therefore, for $\alpha > c$, it holds

$$\int e^{\lambda f}(-L(\lambda f))d\mu \leq \frac{1}{\alpha - c}\int e^{\lambda f}d\mu \log\int e^{-\alpha\lambda Lf}d\mu.$$

Applying the modified log-Sobolev inequality we obtain

$$\mathrm{Ent}_\mu(e^{\lambda f}) \leq c\int e^{\lambda f}(-L(\lambda f))d\mu \leq \frac{c}{\alpha - c}\int e^{\lambda f}d\mu \log\int e^{-\alpha\lambda Lf}d\mu$$

The rest of the proof goes as for the proof of Theorem 3.9. Details are left to the reader. □

Finally we specialized to the Poisson measure $\pi(k) = e^{-\lambda}\frac{\lambda^k}{k!}$, on the integers $k = 0, 1 \ldots$ and the $M/M/\infty$ queuing process. We will use our convexity argument to obtain some new exponential inequalities.

Set $\nabla f(k) := f(k+1) - f(k)$, $k = 0, 1 \ldots$ for the discrete gradient of a function f on the integers.

Theorem 5.2. *Let $\lambda > 0$ and for k integers, define $\pi(k) = e^{-\lambda}\frac{\lambda^k}{k!}$. Define*

$$G(x) := x\mathbb{1}_{[0,1]} + \exp\left\{\lambda\left(x + \left(1 + \frac{2}{x}\right)\log(\lambda x)\right)e^x\right\}\mathbb{1}_{(1,\infty)}, \qquad x > 0,$$

Then, there exists a constant c (that depends only on λ) such that for any $f\colon \mathbb{N} \to \mathbb{R}$ with $f(0) = 0$, it holds that

$$\log\left(\int e^f d\pi\right) \leq c + \int G(|\nabla f|)d\pi$$

and a constant d (that depends only on λ) such that for f with π-mean zero, i.e. $\int f d\pi = 0$, the following holds

$$\log\left(\int e^f d\pi\right) \leq c + d\int G(|\nabla f|)d\pi.$$

Remark 5.3. *The above integrals are understood on $\mathbb{N}$, i.e. $\int g d\pi = \sum_{n=0}^{\infty} g(n)\pi(n)$.*

The form of G is devised to ensure that $x \leq CG(x)$ for some constant $C > 0$. In fact, when x tends to 0, $\exp\left\{\lambda\left(x + \left(1 + \frac{2}{x}\right)\log(\lambda x)\right)e^x\right\}$ is much smaller that x.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$. By duality, we have

$$\begin{aligned}|f(n)| &= |\sum_{k=0}^{n-1} \nabla f(k)| \\ &\leq \sum_{k=0}^{n-1} |\nabla f(k)| \frac{1}{\pi(k)}\pi(k) \\ &\leq \sum_{k=0}^{n-1} G(|\nabla f(k)|)\pi(k) + \sum_{k=0}^{n-1} G^*\left(\frac{1}{\pi(k)}\right)\pi(k)\end{aligned}$$

where we set:
$G(x) := x\mathbb{1}_{[0,1]} + \exp\left\{\lambda\left(x + \left(1 + \frac{2}{x}\right)\log(\lambda x)\right)e^x\right\}\mathbb{1}_{(1,\infty)}$ and $G^*(y) = \sup_{x>0}\{xy - G(x)\}$, for all $y \geq 0$. Therefore,

$$\log\left(\int e^f d\pi\right) \leq \sum_{n=0}^{\infty} G(|\nabla f(n)|)\pi(n) + \log\left(\pi(0) + \sum_{n=1}^{\infty} \exp\left\{\sum_{k=0}^{n-1} G^*\left(\frac{1}{\pi(k)}\right)\pi(k)\right\}\pi(n)\right)$$

and we are left with proving that the second term in the right hand side of the latter is bounded. In fact we need to prove that the sum is convergent. By (27), we have, for n large enough

$$\begin{aligned}\sum_{k=0}^{n-1} G^*\left(\frac{1}{\pi(k)}\right)\pi(k) &\leq \sum_{k=0}^{k_o} G^*\left(\frac{1}{\pi(k)}\right)\pi(k) + \mathbb{1}_{n\geq k_o}\sum_{k=k_o}^{n-1} \log k - \log\lambda - \frac{1}{k} + \frac{3}{k\log k} \\ &\leq C + (\log(n-1)! - n\log\lambda - \log n + D\log_2 n)\mathbb{1}_{n\geq k_o}\end{aligned}$$

