RICHTER'S LOCAL LIMIT THEOREM, ITS REFINEMENT, AND RELATED RESULTS

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Abstract. We give a detailed exposition of the proof of Richter's local limit theorem in a refined form, and discuss related quantitative bounds for characteristic functions and Laplace transforms.

1. **Introduction**

Let $(X_n)_{n \geq 1}$ be independent copies of a random variable X with mean $\mathbb{E}X = 0$ and variance $\mathrm{Var}(X) = 1$. Suppose that the normalized sum

$$Z_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$$

has a bounded density p_{n_0} for some $n = n_0$, that is, $p_{n_0}(x) \leq M$ for all $x \in \mathbb{R}$ with some constant M. Then all Z_n with $n \geq 2n_0$ have continuous bounded densities $p_n(x)$. An asymptotic behavior of these densities describing their closeness to the normal density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R},$$

is governed by several local limit theorems. First of all, there is a uniform local limit theorem due to Gnedenko

$$\sup_x |p_n(x) - \varphi(x)| \to 0 \quad \text{as } n \to \infty.$$

Under higher order moment assumptions, say if $\mathbb{E}\,|X|^m < \infty$ for an integer $m \geq 3$, this statement may be considerably sharpened in the form of a non-uniform local limit theorem

$$\sup_x\, (1 + |x|^m)\, |p_n(x) - \varphi_m(x)| = o\big(n^{-\frac{m-2}{2}}\big), \tag{1.1}$$

where φ_m denotes the Edgeworth correction of φ of order m (cf. [10], [15], [16]). In various applications, this relation is typically effective in the range $|x| \leq \sqrt{c \log n}$, since then the ratio $p_n(x)/\varphi(x)$ remains close to 1 (for a suitable c). For example, (1.1) is crucial in the study of rates in the entropic central limit theorem, rates for Rényi divergences of finite orders and for the relative Fisher information ([4]-[6]).

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As for larger regions, the asymptotic behavior of $p_n(x)$ is governed by the following remarkable theorem due to Richter [17], assuming the finiteness of an exponential moment for the random variable X.

Theorem 1.1. *Suppose that, for some $b > 0$,*

$$\mathbb{E}\, e^{b|X|} < \infty. \tag{1.2}$$

Then, for $x = o(\sqrt{n})$, the densities of Z_n admit the representation

$$\frac{p_n(x)}{\varphi(x)} = \exp\Big\{\frac{x^3}{\sqrt{n}}\,\lambda\Big(\frac{x}{\sqrt{n}}\Big)\Big\}\,\Big(1 + O\Big(\frac{1+|x|}{\sqrt{n}}\Big)\Big). \tag{1.3}$$

Here, $\lambda(\tau)$ represents an analytic function in some neighborhood of zero.

It was shown by Amosova [1] that the condition (1.2) is necessary for the existence of a representation like (1.3) in the region $|x| = o(\sqrt{n})$ with some analytic function λ.

The function λ in (1.3) is representable as a power series, called the Cramér series,

$$\lambda(\tau) = \sum_{k=0}^{\infty} \lambda_k \tau^k, \tag{1.4}$$

which is absolutely convergent in some disc $|\tau| < \tau_0$ of the complex plane. It has appeared in the work by Cramér [9] in a similar representation for the ratio of the tails of distribution functions of Z_n and the standard normal law (cf. also [11], [13], [14]).

Let us also mention that (1.3) is stated in Richter's work in a slightly different form with $O(\frac{|x|}{\sqrt{n}})$ in the last brackets and for $|x| > 1$. A similar result is proved in the book by Ibragimov and Linnik [12] under the assumption that X has a bounded continuous density.

As a consequence of (1.3), one immediately obtains, for example, that

$$\frac{p_n(x)}{\varphi(x)} \to 1 \quad \text{as } n \to \infty \tag{1.5}$$

uniformly in the region $|x| = o(n^{1/6})$. In the region $c_0 n^{1/6} \le |x| \le c_1 n^{1/2}$, the bahavior may be quite different, and in order to describe it, the appearence of the term $O(\frac{1+|x|}{\sqrt{n}})$ in (1.3) is non-desirable. The purpose of this paper is to give a detailed exposition of the proof of Theorem 1.1, clarifying the meaning of the leading coefficient in (1.4) and replacing this term with an n-depending quantity. We basically follow the presentation of [12] and derive the following refinement.

Theorem 1.2. *Let the conditions of Theorem* 1.1 *be fulfilled, and $n \ge 2n_0$. There is a constant $\tau_0 > 0$ with the following property. With $\tau = x/\sqrt{n}$, we have for $|\tau| \le \tau_0$*

$$\frac{p_n(x)}{\varphi(x)} = e^{n\tau^3\lambda(\tau) - \mu(\tau)}\,\big(1 + O(n^{-1}(\log n)^3)\big), \tag{1.6}$$

where $\mu(\tau)$ is an analytic function in $|\tau| \le \tau_0$ such that $\mu(0) = 0$.

As we will see in Section 7,

$$\begin{aligned}\lambda(\tau) &= \frac{1}{m!}\,\gamma_m \tau^{m-3} + O(|\tau|^{m-2}),\\ \mu(\tau) &= \frac{1}{2(m-2)!}\,\gamma_m \tau^{m-2} + O(|\tau|^{m-1}),\end{aligned}$$

where γ_m ($m \geq 3$) is the first non-zero cumulant of the random variable X (assuming that it is not normal). Equivalently, m is the smallest integer such that $\mathbb{E}X^m \neq \mathbb{E}Z^m$, where Z is a standard normal random variable, in which case

$$\gamma_m = \mathbb{E}X^m - \mathbb{E}Z^m.$$

With this refinement, it should be clear that the relation (1.5) holds true uniformly over all x in the potentially larger region

$$|x| \leq \varepsilon_n\, n^{\frac{1}{2}-\frac{1}{m}} \quad (\varepsilon_n \to 0).$$

For example, if the distribution of X is symmetric about the origin, then $\gamma_3 = 0$, so that necessarily $m \geq 4$.

Another consequence of (1.4), which does not follow from (1.3), is needed in the study of the rate of convergence in the central limit theorem with respect to the Rényi divergence of infinite order (which we do not discuss here).

Corollary 1.3. *Under the conditions of Theorem* 1.1, *suppose that m is even and $\gamma_m < 0$. There exist constants $\tau_0 > 0$ and $c > 0$ with the following property. If $|\tau| \leq \tau_0$, $\tau = x/\sqrt{n}$, then*

$$\frac{p_n(x)}{\varphi(x)} \leq 1 + \frac{c(\log n)^3}{n}. \tag{1.7}$$

Here, the conditions about cumulants are fulfilled, for example, when the random variable X is strongly subgaussian in the sense that $\mathbb{E}\, e^{tX} \leq e^{t^2/2}$ for all $t \in \mathbb{R}$. This interesting class of probability distributions is rather rich, and we refer the reader to [8] for discussions and various examples.

