SECOND ORDER CONCENTRATION ON THE SPHERE

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ABSTRACT. Sharpened forms of the concentration of measure phenomenon for classes of functions on the sphere are developed in terms of Hessians of these functions.

1. Introduction

Let σ_{n-1} denote the normalized Lebesgue measure on the unit sphere

$$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}, \qquad n \geq 2,$$

in the Euclidean n-space which is equipped with the canonical inner product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|$. The spherical concentration phenomenon asserts in particular that mean zero smooth functions f on S^{n-1} are of order at most $\frac{1}{\sqrt{n}}$ on a large part of the sphere in the sense of σ_{n-1}. This follows already from the Poincaré inequality

$$\int f^2 \, d\sigma_{n-1} \leq \frac{1}{n-1} \int |\nabla_S f|^2 \, d\sigma_{n-1}, \tag{1.1}$$

where $\nabla_S f$ stands for the spherical gradient of f (cf. e.g. [L1]). Hence, if the integral on the right-hand side is of order 1, the L^2-norm of f will be of order at most $\frac{1}{\sqrt{n}}$. Moreover, in case $|\nabla_S f| \leq 1$, there is a considerably stronger property

$$\int e^{(n-1)\, f^2/c} \, d\sigma_{n-1} \leq 2$$

involving some absolute constant $c > 0$. Using a standard normal random variable Z, it may be stated informally as stochastic dominance

$$|f| \preceq c \, \frac{|Z|}{\sqrt{n}}, \tag{1.2}$$

which means a corresponding inequality for the measures/probabilities of the tail sets $|f| \geq r$ and $\frac{c}{\sqrt{n}} \, |Z| \geq r$ for all $r > 0$. This property was first emphasized in the early 70's by V. D. Milman in the context of the local theory of Banach spaces and led him to the

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understanding of the concentration of measure phenomenon in a much broader sense; cf. V. D. Milman, G. Schechtman [M-S], subsequent works by M. Talagrand [T1-2], and the book by M. Ledoux [L2] for an account of basic ideas and results in this direction up to the end of 90's.

Returning to the sphere, in certain problems one deals however with smooth functions that turn out to be of a much smaller order than $\frac{1}{\sqrt{n}}$. This cannot be guaranteed just by the Lipschitz condition $|\nabla_S f| \leq 1$, even if f is orthogonal to linear functions in $L^2(S^{n-1}, \sigma_{n-1})$ (which play an extremal role in (1.1)). Hence, conditions on higher derivatives of f are required. The aim of this note is to study corresponding conditions in terms of the Hessian of f''_S of f by involving both the operator norms $\|f''_S(\theta)\|$ and the Hilbert-Schmidt norms $\|f''_S(\theta)\|_{\rm HS}$ of the matrices $f''_S(\theta)$ $(\theta \in S^{n-1})$.

Orthogonality of functions on the unit sphere will be understood as orthogonality in the Hilbert space $L^2(S^{n-1}, \sigma_{n-1})$. Restrictions of affine, linear and quadratic functions on $\mathbf{R}^n$ to the sphere S^{n-1} will be again called affine, linear and quadratic functions respectively on the sphere.

Theorem 1.1. *Assume that f is a C^2-smooth function on S^{n-1} which is orthogonal to all affine functions. If $\|f''_S\| \leq 1$ at all points on the sphere and $\int \|f''_S\|_{\rm HS}^2\, d\sigma_{n-1} \leq b^2$, then*

$$\int \exp\Big\{\frac{n-1}{2(1+b^2)}\,|f|\Big\}\, d\sigma_{n-1} \leq 2. \tag{1.3}$$

By Chebyshev's inequality, (1.3) provides bounds on tails, which may be written similarly to (1.2) as

$$|f| \,\preceq\, c_b \Big(\frac{Z}{\sqrt{n}}\Big)^2,$$

however – with the right-hand side behaving like $\frac{1}{n}$ with respect to the dimension (provided that b is of order 1).

We refer to Theorem 1.1 as (a variant of) the second order concentration on the sphere. It is consistent with a second order Poincaré-type inequality

$$\int f^2\, d\sigma_{n-1} \leq \frac{1}{2n(n+2)} \int \|f''_S\|_{\rm HS}^2\, d\sigma_{n-1},$$

valid for all smooth f on S^{n-1} that are orthogonal to affine functions (with equality attainable for all quadratic spherical harmonics). This inequality can be derived using the spectral decomposition of f in spherical harmonics by means of the identity

$$\int \|f''_S\|_{\rm HS}^2\, d\sigma_{n-1} = \int f\,\big(\Delta_S(\Delta_S f) + (n-2)\Delta_S f\big)\, d\sigma_{n-1}. \tag{1.4}$$

Here and in the sequel $\Delta_S = {\rm Tr}\, f''_S$ denotes the Laplacian operator on S^{n-1} which acts diagonally on all homogeneous spherical harmonics. Although typically $\Delta_S f$ behaves in a more "chaotic" (oscillatory) way than f, the average in (1.4) captures and cancels such potentially large oscillations.

The conditions on the spherical second derivative in Theorem 1.1 are fulfilled, for example, when $\|f''_S\|_{\rm HS} \leq b$ on S^{n-1}. However, in applications, one might prefer to deal with functions on the sphere induced by smooth functions in $\mathbf{R}^n$ or at least in a neighbourhood of the sphere via restriction and using the Euclidean derivatives of such functions, rather than intrinsic derivatives on S^{n-1}. Using this Euclidean setup, we may formulate a related statement as follows.

In the sequel we denote by $f''(x) = \big(\partial_{ij} f(x)\big)_{i,j=1}^n$ the matrix of partial derivatives of f of second order at the point x, and by I_n the identity $n \times n$ matrix.

Theorem 1.2. *Let f be defined and C^2-smooth in some open neighbourhood of S^{n-1}. Assume that it is orthogonal to all affine functions and satisfies $\|f'' - aI_n\| \leq 1$ on S^{n-1} together with*

$$\int \|f'' - aI_n\|_{\rm HS}^2 \, d\sigma_{n-1} \leq b^2, \tag{1.5}$$

for some $a \in \mathbf{R}$ and $b \geq 0$. Then

$$\int \exp\Big\{\frac{n-1}{2(1+4b^2)}\,|f|\Big\}\, d\sigma_{n-1} \leq 2. \tag{1.6}$$

In Theorems 1.1-1.2 one may also start with an arbitrary C^2-smooth function f, but apply the hypotheses and the conclusions (1.3)/(1.6) to the projection Tf of f to the orthogonal complement of the space of all affine functions on the sphere in $L^2(S^{n-1},\sigma_{n-1})$. The "affine" part of f may be described as $l(\theta) = m + \langle v,\theta\rangle$ with

$$m = \int f(x)\, d\sigma_{n-1}(x), \quad v = n\int x f(x)\, d\sigma_{n-1}(x),$$

so $Tf(\theta) = f(\theta) - l(\theta)$. For example, if f is even, i.e. $f(-\theta) = f(\theta)$ for all $\theta \in S^{n-1}$, then $Tf = f - m$.