for some constant C that depends only on λ and some universal constant D. Above k_o is determined by Lemma 5.5. As a consequence, considering a bigger constant C if needed,

$$\begin{aligned}\sum_{n=1}^{\infty} \exp\left\{\sum_{k=0}^{n-1} G^*\left(\frac{1}{\pi(k)}\right)\pi(k)\right\}\pi(n) &\leq e^C\left(1+\sum_{n=2}^{\infty}(n-1)!\lambda^{-n}\frac{(\log n)^D}{n}\pi(n)\right)\\ &= e^C\left(1+e^{-\lambda}\sum_{n=2}^{\infty}\frac{(\log n)^D}{n^2}\right)\\ &< \infty.\end{aligned}$$

This proves the first part of the theorem.

Consider a function f with mean 0 with respect to the Poisson measure, *i.e.* such that $\pi(f) \coloneqq \sum_{n=0}^{\infty} f(n)\pi(n) = 0$. Then,

$$\begin{aligned}\log\left(\sum_{n=0}^{\infty} e^{|f(n)|}\pi(n)\right) &= \log\left(\sum_{n=0}^{\infty} e^{|f(n)-f(0)+f(0)-\pi(f)|}\pi(n)\right)\\ &\leq \log\left(\sum_{n=0}^{\infty} e^{|f(n)-f(0)|}\pi(n)\right) + |f(0)-\pi(f)|. \end{aligned} \tag{25}$$

The first term in the right hand side can be bounded using the first part of the theorem. We need to bound the second term $|f(0)-\pi(f)| \leq \sum_{n=0}^{\infty}|f(n)-f(0)|\pi(n)$. To that purpose we may use a direct computation. Indeed, we have

$$\begin{aligned}\sum_{n=0}^{\infty}|f(n)-f(0)|\pi(n) &= \sum_{n=1}^{\infty}|\sum_{k=0}^{n-1}\nabla f(k)|\pi(n)\\ &\leq \sum_{k=0}^{\infty}|\nabla f(k)|\pi(k)\sum_{n=k+1}^{\infty}\frac{\pi(n)}{\pi(k)}\\ &\leq A\sum_{k=0}^{\infty}|\nabla f(k)|\pi(k)\end{aligned}$$

where

$$A \coloneqq \sup_{k=0}^{\infty}\frac{1}{\pi(k)}\sum_{n=k+1}^{\infty}\pi(n).$$

The above inequality belongs to the family of Hardy's inequality[2]. It should be noticed that the constant A is best possible as one can convince oneself by considering the test function $f = \mathbb{1}_{[k,\infty)}$. It is easy to check that A, that depends only on λ, is finite. From (25) we have

$$\log\int e^f d\pi \leq c + \int G(|\nabla f|)d\pi + A\int|\nabla f|d\pi \leq c + d\int G(|\nabla f|)d\pi.$$

where in the last inequality we used the fact that $x \leq CG(x)$, for some constant C that depends only on λ, and all $x \geq 0$. □

In the proof above we used some technical lemmas. Recall the following well-known non-asymptotic version of the Stirling formula (see *e.g.* [36])

$$e^{\frac{1}{1+12n}} \leq \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \leq e^{\frac{1}{12n}}, \qquad n \geq 1 \tag{26}$$

[2]After [35], Miclo [34] developed Hardy-type inequalities in the discrete setting related to functional inequalities, see also [10] and [7, 6].

For simplicity we denote $\log_2 = \log(\log)$ and $\log_3 = \log_2(\log)$ for the iterated logarithms. Given a function F we denote F^{-1} its inverse function, when it exists.

Lemma 5.4. *Define $\Phi_o(x) = x\log x$ for $x \geq e$, and $\Phi_1(x) = (1+\frac{1}{\log_2 x})x\log x$ for $x \geq e^2$. Then, Φ_o and Φ_1 are one to one non-decreasing on $[e,\infty)$ onto itself and on $[e^2,\infty)$ onto $[4e^2,\infty)$ respectively, their inverse functions Φ_o^{-1} and Φ_1^{-1} satisfy, for any $x \geq e$ (respectively any $x \geq e^2$)*

$$\frac{x}{\log x} \leq \Phi_o^{-1}(x) \qquad \textit{and} \qquad \Phi_1^{-1}(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}\right).$$