In most cases, the involved constants such as τ_0 in Theorem 1.2 may only depend on the parameters n_0, M, b, and the value of the integral in (1.2). In order to clarify the character of this dependence and make the proofs/arguments more transparent and self-contained, we include a short review of various related results – partly technical, but often interesting in themselves – about maxima of densities, analytic characteristic functions and log-Laplace transforms. With this in mind, we use the following plan:

Contents:

2. Maximum of Convolved Densities

The convolved densities are known to have improved smoothing properties. First, let us emphasize the following general fact (which explains the condition $n \geq 2n_0$ mentioned before Theorem 1.1).

Proposition 2.1. *If independent random variables ξ_1 and ξ_2 have bounded densities, then the sum $\xi = \xi_1 + \xi_2$ has a bounded uniformly continuous density vanishing at infinity.*

Proof. Denote by q_k the densities of ξ_k and assume that $q_k(x) \leq M_k$ for all $x \in \mathbb{R}$ with some constants M_k ($k = 1, 2$). By the Plancherel theorem, for the characteristic functions $g_k(t) = \mathbb{E}\, e^{it\xi_k}$, we have

$$\int_{-\infty}^{\infty} |g_k(t)|^2 \, dt = 2\pi \int_{-\infty}^{\infty} q_k(x)^2 \, dx \leq 2\pi \int_{-\infty}^{\infty} M_k q_k(x) \, dx = 2\pi M_k.$$

Hence, by Cauchy's inequality, the characteristic function $g(t) = g_1(t) g_2(t)$ of ξ is integrable and has L^1-norm

$$\begin{aligned} \int_{-\infty}^{\infty} |g(t)| \, dt \ &\leq \ \Big(\int_{-\infty}^{\infty} |g_1(t)|^2 \, dt \Big)^{1/2} \Big(\int_{-\infty}^{\infty} |g_2(t)|^2 \, dt \Big)^{1/2} \\ &\leq \ 2\pi \sqrt{M_1 M_2} < \infty. \end{aligned} \tag{2.1}$$

One may conclude that the random variable ξ has a bounded, uniformly continuous density expressed by the inversion Fourier formula

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} g(t) \, dt, \quad x \in \mathbb{R}. \tag{2.2}$$

Since g is integrable, it also follows that $q(x) \to 0$ as $|x| \to \infty$. $\square$

Consider the functional

$$M(\xi) = \operatorname{ess\,sup}_x \, q(x),$$

where ξ is a random variable with density q. Since, by (2.2),

$$q(x) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)| \, dt$$

for all $x \in \mathbb{R}$, the inequality (2.1) also implies that

$$M(\xi_1 + \xi_2) \leq \sqrt{M(\xi_1) M(\xi_2)}.$$

Using the Hausdorff-Young inequality, this relation may be extended to several independent summands as

$$M(S_m) \leq (M(\xi_1) \dots M(\xi_m))^{1/m}, \tag{2.3}$$

where $S_m = \xi_1 + \dots + \xi_m$. This show in particular that $M(\xi)$ may not increase by adding to ξ an independent random variable. However, the relation (2.3) does not correctly reflect the behavior of $M(S_m)$ with respect to the growing parameter m, especially in the i.i.d. situation. A more precise statement is described in the following relation, where the geometric mean of maxima is replaced with the harmonic mean.

Proposition 2.2. *Given independent random variables ξ_k, $1 \leq k \leq m$, one has*

$$\frac{1}{M(S_m)^2} \geq \frac{1}{2} \sum_{k=1}^{m} \frac{1}{M(\xi_k)^2}. \tag{2.4}$$

This bound may be viewed as a counterpart of the entropy power inequality in Information Theory. It may be obtained by combining Rogozin's maximum-of-density theorem with Ball's bound on the volume of slices of the cube. Namely, it was shown in [18] that, if the values $M_k = M(\xi_k)$ are fixed, $M(S_m)$ is maximized for ξ_k uniformly distributed in the intervals of length $1/M_k$. Of course, in this case $M(S_m)$ has a rather complicated structure as a function of m variables M_k's. On the other hand, if $T_m = a_1\eta_1 + \dots + a_m\eta_m$, where η_k are independent and uniformly distributed in $(0,1)$, and the coefficients satisfy $a_1^2 + \dots + a_m^2 = 1$, then

$$1 \leq M(T_m) \leq \sqrt{2},$$

cf. [2]. In geometric language, this is the same as saying that $1 \leq |Q \cap H| \leq \sqrt{2}$, where $Q = (0,1)^m$ is the unit cube, H is an arbitary hyperplane in $\mathbb{R}^m$ passing through the center of the cube, and $|\cdot|$ stands for the $(m-1)$-dimensional volume.

With this argument, the relation (2.4) is mentioned in [3], where its multidimensional analog is derived by applying the Hausdorff-Young inequality with best constants (due to Beckner and Lieb).

3. L^p-Norms of Characteristic Functions

One useful consequence of (2.4) is the next bound on L^{2m}-norms of characteristic functions.

Proposition 3.1. *If $g(t)$ is the characteristic function of a random variable ξ, then for any integer $m \geq 1$,*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)|^{2m}\,dt \leq \frac{1}{\sqrt{m}}\,M(\xi). \tag{3.1}$$

Proof. We apply Proposition 2.2 to $2m$ summands $\xi_1, -\xi_1', \dots, \xi_m, -\xi_m'$, assuming that ξ_k' are independent copies of ξ_k, being independent of all ξ_j. Introduce the symmetrized random variable $\tilde{S}_m = S_m - S_m'$, where S_m' is an independent copy of S_m. By (2.4), we then get

$$M(\tilde{S}_m) \leq \frac{1}{\sqrt{m}}\,M(\xi).$$

In addition, $\tilde{S}_m$ has characteristic function $|g(t)|^{2m}$. If $M(\xi)$ is finite, one may apply Proposition 2.1 and conclude that $\tilde{S}_m$ has a bounded continuous density $q_m(x)$ which is maximized at $x = 0$. Moreover, its value at this point is described by the inversion formula (2.2) which gives

$$M(\tilde{S}_m) = q_m(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)|^{2m}\,dt.$$

□

Using (2.3), one can obtain a similar relation, but without the factor $1/\sqrt{m}$ in (3.1).

When $M(\xi)$ is finite and m is large, this bound may be considerably sharpened asymptotically with respect to m when restricting the integration to the regions $|t| \geq \varepsilon > 0$. Before making this precise, first let us note that, since the randon variable ξ has a density, we have

$$\delta_g(\varepsilon) = \max_{|t|\geq\varepsilon} |g(t)| < 1 \tag{3.2}$$

for all $\varepsilon > 0$. This holds by continuity of g, and since $|g(t)| < 1$ for all $t \neq 0$ (which is true for any non-lattice distribution), while $g(t)$ tends to zero as $t \to \infty$, by the Riemann-Lebesgue lemma. By the way, this property remains to hold in the more general situation, where the m-fold convolution of the distribution of ξ with itself has a density (while the distribution of ξ might be not absolutely continuous). Indeed, in that case, (3.2) may be applied to g^m, and it remains to notice that this relation does not depend on m.

The property (3.2) may be quantified using, for example, the following observation due to Statuljavičus [19].