In the setting of Theorem 1.2, the functions Tf and f have identical Euclidean second derivatives. Hence, if we want to obtain an inequality similar to (1.6) without the orthogonality assumption (still assuming conditions on the Euclidean second derivative), we need to verify that the affine part l is of order $\frac{1}{n}$. This may be achieved by estimating the L^2-norm of l and using the well-known fact that the linear functions on the sphere behave like Gaussian random variables. If, for definiteness, f has mean zero, then

$$\|l\|_{L^2}^2 = \frac{1}{n}\,|v|^2 = nI, \quad \text{where} \ \ I = \iint \langle x,y\rangle\, f(x)f(y)\, d\sigma_{n-1}(x)d\sigma_{n-1}(y).$$

Therefore, a natural requirement would be a bound $I \leq \frac{b_0}{n^3}$ with b_0 of order 1. This leads to a variant of Theorem 1.2 which is more flexible in applications.

Theorem 1.3. *Let f be defined and C^2-smooth in some open neighbourhood of S^{n-1}. Assume that it has mean zero and*

$$\int\!\!\int \langle x,y\rangle\ f(x)f(y)\,d\sigma_{n-1}(x)d\sigma_{n-1}(y) \leq \frac{b_0}{n^3}, \qquad b_0 \geq 0.$$

If $\|f'' - aI_n\| \leq 1$ holds on S^{n-1} together with (1.5)*, then*

$$\int \exp\Big\{\frac{n-1}{4(1+b_0^2+4b^2)}\,|f|\Big\}\,d\sigma_{n-1} \leq 2.$$

We believe that the second order concentration on the sphere may indeed be useful in various applications. One motivating example has been the question of optimal rates of approximation in the central limit theorem for linear forms $X_\theta = \langle X,\theta\rangle$, where $X = (X_1,\dots,X_n)$ is a given random vector in $\mathbf{R}^n$ whose components are not necessarily independent. If the covariance matrix of X has a bounded spectral radius, a celebrated result of Sudakov [S] indicates that, for n large, the distributions F_θ of X_θ are concentrated for most of θ (in the sense of σ_{n-1}) around a certain typical measure F on the real line, which may or may not be Gaussian. Many authors studied various aspects of this interesting phenomenon, and we omit references. Let us mention only that one can study the deviations F_θ from F in terms of the Fourier-Stieltjes transforms

$$f_t(\theta) = \mathbf{E}\,e^{it\langle\theta,X\rangle} = \int_{-\infty}^{\infty} e^{it\langle\theta,x\rangle}\,dF_\theta(x) \qquad (t\in\mathbf{R},\ \theta\in\mathbf{R}^n),$$

which are naturally defined as smooth functions on the whole space $\mathbf{R}^n$. By the direct differentiation in θ,

$$\langle f_t''(\theta)v,w\rangle = -t^2\,\mathbf{E}\,\langle v,X\rangle\,\langle w,X\rangle\,e^{it\langle\theta,X\rangle}.$$

Here, condition (1.5) leads to a certain correlation-type condition for products X_jX_k, such that (1.6) will ensure $\frac{1}{n}$-bounds for typical deviations of F_θ from F (in contrast with $\frac{1}{\sqrt{n}}$-bounds in the classical Berry-Esseen theorem). Such improving effects have recently been shown in the work of B. Klartag and S. Sodin in case of independent summands ([K-S], cf. also [K]). As for the general setting, this concentration problem will be dealt with in a separate paper and hence will not be discuss it here further.

The proof of Theorems 1.1-1.2 is based on the application of the logarithmic Sobolev inequality on the sphere and requires derivation of bounds on the integrals

$$\int |\nabla_S f|^2\,d\sigma_{n-1}, \qquad \int |\nabla f|^2\,d\sigma_{n-1} \tag{1.7}$$

in terms of the second derivatives. Basic tools leading to exponential bounds under logarithmic Sobolev inequalities are rather universal and can be developed in the setting of abstract metric spaces, cf. Section 2. Then we turn to the case of the sphere and sharpen the Poincaré inequality by involving the norm $\|f_S''\|$ (Section 3). Sections 5-6 are devoted to the estimation of the integrals (1.7). As a preliminary step, the identity

(1.4) is derived separately in Section 4. The proofs of Theorem 1.1 and Theorems 1.2-1.3 are completed in Sections 5 and 7, respectively. In the extended version of this note we add an Appendix providing for the readers convenience more details on the underlying computations in spherical calculus, cf. [B-C-G].

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2. **Logarithmic Sobolev Inequalities on Metric Spaces**

Assume that a metric space (M, ρ) is equipped with a Borel probability measure μ. The triple (M, ρ, μ) is said to satisfy a logarithmic Sobolev inequality with constant $\sigma^2 < \infty$, if

$$\mathrm{Ent}_\mu(f^2) \leq 2\sigma^2 \int |\nabla f|^2 \, d\mu \tag{2.1}$$

for any bounded function f on M with finite Lipschitz semi-norm $\|f\|_{\mathrm{Lip}}$. The optimal value of σ^2 is then called the logarithmic Sobolev constant.

Here

$$\mathrm{Ent}_\mu(u) = \int u \log u \, d\mu - \int u \, d\mu \log \int u \, d\mu \qquad (u \geq 0)$$

is the entropy functional defined for non-negative measurable functions on M. As for the modulus of the gradient in (2.1), it may be understood in the generalized sense as

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(x) - f(y)|}{\rho(x,y)} \qquad (x \in M). \tag{2.2}$$

This function is always Borel measurable, whenever f is continuous. In this abstract setting, (2.1) actually extends to the larger class of all f that have a finite Lipschitz semi-norm on every ball in M; such functions will be called locally Lipschitz.

Now, define the function

$$|\nabla^2 f(x)| = |\nabla \, |\nabla f(x)| \, | = \limsup_{y \to x} \frac{| \, |\nabla f(x)| - |\nabla f(y)| \, |}{\rho(x,y)}, \tag{2.3}$$

which we call a second order modulus of the gradients of f.