Proof. The maps Φ_o, Φ_1 are clearly non-decreasing therefore one to one. For the lower bound observe that, for $x \geq e$,

$$\Phi_o\left(\frac{x}{\log x}\right) = x\frac{\log x - \log_2 x}{\log x} \leq x$$

while for the upper bound (using that $\log(1+\frac{1}{\log x}) \geq 0$ and that $-\log_2 x + \log(1+\frac{1}{\log x} \leq 0)$ we have for $x \geq e^2$

$$\begin{aligned}\Phi_1\left(\frac{x}{\log x}\left(1+\frac{1}{\log x}\right)\right) &= x\left(1+\frac{1}{\log x}\right)\frac{\log x - \log_2 x + \log[1+\frac{1}{\log x}]}{\log x}\times \\ &\quad\times\left(1+\frac{1}{\log\left(\log x - \log_2 x + \log(1+\frac{1}{\log x})\right)}\right) \\ &\geq x\left(1+\frac{1}{\log x}\right)\left(1-\frac{\log_2 x}{\log x}\right)\left(1+\frac{1}{\log_2 x}\right).\end{aligned}$$

Now it is not difficult to check that the product of the three brackets in the right hand side of the latter is greater than or equal to 1 for any $x \geq e^2$, proving that $\Phi_1^{-1}(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}\right)$ as announced. $\square$

Lemma 5.5. *For $\lambda > 0$ define*

$$G(x) := x\mathbb{1}_{[0,1]} + \exp\left\{\lambda\left(x+\left(1+\frac{2}{x}\right)\log(\lambda x)\right)e^x\right\}\mathbb{1}_{(1,\infty)}, \qquad x > 0,$$

and $\psi(x) := \log_2 G(x)$, $x \geq \max(1, 1/\lambda)$. Then there exists x_o (that depends only on λ) such that for any $x \geq x_o$, the following holds

$$\psi^{-1}(x) \leq x - \log(\lambda x).$$

Furthermore,

$$G^*(x) := \sup_{y>0}\{xy - G(y)\}, \qquad x \geq 0$$

satisfies the following inequality

$$G^*(x) \leq x\psi^{-1}(\log_2 \Phi_1^{-1}(x)), \qquad x \geq x_o$$

where Φ_1^{-1} is the inverse function of $\Phi_1 \colon x \mapsto x\log x(1+\frac{1}{\log_2 x})$ studied in Lemma 5.4. In particular, for any integer k large enough (how large depends only on λ), $\pi(k) := \frac{e^{-\lambda}\lambda^k}{k!}$ satisfies

$$\pi(k)G^*(1/\pi(k)) \leq \log k - \log\lambda - \frac{1}{k} + \frac{3}{k\log k}. \tag{27}$$

Proof. Note that ψ is non-decreasing and therefore one to one from $(\max(1,1/\lambda),\infty)$ onto its image. Its inverse function is well defined and we have, for x large enough

$$\begin{aligned}\psi\left(x-\log(\lambda x)\right) &= x+\log\left(\frac{1}{x}\left[x-\log(\lambda x)+\left(1+\frac{2}{x-\log(\lambda x)}\right)\log(\lambda(x-\log(\lambda x)))\right]\right)\\ &= x+\log\left(1+\frac{1}{x}\left[\log(1-\frac{\log(\lambda x)}{x})+\frac{2}{x-\log(\lambda x)}\log(\lambda(x-\log(\lambda x)))\right]\right)\geq x.\end{aligned}$$

The first statement on ψ^{-1} follows.

On $(1,\infty)$,

$$G'(x)=G(x)\log G(x)\left(1+\frac{2+x+x^2-\log(\lambda x)}{x(x^2+(2+x)\log(\lambda x))}\right)$$

so that G' is non-decreasing for $x\geq\max(1,1/\lambda)$ with inverse function we denote by ${G'}^{-1}$. In particular, for x large enough,

$$G^*(x)=x{G'}^{-1}(x)-G({G'}^{-1}(x))\leq x{G'}^{-1}(x).$$

Moreover, for any x large enough,

$$G'(x)\geq G(x)\log G(x)(1+\frac{1}{\log_2 G(x)})=\Phi_1(G(x))$$

with $\Phi_1(x):=x\log x(1+\frac{1}{\log_2 x})$. Therefore, recall that $\psi:=\log_2 G$, for x large enough it holds

$${G'}^{-1}(x)\leq G^{-1}(\Phi_1^{-1}(x))=\psi^{-1}(\log_2\Phi_1^{-1}(x))$$

leading to the announced upper bound on G^*.