Proposition 3.2. *If a random variable ξ has a bounded density with $M = M(\xi)$ and finite variance $\sigma^2 = \mathrm{Var}(\xi)$, $\sigma > 0$, then its characteristic function g satisfies, for all $\varepsilon > 0$,*

$$\delta_g(\varepsilon) \leq \exp\Big\{-\frac{\varepsilon^2}{96\, M^2\,(2\sigma\varepsilon + \pi)^2}\Big\}. \tag{3.3}$$

Note that the functional $\xi \to M\sigma$ is affine invariant. It is well-known that $M\sigma \geq \frac{1}{12}$, and an equality is attained for the uniform distribution on any bounded interval.

This relation may be extended to non-bounded densities q, in which case the parameter M should be replaced with quantiles of the random variable $q(\xi)$. The moment condition may also be removed, and instead it is sufficient to deal with quantiles of $|\xi - \xi'|$, where ξ' is an independent copy of ξ; cf. [4] for details.

Returning to (3.1) and applying (3.3), we then have

$$\int_{|t|\geq\varepsilon} |g(t)|^{4m}\,dt \,\leq\, \delta_g(\varepsilon)^{2m} \int_{-\infty}^{\infty} |g(t)|^{2m}\,dt \,\leq\, \frac{2\pi M}{\sqrt{m}}\,\exp\Big\{-\frac{m\varepsilon^2}{CM^2}\Big\}$$

with some absolute constant C. Thus, the resulting bound decays asymptotically fast in m.

Let us derive a similar bound in the scheme of independent copies $(X_n)_{n\geq 1}$ of the random variable X with $\mathrm{Var}(X) = 1$, assuming that the normalized sum Z_n has a bounded density for $n = n_0$ with $M = M(Z_{n_0})$. Consider the characteristic function $f(t) = \mathbb{E}\, e^{itX}$. We apply Propositions 3.1-3.2 with $\xi = X_1 + \dots + X_{n_0}$ in which case $g(t) = f(t)^{n_0}$ and $M(\xi) = \frac{1}{\sqrt{n_0}}\, M$. Then, for any $1 \leq m \leq n/2n_0$, by (3.1),

$$\begin{aligned}\int_{|t|\geq\varepsilon} |f(t)|^n\,dt \ &= \ \int_{|t|\geq\varepsilon} |f(t)|^{n-2mn_0}\,|g(t)|^{2m}\,dt \\ &\leq \ \delta_f(\varepsilon)^{n-2mn_0} \int_{-\infty}^{\infty} |g(t)|^{2m}\,dt \,\leq\, \frac{2\pi M}{\sqrt{mn_0}}\,\delta_f(\varepsilon)^{n-2mn_0}.\end{aligned}$$

If $n \geq 4n_0$, let us choose $m = [\frac{n}{4n_0}]$. Then $n - 2mn_0 \geq \frac{n}{2}$, while $m \geq \frac{n}{8n_0}$, and we arrive at

$$\int_{|t|\geq\varepsilon} |f(t)|^n\,dt \leq \frac{8\pi M}{\sqrt{n}}\,\delta_f(\varepsilon)^{n/2}.$$

By (3.3) with $\varepsilon \leq 1$, we also have

$$\begin{aligned}\delta_f(\varepsilon) &\leq \exp\Big\{-\frac{\varepsilon^2}{96\,M^2\,(2\varepsilon\sqrt{n_0}+\pi)^2}\Big\}\\ &\leq \exp\Big\{-\frac{\varepsilon^2}{96\,(2+\pi)^2 n_0 M^2}\Big\}.\end{aligned}$$

Combining the two bounds, one may summarize.

Corollary 3.3. *Let* $\mathrm{Var}(X) = 1$, *and suppose that* Z_n *has a density for* $n = n_0$ *bounded by* M. *Then, for all* $0 < \varepsilon \leq 1$ *and* $n \geq 4n_0$, *the characteristic function* f *of* X *satisfies*

$$\int_{|t|\geq\varepsilon} |f(t)|^n\,dt \leq \frac{8\pi M}{\sqrt{n}}\,\exp\Big\{-\frac{n\varepsilon^2}{Cn_0M^2}\Big\}, \quad C = 5200. \tag{3.4}$$

4. Exponential Moments and Orlicz Norms

Under the condition (1.2), the characteristic function

$$f(z) = \mathbb{E}\,e^{izX}, \quad z = t + iy, \quad t, y \in \mathbb{R},$$

is well-defined and analytic in the strip $|y| = |\mathrm{Re}(z)| < b$ of the complex plane. One may quantify its behavior near zero, assuming that, for some $\alpha > 0$,

$$\mathbb{E}\,e^{\alpha|X|} \leq 2. \tag{4.1}$$

We discuss this issue in the next section, and here make a few preliminary remarks about the conditions (1.2) and (4.1).

When X has a finite exponential moment, and α is optimal, then (4.1) becomes an equality. In this case, the quantity $\frac{1}{\alpha}$ represents the Orlicz norm of the random variable X generated by the Young function $\psi(x) = e^{|x|} - 1$, $x \in \mathbb{R}$:

$$\|X\|_\psi = \inf\big\{\lambda > 0 : \mathbb{E}\,\psi(X/\lambda) \leq 1\big\}.$$

If $\mathbb{E}X^2 = 1$, the parameter α may not be large, since the L^2-norm is dominated by the L^ψ-norm. More precisely, using $x^2e^{-x} \leq 4e^{-2}$ $(x \geq 0)$, we have

$$\alpha^2 = \mathbb{E}\,(\alpha X)^2 \leq 4e^{-2}\,\mathbb{E}\,e^{\alpha|X|} = 8e^{-2},$$

implying $\alpha \leq 2e^{-1}\sqrt{2} < 1.05$. In fact, this bound may be sharpened.

Lemma 4.1. *If* $\mathbb{E}X^2 = 1$, *and* (4.1) *holds, then* $\alpha < 1$.

Proof. We may assume that $X \geq 0$ and then we need to show that $\mathbb{E}\,e^X > 2$. It is easy to check that $x + \frac{1}{6}\,x^3 \geq ax^2$ for all $x \geq 0$ with the optimal constant $a = \frac{2}{\sqrt{6}}$. Since

$\mathbb{E}X^k \geq (\mathbb{E}X^2)^{k/2} = 1$ for $k \geq 2$, we get

$$\begin{aligned} \mathbb{E}\, e^X &= 1 + \frac{1}{2}\,\mathbb{E}X^2 + \mathbb{E}\left(X + \frac{1}{6}\,X^3\right) + \sum_{k=4}^\infty \frac{1}{k!}\,\mathbb{E}X^k \\ &\geq \frac{3}{2} + a + \sum_{k=4}^\infty \frac{1}{k!} \ = \ e - \frac{7}{6} + \frac{2}{\sqrt{6}} \ > \ 2.36. \end{aligned}$$