The Lipschitz property $\|f\|_{\mathrm{Lip}} \leq 1$ implies that $|\nabla f(x)| \leq 1$ for all $x \in M$. The converse is also true, at least when M is a (connected) Riemannian manifold. In this case, the assumption $|\nabla^2 f(x)| \leq 1$ for every x in M means that the function $|\nabla f|$ is Lipschitz. If $|\nabla f|$ is locally Lipschitz, then f is of course locally Lipschitz as well.

The next statement indicates how the definition (2.3) could be used in applications.

Proposition 2.1. *Assume that a metric probability space (M, ρ, μ) satisfies a logarithmic Sobolev inequality with constant σ^2. Then, for any locally Lipschitz function*

f on M with μ-mean zero, such that $|\nabla f|$ is locally Lipschitz and $|\nabla^2 f| \leq 1$ on the support of μ, we have

$$\int \exp\Big\{\frac{1}{2\sigma^2}\, f\Big\}\, d\mu \leq \exp\Big\{\frac{1}{2\sigma^2} \int |\nabla f|^2\, d\mu\Big\}. \tag{2.4}$$

Proof. The argument is based on two general results that relate (2.1) to the exponential integrability of Lipschitz functions. Namely, for any locally Lipschitz μ-integrable function u on M,

$$\int e^{u - \int u\, d\mu}\, d\mu \leq \int e^{\sigma^2 |\nabla u|^2}\, d\mu. \tag{2.5}$$

In addition, if $|\nabla u| \leq 1$ on the support of μ, say M_1, then for all $0 \leq t < \frac{1}{2\sigma^2}$,

$$\int e^{tu^2}\, d\mu \leq \exp\Big\{\frac{t}{1 - 2\sigma^2 t} \int u^2\, d\mu\Big\}. \tag{2.6}$$

On the basis of (2.1), the inequality (2.5) was derived in [B-G], cf. also [L1-2]. The second inequality, (2.6), is a classical result of Aida, Masuda and Shigekawa [A-M-S]. We refer to [B-G] for a detailed discussion.

We apply (2.6) with $t = \sigma^2 \lambda^2$ to the locally Lipschitz function $u = |\nabla f|$. Since the condition $|\nabla u| \leq 1$ is assumed to hold on M_1, we get that

$$\int e^{\sigma^2 \lambda^2 |\nabla f|^2}\, d\mu \leq \exp\Big\{\frac{\sigma^2 \lambda^2}{1 - 2\sigma^4 \lambda^2} \int |\nabla f|^2\, d\mu\Big\}, \qquad \lambda^2 < \frac{1}{2\sigma^4}.$$

On the other hand, since f is locally Lipschitz and has μ-mean zero, one may apply (2.5), which gives

$$\int e^{\lambda f}\, d\mu \leq \int e^{\sigma^2 \lambda^2 |\nabla f|^2}\, d\mu.$$

Hence, the combination of these two bounds yields

$$\int e^{\lambda f}\, d\mu \leq \exp\Big\{\frac{\sigma^2 \lambda^2}{1 - 2\sigma^4 \lambda^2} \int |\nabla f|^2\, d\mu\Big\}.$$

Here one may choose $\lambda = \frac{1}{2\sigma^2}$, and then we arrive at the required inequality (2.4). □

When M is an open region in $\mathbf{R}^n$ (with the Euclidean distance), the definition (2.1) leads to the usual notion of a logarithmic Sobolev inequality, holding for all locally Lipschitz functions on M. To avoid possible confusion about being locally Lipschitz, let us emphasize that, when f is differentiable at a given point x, (2.2) does coincide with the modulus (the length) of the Euclidean gradient. The same remark applies to the sphere $M = S^{n-1}$ with the geodesic or induced Euclidean distances, in which case (2.2) defines $|\nabla_S f(x)|$, the length of the spherical gradient of f.

The second order modulus of the gradients may also be related to the usual (Euclidean) derivatives. Namely, if f is C^2-smooth in the open set M in $\mathbf{R}^n$, the function $|\nabla f|$ will be locally Lipschitz, and

$$|\nabla^2 f(x)| = |\nabla f(x)|^{-1} |f''(x) \nabla f(x)|, \qquad x \in M. \tag{2.7}$$

Here the ratio should be understood as $\|f''(x)\|$ in case $|\nabla f(x)| = 0$. In particular,

$$|\nabla^2 f(x)| \leq \|f''(x)\|. \tag{2.8}$$

For example, for the quadratic function $f(x) = \frac{1}{2} \sum_{i=1}^n \lambda_i x_i^2$, $x = (x_1, \dots, x_n)$,

$$|\nabla^2 f(x)| \ = \ \frac{\sqrt{\sum_{i=1}^n \lambda_i^4 x_i^2}}{\sqrt{\sum_{i=1}^n \lambda_i^2 x_i^2}} \ \leq \ \max_i |\lambda_i|.$$

The identity (2.7) is easily obtained by the direct differentiation. Thus, in the Euclidean setup Proposition 2.1 may be simplified by using the inequality (2.8) as follows.

Corollary 2.2. *Let a probability measure μ on $\mathbf{R}^n$ satisfy a logarithmic Sobolev inequality with constant σ^2, and let a function f be C^2-smooth in an open neighbourhood of the support of μ. If it has μ-mean zero and $\|f''\| \leq 1$ on the support of μ, then*

$$\int \exp\Big\{\frac{1}{2\sigma^2}\, f\Big\}\, d\mu \leq \exp\Big\{\frac{1}{2\sigma^2} \int |\nabla f|^2\, d\mu\Big\}.$$

3. Logarithmic Sobolev Inequality on the Sphere

An important result due to Mueller and Weissler [M-W] sharpens the Poincaré inequality (1.1) in terms of the logarithmic Sobolev inequality. Namely, the logarithmic Sobolev constant of the unit sphere S^{n-1}, which is equipped with the geodesic metric ρ and the uniform measure σ_{n-1}, coincides with the Poincaré constant $\sigma^2 = \frac{1}{n-1}$. That is, for any C^1-smooth function $f : S^{n-1} \rightarrow \mathbf{R}$,

$$\mathrm{Ent}_{\sigma_{n-1}}(f^2) \leq \frac{2}{n-1} \int |\nabla_S f|^2\, d\sigma_{n-1}. \tag{3.1}$$

To see the connection of (3.1) with the concentration phenomenon on the sphere in the form (1.2), one may apply (2.6) with $u = f$ and $t = \frac{n-1}{4}$.