Now by Lemma 5.4, we deduce that, for x small enough, setting $y=\log(1/x)$,

$$\begin{aligned}xG^*(1/x) &\leq \psi^{-1}(\log_2\Phi^{-1}((1/x)))\\ &\leq \psi^{-1}\left(\log_2\left(\frac{1}{x\log(1/x)}\left[1+\frac{1}{\log(1/x)}\right]\right)\right)\\ &= \psi^{-1}\left(\log\left[y-\log y+\log\left(1+\frac{1}{y}\right)\right]\right).\end{aligned}$$

Since $\log(1+z)\leq z$, it holds

$$\log\left[y-\log y+\log\left(1+\frac{1}{y}\right)\right]\leq\log y-\frac{\log y}{y}+\frac{1}{y^2}.$$

Therefore, using the upper bound on ψ^{-1}, in the large we obtain that

$$\begin{aligned}xG^*(1/x) &\leq \psi^{-1}\left(\log y-\frac{\log y}{y}+\frac{1}{y^2}\right)\\ &\leq \log y-\frac{\log y}{y}+\frac{1}{y^2}-\log\lambda-\log\left(\log y-\frac{\log y}{y}+\frac{1}{y^2}\right)\\ &\leq \log y-\log_2 y-\log\lambda-\frac{\log y}{y}+\frac{2}{y}.\end{aligned}$$

It follows that (recall that $y=\log(1/x)$), for k large enough,

$$\pi(k)G^*(1/\pi(k))\leq\log_2\frac{1}{\pi(k)}-\log_3\frac{1}{\pi(k)}-\log\lambda-\frac{\log_2\frac{1}{\pi(k)}}{\log\frac{1}{\pi(k)}}+\frac{2}{\log(1/\pi(k))}.$$

By the approximation of the Stirling formula (26), for k large enough

$$\begin{aligned}\log \frac{1}{\pi(k)} &\leq k\log k - k\log\lambda - k + \frac{1}{2}\log k + \lambda + \frac{1}{2}\log(2\pi) + \frac{1}{12k} \\ &\leq k\log k\end{aligned}$$

and

$$\begin{aligned}\log \frac{1}{\pi(k)} &\geq k\log k - k\log\lambda - k + \frac{1}{2}\log k + \lambda + \frac{1}{2}\log(2\pi) \\ &\geq \frac{2}{3}k\log k.\end{aligned}$$

Therefore, for k large enough

$$\log_2 \frac{1}{\pi(k)} \leq \log k + \log_2 k.$$

On the other hand, using again (26), for k large enough

$$\begin{aligned}\log_2 \frac{1}{\pi(k)} &\geq \log\left(k\log k - k\log\lambda - k + \frac{1}{2}\log k + \lambda + \frac{1}{2}\log(2\pi)\right) \\ &\geq \log k + \log_2 k - 1 \\ &\geq \log k\end{aligned}$$

so that

$$\log_3 \frac{1}{\pi(k)} \geq \log_2 k.$$

It follows that

$$\pi(k)G^*(1/\pi(k)) \leq \log k - \log\lambda - \frac{1}{k} + \frac{3}{k\log k}.$$

This is the desired thesis. □

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References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[2] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on probability theory (Saint-Flour, 1992)*, volume 1581 of *Lecture Notes in Math.*, pages 1–114. Springer, Berlin, 1994.

[3] D. Bakry, F. Bolley, and I. Gentil. Dimension dependent hypercontractivity for Gaussian kernels. *Probab. Theory Related Fields*, 154(3-4):845–874, 2012.

[4] D. Bakry and M. Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.

[5] D. Bakry and M. Ledoux. Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator. *Invent. Math.*, 123(2):259–281, 1996.

[6] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iberoam.*, 22(3):993–1067, 2006.

[7] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. volume 159, pages 481–497. 2003. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday (Polish).

[8] F. Barthe and C. Roberto. Modified logarithmic Sobolev inequalities on $\mathbb{R}$. *Potential Anal.*, 29(2):167–193, 2008.