$\square$

Note that if we start with a more general condition $B = \mathbb{E}e^{b|X|} < \infty$ as in Theorem 1.1, (4.1) is fulfilled for a certain constant $\alpha > 0$. Indeed, if $B \leq 2$, then one may take $\alpha = b$. Otherwise,

$$\mathbb{E}\, e^{\varepsilon b|X|} \leq (\mathbb{E}\, e^{b|X|})^\varepsilon \leq B^\varepsilon = 2$$

for $\varepsilon = \frac{1}{\log_2(B)}$. Hence $\alpha = \varepsilon b = \frac{b}{\log_2(B)}$ works as well. The two cases may be united by taking

$$\alpha = \frac{b}{\log_2(\max(B,2))}.$$

5. Behavior of Characteristic Functions near Zero

To start with, first let us quantify the closeness of the characteristic function $f(z)$ of the random variable X to 1 for small values of $|z|$. We assume throughout that

$$\mathbb{E}\, e^{\alpha|X|} \leq 2 \quad (\alpha > 0). \tag{5.1}$$

Using $xe^{-x} \leq 2e^{-1}$ $(x \geq 0)$ and writing $z = t + iy$ with $|y| \leq \frac{\alpha}{2}$, we then have

$$\begin{aligned} |f'(z)| &= |\,\mathbb{E}\, Xe^{izX}| \ \leq \ \mathbb{E}\,|X|\, e^{|yX|} \ \leq \ \mathbb{E}\,|X|\, e^{\alpha|X|/2} \\ &= \mathbb{E}\,|X|\, e^{-\alpha|X|/2}\, e^{\alpha|X|} \ \leq \ \frac{4}{\alpha e}. \end{aligned}$$

Hence $|f(z) - 1| \leq \frac{4}{\alpha e}\,|z|$ (since $f(0) = 1$). Thus, we obtain:

Lemma 5.1. *For all complex numbers z in the disc $|z| \leq \frac{\alpha}{2}$,*

$$|f'(z)| \leq \frac{4}{\alpha e}, \quad |f(z) - 1| \leq \frac{2}{e}.$$

This allows one to consider the log-Laplace transform

$$K(z) = \log \mathbb{E}\, e^{zX} = \log f(-iz)$$

as an analytic function in the disc $|z| \leq \frac{\alpha}{2}$. Since it has derivative $K'(z) = -i\,\frac{f'(-iz)}{f(-iz)}$, from Lemma 5.1 we get that in this disc

$$|K'(z)| \leq \frac{6}{\alpha}, \quad |K(z)| \leq 3. \tag{5.2}$$

One may also bound the derivatives of all orders, using Cauchy's formula

$$K^{(k)}(z) = \frac{k!}{2\pi} \int_{|w-z|=r} \frac{K(w)}{w^{k+1}}\, dw,$$

where we need to assume that $|z| + r \le \frac{\alpha}{2}$. Choosing $r = \frac{\alpha}{4}$ and applying (5.2), we obtain:

Lemma 5.2. *For all complex numbers z in the disc $|z| \le \frac{\alpha}{4}$,*

$$|K^{(k)}(z)| \le 3k! \left(\frac{4}{\alpha}\right)^k, \quad k = 1, 2, \dots \tag{5.3}$$

Thus, these derivatives have at most a factorial growth in absolute value with respect to the growing parameter k. For the particular orders $k = 2$ and $k = 3$, and under our standard moment assumptions, the bound (5.3) may be refined (in a smaller disc).

Lemma 5.3. *If $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, then for all complex numbers z in the disc $|z| \le \frac{\alpha}{16}$,*

$$|K'''(z)| \le \frac{8}{\alpha^3}. \tag{5.4}$$

As a consequence,

$$|K''(z) - 1| \le \frac{1}{2}, \quad |z| \le \frac{\alpha^3}{16}. \tag{5.5}$$

Proof. In terms of the Laplace transform $L(z) = \mathbb{E}\, e^{zX}$, we have $K' = L'/L$ and

$$K''' = \frac{L'''}{L} - 3\,\frac{L''L'}{L^2} + 2\,\frac{L'^3}{L^3}.$$

For $x \ge 0$ and $p = 1, 2, 3$, we use the elementary inequality $x^p e^{-x} \le (p/e)^p$. Suppose that $|z| \le (1-c)\,\alpha$ with $0 < c < 1$. Since $L^{(p)}(z) = \mathbb{E}\, X^p\, e^{zX}$, we then have

$$|L^{(p)}(z)| \le \mathbb{E}\,|X|^p\, e^{(1-c)\,\alpha|X|}.$$

Hence, by (5.1),

$$|L^{(p)}(z)| \le\ \mathbb{E}\,|X|^p\, e^{-c\alpha|X|}\, e^{\alpha|X|}\ \le\ 2\left(\frac{p}{c\alpha e}\right)^p.$$

In particular,

$$|L'(z)| \le \frac{2}{c\alpha e}, \quad |L''(z)| \le \frac{8}{(c\alpha e)^2}, \quad |L'''(z)| \le \frac{54}{(c\alpha e)^3},$$

so

$$|L(z) - 1| \le \frac{2}{c\alpha e}\,|z| \le \frac{2(1-c)}{ce}, \qquad |L(z)| \ge 1 - \frac{2(1-c)}{ce}.$$

Putting $q^{-1} = 1 - \frac{2(1-c)}{ce}$, it follows that

$$|K'''(z)| \le (c\alpha e)^{-3}\,(54\,q + 48\,q^2 + 16\,q^3).$$

Choosing $c = 15/16$, we have $q = (1 - \frac{2}{15\,e})^{-1} < 1.06$, and the last expression becomes smaller than $8\alpha^{-3}$. Hence

$$|K'''(z)| \le \frac{8}{\alpha^3}, \quad |K''(z) - 1| \le \frac{8}{\alpha^3}\,|z|,$$

for $|z| \le \frac{\alpha}{16}$, where we used $K''(0) = 1$. The last inequality readily implies (5.5). $\square$

We shall now show that $|f(z)|$ is bounded away from 1 in a certain region near zero.

Lemma 5.4. *Let $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$. For all complex numbers $z = t + iy$ with $|t| \leq \frac{\alpha^3}{18}$, $|y| \leq \frac{1}{2}\,|t|$, we have*

$$|f(z)| \leq e^{-t^2/5}. \tag{5.6}$$

Proof. One may start with an integral Taylor formula

$$f(z) = 1 - \frac{1}{2}\,z^2 + \frac{1}{2}\,z^3 \int_0^1 f'''(sz)(1-s)^2\,ds,$$

where we used $f'(0) = 0$, $f''(0) = -1$. Here, by Lemma 5.3, cf. (5.4),

$$|f'''(sz)| \leq \frac{8}{\alpha^3} \quad \text{for } |z| \leq \frac{\alpha}{16}.$$

Hence, by the triangle inequality, in this disc

$$|f(z)| \leq \Big|1 - \frac{1}{2}\,z^2\Big| + \frac{4}{3\alpha^3}\,|z|^3. \tag{5.7}$$

Suppose that $|y| \leq \frac{1}{2}\,|t|$ and $|t| \leq \frac{\alpha^3}{18}$. Then $|z| \leq \frac{\alpha^3}{18}\sqrt{5/4} < \frac{\alpha}{16}$, so that the above bound is applicable. Next, we use

$$t^2 - y^2 \geq \frac{3}{4}\,t^2, \quad |ty| \leq \frac{1}{2}\,t^2, \quad |z|^3 \leq \Big(\frac{5}{4}\Big)^{3/2}\,|t|^3.$$