We are also in the position to apply the abstract Proposition 2.1 to $(S^{n-1}, \rho, \sigma_{n-1})$ and thus involve the second order modulus of the gradients, $|\nabla_S^2 f|$. On the unit sphere it is defined according to (2.2)-(2.3) with $|\nabla f|$ replaced by

$$|\nabla_S f(\theta)| \ = \ \limsup_{\theta' \rightarrow \theta} \frac{|f(\theta) - f(\theta')|}{\rho(\theta, \theta')} \qquad (\theta, \theta' \in S^{n-1}).$$

Note that both the geodesic and Euclidean metrics on S^{n-1} may equivalently be used for computing the modulus of the gradient of first and second orders.

For example, the Euclidean derivatives of the linear function $f(x) = \langle v, x\rangle$ are just $\nabla f(x) = v$ and $f'' = 0$. As for the first and second order modulus of its spherical gradient, we have

$$|\nabla_S f(\theta)| = \sqrt{|v|^2 - \langle v,\theta\rangle^2} \qquad (|v| = 1),$$

and, by the chain rule,

$$\begin{aligned} \nabla_S |\nabla_S f(\theta)| \ &= \ -\frac{1}{2\sqrt{|v|^2 - \langle v,\theta\rangle^2}}\, \nabla_S \left(\langle v,\theta\rangle^2\right) \\ &= \ -\frac{1}{\sqrt{|v|^2 - \langle v,\theta\rangle^2}}\, \langle v,\theta\rangle\, \nabla_S \langle v,\theta\rangle \qquad (\theta \neq v). \end{aligned}$$

Hence, $|\nabla_S^2 f(\theta)| = |\langle v,\theta\rangle|$ in contrast with $|\nabla^2 f(\theta)| = 0$.

To simplify the condition $|\nabla_S^2 f| \leq 1$, one may use the following equality which is a full analog of the formula (2.7) mentioned before for the case of open regions in $\mathbf{R}^n$. It is stated below without proof (cf. [B-C-G]).

Lemma 3.1. *Given a C^2-smooth function f on S^{n-1}, $|\nabla_S f|$ has a finite Lipschitz semi-norm and, for all $\theta \in S^{n-1}$,*

$$|\nabla_S^2 f(\theta)| \ = \ |\nabla_S f(\theta)|^{-1}\, |f_S''(\theta) \nabla_S f(\theta)|,$$

where the right-hand side is understood as $\|f_S''(\theta)\|$ in case $|\nabla_S f(\theta)| = 0$. In particular, $|\nabla_S^2 f(\theta)| \leq \|f_S''(\theta)\|$.

Thus, in order to bound exponential moments of f similarly to (2.4), one may require the condition $\|f_S''\| \leq 1$. There is however an alternative way based on the application of Corollary 2.2; the latter would allow us to work with Euclidean derivatives. Let us state both consequences of the logarithmic Sobolev inequality (3.1). Henceforth we shall always understand the mean of functions on the unit sphere to be taken with respect to the measure σ_{n-1}.

Corollary 3.2. *Let f be a C^2-smooth function on S^{n-1} with mean zero. If $\|f_S''\| \leq 1$, then*

$$\log \int \exp\Big\{\frac{n-1}{2}\, f\Big\}\, d\sigma_{n-1} \leq \frac{n-1}{2} \int |\nabla_S f|^2\, d\sigma_{n-1}. \tag{3.2}$$

Moreover, if f is C^2-smooth in an open neighbourhood of the unit sphere with $\|f''\| \leq 1$ on S^{n-1}, then

$$\log \int \exp\Big\{\frac{n-1}{2}\, f\Big\}\, d\sigma_{n-1} \leq \frac{n-1}{2} \int |\nabla f|^2\, d\sigma_{n-1}. \tag{3.3}$$

Applying (3.2) to functions εf with $\varepsilon \to 0$, this inequality returns us to (1.1) with an additional factor 2. The condition $\|\varepsilon f_S''\| \leq 1$ is fulfilled for all ε small enough, so any constraint on the second derivative may be removed from the conclusion. In this sense, Corollary 3.2 provides a sharper form of the Poincaré inequality.

4. **Second Derivative and Laplacian**

In order to estimate the integral appearing on the right-hand side in (3.2), we first derive the formula (1.4), involving the square of the spherical Laplacian, i.e. the operator $\Delta_S^2 f = \Delta_S \Delta_S f$. Given a point $\theta \in S^{n-1}$, it will be convenient to work with the spherical second derivative $f_S''(\theta)$ as a symmetric $n \times n$ matrix, i.e. as a linear operator on $\mathbf{R}^n$, rather than as a linear operator on the tangent space $\theta^\perp$. More precisely, we extend the usual Hessian of f at θ to the whole space by putting $f_S''(\theta)\theta = 0$ (in particular, both the operator norm and the Hilbert-Schmidt norm will not increase for the extended matrix). The extended Hessian $f_S''(\theta)$ may also be defined as the $n \times n$ matrix B with the smallest Hilbert-Schmidt norm, satisfying the Taylor expansion

$$\begin{aligned} f(\theta') &= f(\theta) + \langle \nabla_S f(\theta), \theta' - \theta \rangle \\ &\quad + \frac{1}{2} \langle B(\theta' - \theta), \theta' - \theta \rangle + o\big(|\theta' - \theta|^2\big) \qquad (\theta' \to \theta, \ \ \theta' \in S^{n-1}). \end{aligned}$$

When f is C^2-smooth in an open region containing the unit sphere, the spherical second derivative is related to the Euclidean derivatives by

$$f_S''(\theta) = P_{\theta^\perp} B P_{\theta^\perp}, \qquad B = f''(\theta) - \langle \nabla f(\theta), \theta \rangle \, I_n,$$

where $P_{\theta^\perp}$ is the projection operator from $\mathbf{R}^n$ to the space $\theta^\perp$ orthogonal to θ. Also, recall that $\nabla_S f(\theta) = P_{\theta^\perp} \nabla f(\theta)$.

Proposition 4.1. *For any C^4-smooth function f on S^{n-1},*

$$\int \|f_S''\|_{\mathrm{HS}}^2 \, d\sigma_{n-1} = \int f \, \big(\Delta_S^2 f + (n-2) \, \Delta_S f\big) \, d\sigma_{n-1}. \tag{4.1}$$

One can give a short proof of (4.1) on the basis of the Bochner-Lichnerowicz formula from Riemannian Geometry (cf. Remark 4.6 below). Nevertheless, for reader's convenience, we supply it with a direct argument based on partial integration formulas for the multivariate calculus on the sphere which we state below without proofs and refer to [B-C-G]. The first relation connects the spherical second derivative with the iteration of spherical derivatives. The second one is a formula for the commutator of the Laplacian and the gradient.