[9] S. Bobkov and M. Ledoux. Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution. *Probab. Theory Related Fields*, 107(3):383–400, 1997.
[10] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.*, 163(1):1–28, 1999.
[11] S. G. Bobkov and B. Zegarlinski. Entropy bounds and isoperimetry. *Mem. Amer. Math. Soc.*, 176(829):x+69, 2005.
[12] S.G. Bobkov and P. Tetali. Modified logarithmic Sobolev inequalities in discrete settings. *J. Theoret. Probab.*, 19(2):289–336, 2006.
[13] L.A. Caffarelli. Monotonicity properties of optimal transportation and the FKG and related inequalities. *Communications in Mathematical Physics*, 214(3):547–563, 2000.
[14] L.A. Caffarelli. Erratum: "Monotonicity of optimal transportation and the FKG and related inequalities" [Comm. Math. Phys. **214** (2000), no. 3, 547–563; MR1800860 (2002c:60029)]. *Comm. Math. Phys.*, 225(2):449–450, 2002.
[15] A. Cianchi, V. Musil, and L. Pick. Moser inequalities in Gauss space. *Math. Ann.*, 377(3-4):1265–1312, 2020.
[16] A. Cianchi, V. Musil, and L. Pick. On the Existence of Extremals for Moser-Type Inequalities in Gauss Space. *International Mathematics Research Notices*, 2022(2):1494–1537, 07 2020.
[17] A. Cianchi, V. Musil, and L. Pick. Sharp exponential inequalities for the Ornstein-Uhlenbeck operator. *J. Funct. Anal.*, 281(11):Paper No. 109217, 68, 2021.
[18] A. Cianchi, L. Pick, and L. Slavíková. Higher-order sobolev embeddings and isoperimetric inequalities. *Advances in Mathematics*, 273:568–650, 2015.
[19] A. Ehrhard. Symétrisation dans l'espace de Gauss. *Math. Scand.*, 53(2):281–301, 1983.
[20] A. Ehrhard. Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 17(2):317–332, 1984.
[21] K. D. Elworthy. Stochastic flows on Riemannian manifolds. In *Diffusion processes and related problems in analysis, Vol. II (Charlotte, NC, 1990)*, volume 27 of *Progr. Probab.*, pages 37–72. Birkhäuser Boston, Boston, MA, 1992.
[22] M. Fradelizi and O. Guédon. A generalized localization theorem and geometric inequalities for convex bodies. *Adv. Math.*, 204(2):509–529, 2006.
[23] I. Gentil, A. Guillin, and L. Miclo. Modified logarithmic Sobolev inequalities and transportation inequalities. *Probab. Theory Related Fields*, 133(3):409–436, 2005.
[24] I. Gentil, A. Guillin, and L. Miclo. Modified logarithmic Sobolev inequalities in null curvature. *Rev. Mat. Iberoam.*, 23(1):235–258, 2007.
[25] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Displacement convexity of entropy and related inequalities on graphs. *Probab. Theory Related Fields*, 160(1-2):47–94, 2014.
[26] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.
[27] A. Guionnet and B. Zegarlinski. Lectures on logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 1–134. Springer, Berlin, 2003.
[28] P. Ivanisvili and R. Russell. Exponential integrability for log-concave measures. 2020.
[29] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995.
[30] M. Ledoux. *The Concentration of Measure Phenomenon*. American Mathematical Society, 2001.
[31] L. Lovász and M. Simonovits. Random walks in a convex body and an improved volume algorithm. *Random Structures Algorithms*, 4(4):359–412, 1993.
[32] V. Maz'ya and T.O. Shaposhnikova. *Sobolev Spaces: with Applications to Elliptic Partial Differential Equations*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.
[33] V. G. Maz'ya. Classes of domains and imbedding theorems for function spaces. *Soviet Math. Dokl.*, 1:882–885, 1960.
[34] L. Miclo. An example of application of discrete Hardy's inequalities. *Markov Process. Related Fields*, 5(3):319–330, 1999.
[35] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972.
[36] H. Robbins. A remark on Stirling's formula. *Amer. Math. Monthly*, 62:26–29, 1955.
[37] C. Roberto and B. Zegarliński. Orlicz-Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semi-groups. *J. Funct. Anal.*, 243(1):28–66, 2007.
[38] L. Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.
[39] A.J. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Information and Control*, 2:101–112, 1959.

[40] M. Talagrand. A new isoperimetric inequality and the concentration of measure phenomenon. In *Geometric aspects of functional analysis (1989–90)*, volume 1469 of *Lecture Notes in Math.*, pages 94–124. Springer, Berlin, 1991.

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