This gives

$$\begin{aligned} \Big|1 - \frac{1}{2}\,z^2\Big|^2 &= 1 - (t^2 - y^2) + \frac{1}{4}\,(t^2 - y^2)^2 + (ty)^2 \\ &\leq 1 - \frac{3}{4}\,t^2 + \frac{1}{2}\,t^4 \,\leq\, \Big(1 - \frac{1}{3}\,t^2\Big)^2, \end{aligned}$$

where we used $|t| \leq \frac{1}{18}$. Hence, from (5.7),

$$\begin{aligned} |f(z)| &\leq 1 - \frac{1}{3}\,t^2 + \frac{4}{3\alpha^3}\Big(\frac{5}{4}\Big)^{3/2}|t|^3 \\ &\leq 1 - \frac{1}{3}\,t^2 + \frac{2}{27}\Big(\frac{5}{4}\Big)^{3/2}t^2 \,\leq\, 1 - \frac{1}{5}\,t^2. \end{aligned}$$

$\square$

6. **Saddle Point**

Assume that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\mathbb{E}\,e^{\alpha|X|} \leq 2$ ($\alpha > 0$). Since the log-Laplace transform $K(z) = \log \mathbb{E}\,e^{zX}$ was defined as an analytic function in the disc $|z| \leq \frac{\alpha}{2}$ of the complex plane, it may be expanded as an absolutely convergent power series

$$K(z) = \frac{1}{2}\,z^2 + \sum_{k=3}^\infty \frac{\gamma_k}{k!}\,z^k.$$

Here, the coefficents $\gamma_k = K^{k)}(0)$ are called cumulants of X. Every γ_k represents a certain polynomial in moments of X up to order k. In particular, $\gamma_3 = \mathbb{E}\,X^3$ and $\gamma_4 = \mathbb{E}\,X^4 - 3$.

Similarly,

$$K'(z) = z + \sum_{k=2}^{\infty} \frac{\gamma_{k+1}}{k!}\, z^k.$$

The next object is important for contour integration.

Definition 6.1. Given $\tau \in \mathbb{C}$, a saddle point is a solution $z_0 = z_0(\tau)$ of the equation

$$K'(z) = \tau. \tag{6.1}$$

Thus, a saddle point is the solution of

$$\tau = z + \sum_{k=2}^{\infty} \frac{\gamma_{k+1}}{k!}\, z^k. \tag{6.2}$$

Proposition 6.2. *In the disc $|\tau| \le \frac{\alpha^3}{32}$, the equation* (6.1) *has a unique solution $z_0(\tau)$. Moreover, it represents an injective analytic function satisfying $z_0'(0) = 1$ and*

$$|z_0(\tau)| \le 2\tau \le \frac{\alpha^3}{16}, \quad |\tau| \le \frac{\alpha^3}{32}. \tag{6.3}$$

Proof. Let us use (6.2) as the definition of the analytic function $\tau = K'(z)$. If τ is sufficiently small, say $|\tau| \le \tau_0$, this equality may be inverted as a power series in τ,

$$z = z_0(\tau) = \tau - \frac{\gamma_3}{2}\, \tau^2 + \frac{3\gamma_3^2 - \gamma_4}{6}\, \tau^3 + \dots$$

Let us indicate an explicit expression for τ_0 in the form of a positive function of α.

By Lemma 5.3, cf. (5.4),

$$|\tau'(z) - 1| \le \frac{8}{\alpha^3}\, |z|, \quad |z| \le \frac{\alpha}{16}. \tag{6.4}$$

One may use this relation for $|z| \le \frac{\alpha^3}{16}$, since $\alpha < 1$, by Lemma 4.1. Given two points z_1 and z_2 in the disc $|z| \le \frac{\alpha^3}{16}$, define the path $z_t = (1-t)z_1 + tz_2$ connecting these points. We have

$$\tau(z_2) - \tau(z_1) = (z_2 - z_1)\Big(1 + \int_0^1 (\tau'(z_t) - 1)\, dt\Big), \tag{6.5}$$

implying

$$|\tau(z_2) - \tau(z_1)| \ge |z_2 - z_1|\Big(1 - \int_0^1 |\tau'(z_t) - 1|\, dt\Big).$$

Since $|z_t| \le \frac{\alpha^3}{16}$, it follows from (6.4) that

$$|\tau(z_2) - \tau(z_1)| \ge \frac{1}{2}\, |z_2 - z_1|.$$

As a consequence, the map $z \to \tau(z)$ is injective in the disc $|z| \le \frac{\alpha^3}{16}$. In addition, since $\tau(0) = 0$, we have

$$|\tau(z)| \ge \frac{1}{2}\, |z|. \tag{6.6}$$

Therefore, the image of the circle $|z| = \frac{\alpha^3}{16}$ under this map represents a closed curve on the complex plane outside the circle $|\tau| = \frac{\alpha^3}{32}$. Since the image of the disc $|z| \leq \frac{\alpha^3}{16}$ under τ is a connected set, while $\tau(0) = 0$, this set must contain the disc $|\tau| \leq \frac{\alpha^3}{32}$. Thus, the inverse map $z_0(\tau) = \tau^{-1}$ is well-defined and represents a holomorphic injective function in $|\tau| \leq \frac{\alpha^3}{32}$ satisfying (6.3), by (6.6), and $z_0'(0) = 1$, by (6.4). Hence, one may take $\tau_0 = \frac{\alpha^3}{32}$.

In addition, $z_0(\tau)$ takes real values for real τ. Indeed, since all cumulants are real numbers, $\tau(z)$ is real for real z, so is the inverse function z_0. Also, by (6.5),

$$\tau = z_0(\tau)\Big(1 + \int_0^1 (\tau'(tz_0(\tau)) - 1)\,dt\Big),$$

which shows that $z_0(\tau) > 0$ as long as $0 < \tau \leq \frac{\alpha^3}{32}$ (since the expression under the integral sign is a real-valued function whose absolute value does not exceed 1/2). $\square$

7. Taylor Expansions around Saddle Point

It is natural to determine the leading term in the Taylor expansion for $z_0(\tau)$ when expanding this function as a power series in τ. Assuming that a non-normal random variable X has mean zero, variance one, and a finite exponential moment, let γ_m ($m \geq 3$) be the first non-zero cumulant of X. Then, as $|z| \to 0$,

$$K(z) = \frac{1}{2}\,z^2 + \frac{\gamma_m}{m!}\,z^m + O(|z|^{m+1}),$$

so that

$$K'(z) = z + \frac{\gamma_m}{(m-1)!}\,z^{m-1} + O(|z|^m) \tag{7.1}$$

and

$$K''(z) = 1 + \frac{\gamma_m}{(m-2)!}\,z^{m-2} + O(|z|^{m-1}). \tag{7.2}$$

Since, by Proposition 6.2, $z_0(\tau) = \tau + O(|\tau|^2)$ as $\tau \to 0$, we get from (7.1)

$$\begin{aligned} \tau \ &= \ K'(z_0(\tau)) \ = \ z_0(\tau) + \frac{\gamma_m}{(m-1)!}\,z_0(\tau)^{m-1} + O(|z_0(\tau)|^m) \\ &= \ z_0(\tau) + \frac{\gamma_m}{(m-1)!}\,\tau^{m-1} + O(|\tau|^m). \end{aligned}$$

Therefore,

$$z_0(\tau) = \tau - \frac{\gamma_m}{(m-1)!}\,\tau^{m-1} + O(|\tau|^m). \tag{7.3}$$