Lemma 4.2. *Given a C^2-smooth function f on S^{n-1}, for all $\theta \in S^{n-1}$ and $v \in \mathbf{R}^n$,*

$$f_S''(\theta)v \,=\, \nabla_S \left<\nabla_S f(\theta), v\right> + \left<v, \theta\right> \nabla_S f(\theta). \tag{4.2}$$

Lemma 4.3. *Given a C^3-smooth function f on S^{n-1}, for all $\theta \in S^{n-1}$ and $v \in \mathbf{R}^n$,*

$$\Delta_S \left<\nabla_S f(\theta), v\right> - \left<\nabla_S \Delta_S f(\theta), v\right> \,=\, (n-3)\left<\nabla_S f(\theta), v\right> - 2\left<v, \theta\right> \Delta_S f(\theta).$$

The spherical Laplacian appears when integrating by parts, in particular in the formula

$$\int \left<\nabla_S f, \nabla_S g\right>\, d\sigma_{n-1} = -\int f \Delta_S g\, d\sigma_{n-1}. \tag{4.3}$$

The following analogous identity involves a linear weight.

Lemma 4.4. *For all C^2-smooth functions f, g on S^{n-1} and for any $v \in \mathbf{R}^n$,*

$$\begin{aligned}\int \left<\nabla_S f(\theta), \nabla_S g(\theta)\right> \left<v, \theta\right> d\sigma_{n-1}(\theta) \;&=\; -\int f(\theta) \Delta_S g(\theta) \left<v, \theta\right> d\sigma_{n-1}(\theta) \\ &\quad\; - \int f(\theta) \left<\nabla_S g(\theta), v\right>\, d\sigma_{n-1}(\theta).\end{aligned}$$

Finally, let us mention how to relate the spherical Laplacian to the Euclidean derivatives. The next representation will be used in Section 6.

Lemma 4.5. *If f is C^2-smooth in an open region containing the unit sphere, then for any $\theta \in S^{n-1}$,*

$$\Delta_S f(\theta) = \Delta f(\theta) - (n-1)\left<\nabla f(\theta), \theta\right> - \left<f''(\theta)\theta, \theta\right>.$$

Proof of Proposition 4.1. Using (4.2), one may write

$$\begin{aligned}\int \|f_S''\|_{\mathrm{HS}}^2\, d\sigma_{n-1} \;&=\; n \iint |f_S''(\theta)v|^2\, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v) \\ &=\; n \iint \left|\nabla_S \left<\nabla_S f(\theta), v\right> + \left<v, \theta\right> \nabla_S f(\theta)\right|^2 d\sigma_{n-1}(\theta) d\sigma_{n-1}(v) \\ &=\; n\,(I_1 + 2I_2 + I_3),\end{aligned}$$

where

$$\begin{aligned} I_1 &= \iint |\nabla_S \langle \nabla_S f(\theta), v\rangle|^2 \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v), \\ I_2 &= \iint \langle \nabla_S \langle \nabla_S f(\theta), v\rangle, \nabla_S f(\theta)\rangle \langle v, \theta\rangle \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v), \\ I_3 &= \iint |\nabla_S f(\theta)|^2 \langle v, \theta\rangle^2 \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v). \end{aligned}$$

Integration over v immediately gives

$$I_3 = \frac{1}{n} \int |\nabla_S f|^2 \, d\sigma_{n-1},$$

and according to (4.3),

$$I_1 = -\iint \varphi_v(\theta) \Delta_S \varphi_v(\theta) \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v), \quad \text{where} \quad \varphi_v(\theta) = \langle \nabla_S f(\theta), v\rangle.$$

To continue, we apply Lemma 4.3, so as to develop $\Delta_S \varphi_v(\theta)$ and represent the above integral in the form

$$I_1 = -\big(I_{11} + (n-3)\, I_{12} - 2 I_{13}\big)$$

with

$$\begin{aligned} I_{11} &= \iint \langle \nabla_S f(\theta), v\rangle \langle \nabla_S \Delta_S f(\theta), v\rangle \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v), \\ I_{12} &= \iint \langle \nabla_S f(\theta), v\rangle^2 \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v), \\ I_{13} &= \iint \langle \nabla_S f(\theta), v\rangle \langle v, \theta\rangle \, \Delta_S f(\theta) \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v). \end{aligned}$$

Let us now integrate over v and apply (4.3) with $g = \Delta_S f$ to simplify the first equality as

$$\begin{aligned} I_{11} &= \frac{1}{n} \int \langle \nabla_S f(\theta), \nabla_S \Delta_S f(\theta)\rangle \, d\sigma_{n-1}(\theta) \\ &= -\frac{1}{n} \int f \, \Delta_S(\Delta_S f) \, d\sigma_{n-1} \; = \; -\frac{1}{n} \int f \, \Delta_S^2 f \, d\sigma_{n-1}. \end{aligned}$$

We also have

$$I_{12} = \frac{1}{n} \int |\nabla_S f|^2 \, d\sigma_{n-1}, \qquad I_{13} = \frac{1}{n} \int \langle \nabla_S f(\theta), \theta\rangle \, \Delta_S f(\theta) \, d\sigma_{n-1}(\theta) = 0.$$

This finally gives

$$I_1 \; = \; \frac{1}{n} \int f \, \Delta_S^2 f \, d\sigma_{n-1} - \frac{n-3}{n} \int |\nabla_S f|^2 \, d\sigma_{n-1}.$$

In order to evaluate the integral I_2, we apply Lemma 4.4 with the function $\langle \nabla_S f(\theta), v\rangle$ in place of f and with f in place of g. After integration over θ, we obtain the integral over the remaining variable v, namely,

$$I_2(v) = -\int \langle \nabla_S f(\theta), v\rangle \, \Delta_S f(\theta) \, \langle v, \theta\rangle \, d\sigma_{n-1}(\theta) - \int \langle \nabla_S f(\theta), v\rangle^2 \, d\sigma_{n-1}(\theta).$$

The subsequent integration over v cancels the first integral, since its integrand will contain the inner product $\langle \nabla_S f(\theta), \theta\rangle = 0$ as a factor. As a result,

$$\begin{aligned} I_2 &= \int I_2(v)\, d\sigma_{n-1}(v) \\ &= -\iint \langle \nabla_S f(\theta), v\rangle^2 \, d\sigma_{n-1}(\theta) d\sigma_{n-1}(v) \ = \ -\frac{1}{n} \int |\nabla_S f(\theta)|^2 \, d\sigma_{n-1}(\theta). \end{aligned}$$

It remains to collect these formulas and conclude that

$$n\,(I_1 + 2I_2 + I_3) \ = \ \int f \, \Delta_S^2 f \, d\sigma_{n-1} - (n-2) \int |\nabla_S f|^2 \, d\sigma_{n-1}.$$

Here the last integral can also be written as $-\int f \Delta_S f \, d\sigma_{n-1}$, cf. (4.3). □

Remark 4.6. According to the Bochner-Lichnerowicz formula (cf. e.g. [B-G-L], p. 509), for any smooth function f on the Riemannian manifold (M,g),

$$\frac{1}{2}\, \Delta_g(|\nabla f|^2) = \langle \nabla f, \nabla(\Delta_g f)\rangle + |\nabla \nabla f|^2 + Ric_g(\nabla f, \nabla f), \tag{4.4}$$

where $Ric_g(\nabla f, \nabla f)$ is the Ricci curvature of (M,g) evaluated at ∇f. The unit sphere $M = S^{n-1}$ in $\mathbf{R}^n$ has a constant curvature, namely, in this case

$$Ric_g(\nabla f, \nabla f) = (n-2) |\nabla_S f|^2.$$

Hence, integrating (4.4) over the sphere, we get

$$\int \Big(\frac{1}{2}\, \Delta_S(|\nabla_S f|^2) - \langle \nabla_S f, \nabla_S(\Delta_S f)\rangle\Big) \, d\sigma_{n-1} = \int (|\nabla_S \nabla_S f|^2 + (n-2)|\nabla_S f|^2) \, d\sigma_{n-1}. \tag{4.5}$$

On the other hand, $\int \Delta_S(|\nabla_S f|^2)\, d\sigma_{n-1} = 0$,

$$\int |\nabla_S f|^2 \, d\sigma_{n-1} = -\int f \Delta_S f \, d\sigma_{n-1},$$

(recall (4.3)), and

$$\int \langle \nabla_S f, \nabla_S(\Delta_S f)\rangle \, d\sigma_{n-1} = -\int f \Delta_S^2 f \, d\sigma_{n-1}.$$

Applying these relations in (4.5), we arrive at (4.1).

5. **Expansions in Spherical Harmonics**

Using Proposition 4.1, one may study relations of the form

$$c \int |\nabla_S f|^2 \, d\sigma_{n-1} \leq \int \|f_S''\|_{\rm HS}^2 \, d\sigma_{n-1} \qquad (c > 0) \tag{5.1}$$

by means of the orthogonal expansion in spherical harmonics,

$$f = \sum_{d=0}^{\infty} f_d \qquad (f_d \in H_d). \tag{5.2}$$

As is well-known (cf. e.g. [S-W]), the Hilbert space $L^2(S^{n-1})$ can be decomposed into a sum of orthogonal linear subspaces H_d, $d = 0, 1, 2, \dots$, consisting of all d-homogeneous harmonic polynomials (more precisely - restrictions of such polynomials to the sphere). Any element f_d of H_d represents an eigenfunction of the Laplacian, with the eigenvalue $-d(n+d-2)$. That is,

$$\Delta_S f_d = -d(n+d-2)\, f_d,$$

and hence

$$\Delta_S^2 f_d \,=\, d^2 (n+d-2)^2\, f_d.$$

As a result,

$$\Delta_S f = -\sum_{d=1}^{\infty} d(n+d-2) f_d, \qquad \Delta_S^2 f = \sum_{d=1}^{\infty} d^2 (n+d-2)^2 f_d$$

which should be understood as equalities in L^2 (Note that both $\Delta_S f$ and $\Delta_S^2 f$ are continuous functions, as long as f is C^4-smooth).

According to the representation (4.1), (5.1) is equivalent to

$$\int f\, \Delta_S^2 f \, d\sigma_{n-1} \,\geq\, -(c+n-2) \int f\, \Delta_S f \, d\sigma_{n-1}. \tag{5.3}$$

Moreover, since the spherical harmonics serve as eigenfunctions both for Δ_S and Δ_S^2, the last inequality need to be verified for elements f_d of H_d only. Here, both integrals are vanishing for constant functions, i.e. $d = 0$. If $d \geq 1$, (5.3) becomes

$$c \,\leq\, d^2 + (d-1)(n-2). \tag{5.4}$$

Thus, if we want to involve in (5.1) all C^2-smooth functions f, the optimal value of c is described as the minimum of the right-hand side of (5.4) over all $d \geq 1$. The minimum is achieved for $d = 1$ which leads to the optimal value $c = 1$. However, if we require that f is orthogonal to all linear functions with respect to σ_{n-1}, it means that we only allow the values $d \geq 2$ in (5.4), and then the optimal value is $c = n+2$. As a result, we have proved:

Proposition 5.1. *For any C^2-function f on S^{n-1},*

$$\int |\nabla_S f|^2\, d\sigma_{n-1} \leq \int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1},$$

where equality is attained for all linear functions. Moreover, if f is orthogonal to all linear functions with respect to σ_{n-1}, then

$$\int |\nabla_S f|^2\, d\sigma_{n-1} \leq \frac{1}{n+2} \int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1} \tag{5.5}$$

with equality attainable for all quadratic harmonics.

The expansion (5.2) is commonly used to derive Poincaré-type inequalities such as (1.1). If we require additionally that f should be orthogonal to all linear functions, the constant will slightly improve only, since then

$$\int f^2\, d\sigma_{n-1} \leq \frac{1}{2n} \int |\nabla_S f|^2\, d\sigma_{n-1}.$$

This bound may be combined with (5.5) to get a second order Poincaré-type inequality which was mentioned in the Introduction. But, one can also apply (5.2) directly in the representation (4.1). Indeed, on spherical harmonics f_d of H_d, the inequality of the form $c\int f^2\, d\sigma_{n-1} \leq \int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1}$ becomes

$$c \,\leq\, d(n+d-2)\,\big(d(n+d-2) - (n-2)\big).$$

Since the right-hand side is an increasing function of d, we arrive at:

Proposition 5.2. *For any C^2-function f on S^{n-1} with mean zero,*

$$\int f^2\, d\sigma_{n-1} \leq \frac{1}{n-1} \int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1}, \tag{5.6}$$

where equality is attained for all linear functions. Moreover, if f is orthogonal to all linear functions with respect to σ_{n-1}, then

$$\int f^2\, d\sigma_{n-1} \leq \frac{1}{2n(n+2)} \int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1} \tag{5.7}$$

with equality attainable for all quadratic harmonics.