Also, write down the Taylor expansion around the point $z_0 = z_0(\tau)$:

$$K(z) - \tau z = K(z_0) - \tau z_0 + \sum_{k=2}^{\infty} \frac{\rho_k}{k!}\,(z - z_0)^k, \quad \rho_k = K^{(k)}(z_0). \tag{7.4}$$

Here, we used the property that the function $K(z) - \tau z$ has derivative $K'(z_0) - \tau = 0$ at the saddle point $z = z_0$. Thus, the linear term in (7.4) corresponding to $k = 1$ is vanishing. As for the free term corresponding to $k = 0$, let us recall that that

$$K(z_0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{k!}\,z_0^k$$

and

$$\tau z_0 = z_0 K'(z_0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!}\, z_0^k.$$

Hence

$$K(z_0) - \tau z_0 = -\sum_{k=2}^{\infty} \frac{k-1}{k!}\, \gamma_k z_0^k = -\frac{1}{2}\, z_0^2 - \frac{1}{3}\, \gamma_3 z_0^3 + \dots \tag{7.5}$$

Using (7.3) and (7.5), we actually have

$$\begin{aligned} K(z_0) - \tau z_0 &= -\frac{1}{2}\, z_0^2 - \frac{m-1}{m!}\, \gamma_m z_0^m + \dots \\ &= -\frac{1}{2}\Big(\tau - \frac{\gamma_m}{(m-1)!}\, \tau^{m-1} + O(|\tau|^m)\Big)^2 \\ &\quad - \frac{m-1}{m!}\, \gamma_m \Big(\tau - \frac{\gamma_m}{(m-1)!}\, \tau^{m-1} + O(|\tau|^m)\Big)^m + \dots \end{aligned}$$

which simplifies to

$$\begin{aligned} K(z_0) - \tau z_0 &= -\frac{1}{2}\, \tau^2 + \frac{1}{m!}\, \gamma_m \tau^m + O(|\tau|^{m+1}) \\ &= -\frac{1}{2}\, \tau^2 + \tau^3 \lambda(\tau). \end{aligned} \tag{7.6}$$

Thus, applying Proposition 6.2 and recalling that $K(z)$ is analytic in $|z| \le \frac{\alpha}{2}$ (Lemma 5.1), we obtain:

Proposition 7.1. *The function*

$$\lambda(\tau) = \frac{1}{\tau^3}\Big(K(z_0(\tau)) - \tau z_0(\tau) + \frac{1}{2}\, \tau^2\Big)$$

is well-defined and analytic in the disc $|\tau| \le \frac{\alpha^3}{32}$. Moreover, as $\tau \to 0$,

$$\lambda(\tau) = \frac{1}{m!}\, \gamma_m \tau^{m-3} + O(|\tau|^{m-2}). \tag{7.7}$$

Definition 7.2. Being an analytic function, $\lambda(\tau)$ is represented as a power series in the disc $|\tau| \le \frac{\alpha^3}{32}$. It is called Cramér's series.

Let us also introduce another analytic function which appears in the representation (1.6) of Theorem 1.2.

Proposition 7.3. *The function*

$$\mu(\tau) = \frac{1}{2}\, \log K''(z_0(\tau))$$

is well-defined and analytic in the disc $|\tau| \le \frac{\alpha^3}{32}$. Moreover, as $\tau \to 0$,

$$\mu(\tau) = \frac{1}{2(m-2)!}\, \gamma_m \tau^{m-2} + O(|\tau|^{m-1}). \tag{7.8}$$

Proof. By (6.3), $|z_0(\tau)| \le \frac{\alpha^3}{16}$. Hence, by Lemma 5.3, $K''(z_0(\tau))$ takes values in the disc with center at 1 of radius 1/2. Thus, the principal value of $\log K''(z_0(\tau))$ is well-defined and represents an analytic function in $|\tau| \le \frac{\alpha^3}{32}$. Moreover, by (7.2)-(7.3),

$$\begin{aligned} K''(z_0(\tau)) &= 1 + \frac{\gamma_m}{(m-2)!}\, z_0(\tau)^{m-2} + O(|z_0(\tau)|^{m-1}) \\ &= 1 + \frac{\gamma_m}{(m-2)!}\, \tau^{m-2} + O(|\tau|^{m-1}). \end{aligned}$$

Taking the logarithm of this expression, we arrive at (7.8). □

Let us also mention that the function $K(z)$ is convex and has a positive second derivative on the real line, more precisely – on the interval where it is finite. Hence $\mu(\tau)$ is real-valued for real τ.

8. **Contour Integration**

Let $(X_n)_{n \ge 1}$ be independent copies of a random variable X with $\mathbb{E}X = 0$, $\mathrm{Var}(X) = 1$, and characteristic function $f(t) = \mathbb{E}\, e^{itX}$. Supposed that the normalized sum

$$Z_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$$

has a bounded density for $n = n_0$. As already discussed in Section 2, in this case all Z_n with $n \ge 2n_0$ have continuous bounded densities expressed by the inversion formula

$$p_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f_n(t)\, dt, \quad x \in \mathbb{R},$$

where

$$f_n(t) = f\Big(\frac{t}{\sqrt{n}}\Big)^n$$

denotes the characteristic functions of Z_n. Equivalently,

$$p_n(x) = \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{\infty} e^{-itx\sqrt{n}}\, f(t)^n\, dt. \tag{8.1}$$

Using contour integration, one can cast this formula in a different form envolving the log-Laplace transform $K(z) = \log \mathbb{E}\, e^{zX}$ and the saddle point $z_0 = z_0(\tau)$ for the real value $\tau = x/\sqrt{n}$. This is a preliminary step towards Theorem 1.2.

As before, let $\mathbb{E}\, e^{\alpha |X|} \le 2$ with a parameter $\alpha > 0$.

Lemma 8.1 *Let $n \ge 4n_0$. If $0 < \varepsilon \le \frac{\alpha^3}{18}$ and $|\tau| \le \frac{\varepsilon}{2}$, then*

$$p_n(x) = \frac{\sqrt{n}}{2\pi} \int_{-\varepsilon}^{\varepsilon} \exp\Big\{n(K(z_0 + it) - \tau(z_0 + it))\Big\}\, dt + \theta R_n \tag{8.2}$$

with $|\theta| \le 1$ and

$$R_n = 5M \exp\Big\{-\frac{n\varepsilon^2}{Cn_0 M^2}\Big\}, \quad C = 5200. \tag{8.3}$$

Proof. Applying Corollary 3.3, we get from (8.1) that, for any $\varepsilon \in (0,1]$,

$$\Big| p_n(x) - \frac{\sqrt{n}}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{-itx\sqrt{n}}\, f(t)^n\, dt \Big| \leq 4M \exp\Big\{ - \frac{n\varepsilon^2}{Cn_0 M^2} \Big\} \tag{8.4}$$

with an absolute constant, say $C = 5200$.