An interesting consequence of (5.6) is the statement that the equality $f_S'' = 0$ is possible for constant functions, only (in contrast with the Euclidean Hessian).

Remark 5.3. It is much easier to derive (5.7) with suboptimal, although asymptotically correct constants as n tends to infinity, without appealing to Proposition 4.1. The argument is based on the double application of the Poincaré inequality (1.1). Orthogonality of f to all linear functions ensures that the function $\theta \rightarrow \langle \nabla_S f(\theta), v\rangle$ has mean zero for any $v \in \mathbf{R}^n$. So, using the identity (4.2), we get

$$\begin{aligned}(n-1)\int \langle \nabla_S f(\theta), v\rangle^2\, d\sigma_{n-1}(\theta) \ &\leq\ \int |f_S''(\theta)v - \langle v,\theta\rangle\, \nabla_S f(\theta)|^2\, d\sigma_{n-1}(\theta) \\ &= \ \int |f_S''(\theta)v|^2\, d\sigma_{n-1}(\theta) + \int \langle v,\theta\rangle^2\, |\nabla_S f(\theta)|^2\, d\sigma_{n-1}(\theta) \\ &\quad - 2\int \langle f_S''(\theta)\nabla_S f(\theta), v\rangle\, \langle v,\theta\rangle\ d\sigma_{n-1}(\theta).\end{aligned}$$

The next integration over $d\sigma_{n-1}(v)$ cancels the last integral (due to $f_S''(\theta)\theta = 0$), and we are led to

$$(n-2)\int |\nabla_S f(\theta)|^2\, d\sigma_{n-1}(\theta)\ \leq\ \int \|f_S''(\theta)\|_{\rm HS}^2\, d\sigma_{n-1}(\theta).$$

If f has mean zero, the left integral is estimated from below according to (1.1), and thus

$$\int f^2\, d\sigma_{n-1} \leq \frac{1}{(n-1)(n-2)}\int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1}, \qquad n \geq 3.$$

The constant in this inequality is slightly worse than (5.7), and we loose information about extremal functions.

Remark 5.4. The above argument is also applicable in the Euclidean setup when dealing with a probability measure μ on $\mathbf{R}^n$ satisfying a Poincaré-type inequality

$$\int f^2\, d\mu \leq \sigma^2 \int |\nabla f|^2\, d\mu \qquad \Big(\int f\, d\mu = 0\Big).$$

For example, the standard Gaussian measure with density $\frac{d\mu(x)}{dx} = (2\pi)^{-n/2}\, e^{-|x|^2/2}$ has the Poincaré constant $\sigma^2 = 1$, which yields a second order Poincaré-type inequality

$$\int f^2\, d\mu \leq \frac{1}{2}\int \|f_S''\|_{\rm HS}^2\, d\mu.$$

It holds true in the class of all C^2-smooth functions f on $\mathbf{R}^n$ that are orthogonal to all affine functions in $L^2(\mu)$. A number of interesting results in this direction, including concentration inequalities in terms of higher order derivatives, have been recently obtained by R. Adamczak and P. Wolff, and we refer to [A-W]. Note that, in the general (non-Gaussian) case, orthogonality to linear functions should be replaced with the requirement $\int \nabla f\, d\mu = 0$.

We are now prepared to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us return to the bound (3.2) of Corollary 3.2. Using (5.5), we then get

$$\log \int \exp\Big\{\frac{n-1}{2}\, f\Big\}\, d\sigma_{n-1}\ \leq\ \frac{n-1}{2(n+2)}\int \|f_S''\|_{\rm HS}^2\, d\sigma_{n-1}\ \leq\ \frac{1}{2}\, b^2,$$

and with a similar inequality for the function $-f$,

$$\int e^{\frac{n-1}{2}\,|f|}\,d\sigma_{n-1} \,\le\, \int e^{\frac{n-1}{2}\,f}\,d\sigma_{n-1} + \int e^{-\frac{n-1}{2}\,f}\,d\sigma_{n-1} \,\le\, e^{b^2/2}.$$

It follows that, for any $\lambda \ge 1$,

$$\int e^{\frac{n-1}{2}\,|f|/\lambda}\,d\sigma_{n-1} \,\le\, \Big(\int e^{\frac{n-1}{2}\,|f|}\,d\sigma_{n-1}\Big)^{1/\lambda} \,\le\, (2e^{b^2/2})^{1/\lambda}.$$

It remains to note that $(2e^{b^2/2})^{1/\lambda} = 2$ for $\lambda = 1 + \frac{b^2}{\log 4} \le 1 + b^2$. □

6. Bounds on the L^2-Norm of the Euclidean Gradient

We now turn back to Theorem 1.2 while invoking the second bound of Corollary 3.2. Hence, we need an analog of (5.5) for the modulus of the Euclidean gradient. Assume that a function f is defined and C^2-smooth in some neighbourhood G of S^{n-1}.

Proposition 6.1. *If f is orthogonal to all linear functions with respect to σ_{n-1}, then*

$$\int |\nabla f|^2\,d\sigma_{n-1} \le \frac{5}{n-1} \int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}. \tag{6.1}$$

At the of the proof it will be apparent that for growing dimensions the constant 5 may be asymptotically improved to 2.

Proof. Since the spherical gradient $\nabla_S f(\theta)$ represents the projection of the usual gradient $\nabla f(\theta)$ to the subspace $\theta^\perp$ of $\mathbf{R}^n$ orthogonal to θ, we have

$$|\nabla f|^2 \,=\, |\nabla_S f(\theta)|^2 + \left<\nabla f(\theta),\theta\right>^2.$$

As a preliminary step, first we show that

$$\int |\nabla_S f|^2\,d\sigma_{n-1} \le \frac{1}{n-1} \int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}. \tag{6.2}$$

Write

$$\int |\nabla_S f|^2\,d\sigma_{n-1} \,=\, \int |\nabla f(\theta)|^2\,d\sigma_{n-1}(\theta) - \int \left<\nabla f(\theta),\theta\right>^2\,d\sigma_{n-1}(\theta) \tag{6.3}$$

and represent

$$\int |\nabla f|^2\,d\sigma_{n-1} = n \iint \left<\nabla f(\theta),v\right>^2\,d\sigma_{n-1}(\theta)d\sigma_{n-1}(v). \tag{6.4}$$