Due to the assumption on ε, we may apply Lemma 5.4 which gives

$$|f(\pm\varepsilon + iy)| \leq e^{-\varepsilon^2/5} \quad \text{whenever } |y| \leq \frac{\varepsilon}{2}. \tag{8.5}$$

Assuming for definiteness that $x \geq 0$, we take the rectangle contour

$$L = L_1 + L_2 + L_3 + L_4$$

with segment parts

$L_1 = [-\varepsilon, \varepsilon]$, $L_2 = [\varepsilon, \varepsilon - ih]$,
$L_3 = [\varepsilon - ih, -\varepsilon - ih]$, $L_4 = [-\varepsilon - ih, -\varepsilon]$,

where $h > 0$ is chosen to satisfy $h \leq \frac{\varepsilon}{2}$. With this choice the complex numbers $z = t + iy$ with $|t| \leq \varepsilon$, $|y| \leq h$ lie in the domain of the definition of $K(z)$. Then, by Cauchy's theorem,

$$\int_{L_1} e^{-izx\sqrt{n}}\, f(z)^n\, dz + \int_{L_2} e^{-izx\sqrt{n}}\, f(z)^n\, dz + \int_{L_3} e^{-izx\sqrt{n}}\, f(z)^n\, dz + \int_{L_4} e^{-izx\sqrt{n}}\, f(z)^n\, dz \, = \, 0.$$

Note that in the lower half-plane $z = t - iy$, $0 \leq y \leq h$, we have $|e^{-izx\sqrt{n}}| = e^{-yx\sqrt{n}} \leq 1$. Moreover, $|f(z)|$ is bounded away from 1 on L_2 and L_4 according to (8.5) which gives

$$\Big| \int_{L_2} \Big| + \Big| \int_{L_4} \Big| \leq \varepsilon\, e^{-n\varepsilon^2/5} \leq \frac{1}{18}\, e^{-n\varepsilon^2/5}.$$

To simplify, note that

$$4M \exp\Big\{ - \frac{n\varepsilon^2}{Cn_0 M^2} \Big\} + \frac{1}{18}\, e^{-n\varepsilon^2/5} \leq R_n,$$

where we used $M \geq \frac{1}{12}$. Combining this bound with (8.4), we arrive at

$$p_n(x) = \frac{\sqrt{n}}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{-i(t-ih)\, x\sqrt{n}}\, f(t - ih)^n\, dt + \theta R_n.$$

Using the log-Laplace transform, let us rewrite the above as a contour integral

$$p_n(x) = \frac{\sqrt{n}}{2\pi i} \int_{h-i\varepsilon}^{h+i\varepsilon} \exp\{n(K(z) - \tau z)\}\, dz + \theta R_n$$

with $\tau = x/\sqrt{n}$ and apply it with $h = z_0 = z_0(\tau)$. Due to the requirement $0 \leq \tau \leq \frac{\varepsilon}{2}$, we have $0 \leq \tau \leq \frac{\alpha^3}{32}$ and $0 \leq z_0 \leq \frac{\alpha^3}{16}$, according to (6.3), so that Proposition 6.2 is applicable. After the change of variable, we thus obtain (8.2)-(8.3).

9. **Estimation of the Integral outside a Neighborhood of the Saddle Point**

As a next step, let us show that, at the expense of a small error, the integration in (8.2) may be restricted to the interval $|t| \leq t_n$ with

$$t_n = n^{-1/2}\,\sqrt{8\log n}.$$

Using (7.4) and (7.6) in the representation (8.2), one may rewrite (8.2) as

$$p_n(x) = \frac{\sqrt{n}}{2\pi}\int_{-\varepsilon}^{\varepsilon}\exp\Big\{n\Big(\sum_{k=2}^\infty \rho_k\frac{(it)^k}{k!}\Big)\Big\}\,dt\ e^{n\,(-\frac{1}{2}\,\tau^2+\tau^3\lambda(\tau))} + \theta R_n,$$

where $\tau = x\sqrt{n}$ and $\rho_k = K^{(k)}(z_0)$, assuming for definiteness that $x > 0$. Equivalently

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\,e^{n\tau^3\lambda(\tau)}\int_{-\varepsilon}^{\varepsilon}\exp\Big\{n\Big(\sum_{k=2}^\infty \rho_k\frac{(it)^k}{k!}\Big)\Big\}\,dt + \theta R_n e^{x^2/2}. \tag{9.1}$$

In order to force the new remainder term

$$R_n e^{x^2/2} = 5M\exp\Big\{-\frac{n\varepsilon^2}{Cn_0M^2} + n\tau^2\Big\}, \quad C = 5200,$$

to be exponentially small with respect to n, let us strengthen the assumption $|\tau| \leq \frac{\varepsilon}{2}$ in Lemma 8.1 to $0 \leq \tau \leq \frac{\varepsilon}{80\,n_0M}$ (recall that $M \geq 1/12$). In this case, the expression in the exponent will be still of order at most $-\frac{cn\varepsilon^2}{n_0M^2}$ up to an absolute constant $c > 0$. Thus, (9.1) yields

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\,e^{n\tau^3\lambda(\tau)}\int_{-\varepsilon}^{\varepsilon}\exp\Big\{n\Big(\sum_{k=2}^\infty \rho_k\frac{(it)^k}{k!}\Big)\Big\}\,dt + \theta R_n, \tag{9.2}$$

under the conditions

$$0 \leq \tau \leq \frac{\varepsilon}{80\,n_0M^2}, \quad 0 \leq \varepsilon \leq \frac{\alpha^3}{18}, \tag{9.3}$$

where

$$R_n = 5M\exp\Big\{-\frac{cn\varepsilon^2}{n_0M^2}\Big\}. \tag{9.4}$$

Now, by Lemmas 5.2-5.3,

$$\frac{1}{2} \leq \rho_2 \leq \frac{3}{2}, \qquad |\rho_k| \leq 3k!\,\Big(\frac{4}{\alpha}\Big)^k \quad (k \geq 3). \tag{9.5}$$

It follows that

$$\begin{aligned}\mathrm{Re}\Big(\sum_{k=2}^\infty \rho_k\,\frac{(it)^k}{k!}\Big) &= -\rho_2\,\frac{t^2}{2} + \sum_{k=2}^\infty (-1)^k\rho_{2k}\,\frac{t^{2k}}{(2k)!}\\ &\leq -\frac{1}{4}\,t^2 + 3\sum_{k=2}^\infty \Big(\frac{4t}{\alpha}\Big)^{2k} \leq -\frac{1}{8}\,t^2,\end{aligned}$$

where we used $|t| \leq \varepsilon \leq \frac{\alpha^3}{18}$ and $\alpha < 1$. Hence, when restricted to $|t| \geq t_n$, the absolute value of the integral in (9.3) does not exceed

$$2\int_{t_n}^\infty e^{-nt^2/8}\,dt = \frac{1}{\sqrt{n}}\int_{\frac{t_n}{2}\sqrt{n}}^\infty e^{-s^2/2}\,ds < \frac{\sqrt{2\pi}}{2\sqrt{n}}\,e^{-nt_n^2/8} = \frac{\sqrt{2\pi}}{2n^{3/2}}.$$

As a result, assuming the conditions (9.3),

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\, e^{n\tau^3\lambda(\tau)} \int_{|t|\le t_1} \exp\Big\{n\,\Big(\sum_{k=2}^\infty \rho_k \frac{(it)^k}{k!}\Big)\Big\}\,dt$$
$$+\ \theta_1 n^{-1} e^{n\tau^3\lambda(\tau)} + \theta_2 R_n, \tag{9.6}$$

where $t_n' = \min(t_n,\varepsilon)$, $|\theta_j| \le 1$, R_n is now defined in (9.4).

10. **Proof of Theorem 1.2**

As a final step, we need to explore an asymptotic behavior of the integral in (9.6), where we recall that $\rho_k = K^{(k)}(z_0)$, $z_0 = z_0(\tau)$ being the saddle point for $\tau = x/\sqrt{n}$. In view of the conditions in (9.3), we choose

$$\varepsilon = \frac{\alpha^3}{18}, \quad \tau_0 = \frac{\varepsilon}{80\,n_0 M^2} = \frac{c\,\alpha^3}{n_0 M^2},$$

where $c > 0$ is an absolute constant. Thus, suppose that $x \ge 0$ and $\tau \le \tau_0$.

The integrand in (9.5) may be written as

$$u_n(t) = \exp\Big\{-n\rho_2\,\frac{t^2}{2} + n\rho_3\,\frac{(it)^3}{6} + nv(t)\Big\}$$

with

$$v(t) = \sum_{k=4}^\infty \rho_k\,\frac{(it)^k}{k!}.$$

Let us assume that $n \ge n_1 = \max(4n_0, \varepsilon^{-4})$ which insures that $t_n' = t_n$. As $|t| \le t_n$, from (9.5) it follows that

$$nv(t) = O(nt^4) = \frac{B}{n}\,(\log n)^2,$$

where B denotes a bounded quantity with involved constants depending on α only. We also have $nt^3 = O(n^{-1/2}(\log n)^{3/2})$. Using $e^x = 1 + x + O(x^2)$ as $x \to 0$ with

$$x = n\rho_3\,\frac{(it)^3}{6} + nv(t),$$

we have

$$\begin{aligned} u_n(t) &= e^{-n\rho_2 t^2/2 + x} \\ &= e^{-n\rho_2 t^2/2}\,\big(1 + x + Bx^2\big) \\ &= e^{-n\rho_2 t^2/2}\,\Big(1 + n\rho_3\,\frac{(it)^3}{6} + Bn^{-1}(\log n)^3\Big). \end{aligned}$$

Hence

$$\int_{|t|\le t_n} u_n(t)\,dt = \big(1 + Bn^{-1}(\log n)^3\big)\int_{|t|\le t_n} e^{-n\rho_2 t^2/2}\,dt,$$

and (9.6) simplifies to

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\, e^{n\tau^3\lambda(\tau)}\big(1 + Bn^{-1}(\log n)^3\big)\int_{|t|\le t_0} e^{-n\rho_2 t^2/2}\,dt$$
$$+\ \theta_1 n^{-1} e^{n\tau^3\lambda(\tau)} + \theta_2 R_n, \tag{10.1}$$

Next, one may extend the integration in (10.1) to the whole real line at the expense of an error not exceeding

$$\begin{aligned}\int_{|t|\geq t_n} e^{-n\rho_2 t^2/2}\,dt &= \frac{2}{\sqrt{\rho_2 n}}\int_{t_n\sqrt{\rho_2 n}}^{\infty} e^{-s^2/2}\,ds \\ &< \frac{\sqrt{2\pi}}{\sqrt{\rho_2 n}}\,e^{-n\rho_2 t_n^2/2} < \frac{\sqrt{\pi}}{\sqrt{n}}\,e^{-nt_n^2/4} = \frac{\sqrt{\pi}}{\sqrt{n}}\,n^{-2},\end{aligned}$$

where we used $\frac{1}{2}\leq\rho_2\leq\frac{3}{2}$. Since the integral over the whole real line is equal to $\frac{\sqrt{2\pi}}{\sqrt{\rho_2 n}}$, we obtain from (10.1) that

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}}\,e^{n\tau^3\lambda(\tau)}\big(1+Bn^{-1}(\log n)^3\big)+\theta_1 n^{-1}e^{n\tau^3\lambda(\tau)}+\theta_2 R_n.$$

Here, the first remainder term may be absorbed in the brackets, so that this formula is further simplified to

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}}\,e^{n\tau^3\lambda(\tau)}\Big(1+Bn^{-1}(\log n)^3+2\theta_2 R_n\,e^{-n\tau^3\lambda(\tau)}\Big).$$

Returning to the definition (7.6) and recalling (5.2) and (6.3), let us note that

$$\tau^3\lambda(\tau) = K(z_0)-\tau z_0+\frac{1}{2}\tau^2$$

is bounded in absolute value by $3+\frac{5}{2}\tau^2<4$. Hence, choosing a smaller constant c in (9.4), the last term $R_n\,e^{-n\tau^3\lambda(\tau)}$ will be dominated by the second last term. This leads to

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}}\,e^{n\tau^3\lambda(\tau)}\big(1+Bn^{-1}(\log n)^3\big). \tag{10.2}$$

It remains to recall Proposition 7.3 according to which $\rho_2^{-1/2}=K''(z_0(\tau))^{-1/2}=e^{-\mu(\tau)}$.

Finally, let us note that the case $2n_0\leq n<n_1$ is not interesting, since then $|x|\leq\tau_0 n_1$, and (1.6) holds true by choosing a suitable constant in O in (1.6). Theorem 1.2 is proved.

11. **Proof of Corollary 1.3**

Starting from (7.7) and (7.8), we have

$$\begin{aligned}n\tau^3\lambda(\tau)-\mu(\tau) &= \frac{n}{m!}\,\gamma_m\tau^m+O(|\tau|^{m+1})-\frac{1}{2(m-2)!}\,\gamma_m\tau^{m-2}+O(|\tau|^{m-1}) \\ &= \frac{\gamma_m}{m!}\,\tau^{m-2}\,\Lambda(\tau).\end{aligned}$$

Here

$$\begin{aligned}\Lambda(\tau) &= n\tau^2-\frac{m(m-1)}{2}+nO(\tau^3)+O(\tau) \\ &\geq \frac{1}{2}\Big[n\tau^2-\frac{m(m-1)}{2}\Big],\end{aligned}$$

which is bounded away from zero, if $|\tau|\leq\tau_1$ for some constant $\tau_1>0$ and $n\tau^2=x^2\geq m^2$. In this case, (1.6) immediately yields the desired relation (1.7).

In the remaining bounded interval $|x| \le m$, this argument does not work, and it is better to employ the Chebyshev-Edgeworth expansion for the correction $\varphi_m(x)$ in (1.1) (which depends on n as well). In terms of the first non-zero cumulant, (1.1) may be written more accurately as

$$p_n(x) = \varphi(x) + \frac{\gamma_m}{m!}\, H_m(x)\varphi(x)\, n^{-\frac{m-2}{2}} + \frac{1}{1+|x|^m}\, o\big(n^{-\frac{m-2}{2}}\big),$$

where $H_m(x)$ denotes the Chebyshev-Hermite polynomial of degree m. As a consequence, for any constant $x_0 > 0$,

$$\sup_{|x| \le x_0} \frac{|p_n(x) - \varphi(x)|}{\varphi(x)} = O\big(n^{-\frac{m-2}{2}}\big),$$

which is stronger than (1.7), since m is even (hence $m \ge 4$).

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