The assumption that f is orthogonal to all linear functions is equivalent to the property that every function of the form

$$\left<\nabla_S f(\theta),v\right> = \left<\nabla f(\theta),v\right> - \left<\nabla f(\theta),\theta\right>\left<v,\theta\right>$$

has σ_{n-1}-mean zero (by the integration by parts formula). Hence

$$\int \langle \nabla f(\theta), v\rangle \, d\sigma_{n-1}(\theta) = \int \langle \nabla f(\theta), \theta\rangle \, \langle v, \theta\rangle \, d\sigma_{n-1}(\theta),$$

and, by the Cauchy-Schwarz inequality,

$$\Big(\int \langle \nabla f(\theta), v\rangle \, d\sigma_{n-1}(\theta)\Big)^2 \leq \frac{1}{n} \int \langle \nabla f(\theta), \theta\rangle^2 \, d\sigma_{n-1}(\theta). \tag{6.5}$$

To estimate the L^2-norm of $\langle \nabla f(\theta), v\rangle$, one may apply the Poincaré inequality (1.1). Since $u(x) = \langle \nabla f(x), v\rangle$ has gradient $\nabla u(x) = f''(x)v$, we have, by (6.5),

$$\int \langle \nabla f(\theta), v\rangle^2 \, d\sigma_{n-1}(\theta) \leq \frac{1}{n} \int \langle \nabla f(\theta), \theta\rangle^2 \, d\sigma_{n-1}(\theta) + \frac{1}{n-1} \int |f''(\theta)v|^2 \, d\sigma_{n-1}(\theta).$$

Using this bound in (6.4) and integrating over v, we get

$$\int |\nabla f|^2 \, d\sigma_{n-1} \ \leq \ \int \langle \nabla f(\theta), \theta\rangle^2 \, d\sigma_{n-1}(\theta) + \frac{1}{n-1} \int \|f''(\theta)\|_{\rm HS}^2 \, d\sigma_{n-1}(\theta).$$

It remains to insert this bound in (6.3) which gives (6.2).

Now, rewrite (6.3) as

$$\int |\nabla f|^2 \, d\sigma_{n-1} = \int |\nabla_S f|^2 \, d\sigma_{n-1} + \int \langle \nabla f(\theta), \theta\rangle^2 \, d\sigma_{n-1}(\theta). \tag{6.6}$$

Here, the first integral on the right-hand side is estimated in terms of $\|f''\|_{\rm HS}^2$ by (6.2), and our next task will be to derive a suitable bound on the L^2-norm of the function $\langle \nabla f(\theta), \theta\rangle$. To this aim, we employ the representation of Lemma 4.5 for the spherical Laplacian in terms of the Euclidean derivatives. Since in general (by (4.3)),

$$\int \Delta_S f \, d\sigma_{n-1} = -\int \langle \nabla_S 1, \nabla_S f\rangle \, d\sigma_{n-1} = 0,$$

Lemma 4.5 yields

$$(n-1)\int \langle \nabla f(\theta), \theta\rangle \, d\sigma_{n-1}(\theta) = \int \big(\Delta f(\theta) - \langle f''(\theta)\theta, \theta\rangle\,\big) \, d\sigma_{n-1}(\theta). \tag{6.7}$$

Here the second integrand is equal to

$$I = \sum_{i,j=1}^n \partial_{ij} f(\theta) a_{ij} \quad \text{with} \quad a_{ij} = \delta_{ij} - \theta_i \theta_j.$$

Note that

$$\sum_{i,j=1}^n a_{ij}^2 \ = \ \sum_{i \neq j}^n \theta_i^2 \theta_j^2 + \sum_{i=1}^n (1-\theta_i^2)^2 \ = \ 1 + \sum_{i=1}^n \big((1-\theta_i^2)^2 - \theta_i^4\big) \ = \ n-1.$$

Hence, by Cauchy's inequality,

$$I^2 \le \sum_{i,j=1}^n (\partial_{ij} f(\theta))^2 \sum_{i,j=1}^n a_{ij}^2 = (n-1)\,\|f''(\theta)\|_{\rm HS}^2,$$

and by another application of the Cauchy-Schwarz inequality in (6.7),

$$\Big(\int \langle \nabla f(\theta),\theta\rangle\,d\sigma_{n-1}(\theta)\Big)^2 \le \frac{1}{n-1}\int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}. \qquad (6.8)$$

Next, consider the function $u(x) = \langle \nabla f(x), x\rangle$ and restrict its gradient $\nabla u(x) = \nabla f(x) + f''(x)x$ to the unit sphere. Projecting it to $\theta^\perp$, we obtain the spherical gradient

$$\nabla_S u(\theta) = \nabla_S f(\theta) + P_{\theta^\perp}\big(f''(\theta)\theta\big), \qquad \theta \in S^{n-1}.$$

In particular, by the triangle inequality,

$$|\nabla_S u(\theta)| \le |\nabla_S f(\theta)| + \|f''(\theta)\|.$$

Furthermore, the square of the right-hand side can be estimated by using the elementary inequality $(x+y)^2 \le \frac{\lambda}{\lambda-1}\,x^2 + \lambda y^2$ $(x,y \ge 0,\ \lambda > 1)$, which implies

$$|\nabla_S u(\theta)| \le \frac{\lambda}{\lambda-1}\,|\nabla_S f(\theta)|^2 + \lambda\,\|f''(\theta)\|^2.$$

Hence, using the Poincaré inequality together with (6.8), and increasing the operator norm to the Hilbert-Schmidt norm, we get

$$\begin{aligned}\int u^2\,d\sigma_{n-1} &\le \Big(\int u\,d\sigma_{n-1}\Big)^2 + \frac{1}{n-1}\int |\nabla_S u|^2\,d\sigma_{n-1} \\ &\le \frac{1}{n-1}\int \|f''\|_{\rm HS}^2\,d\sigma_{n-1} + \frac{1}{n-1}\int \Big(\frac{\lambda}{\lambda-1}\,|\nabla_S f|^2 + \lambda\,\|f''\|_{\rm HS}^2\Big)\,d\sigma_{n-1}.\end{aligned}$$

Thus,

$$\int \langle \nabla f(\theta),\theta\rangle^2\,d\sigma_{n-1}(\theta) \le \frac{1}{n-1}\,\frac{\lambda}{\lambda-1}\int |\nabla_S f|^2\,d\sigma_{n-1} + \frac{\lambda+1}{n-1}\int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}.$$

It remains to return to (6.6) and combine the above bound with (6.2). Adding and collecting the coefficients, it gives

$$(n-1)\int |\nabla f|^2\,d\sigma_{n-1} \le \Big(\frac{1}{n-1}\,\frac{\lambda}{\lambda-1} + \lambda + 1\Big)\int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}.$$

The quantity $\frac{1}{n-1}\,\frac{\lambda}{\lambda-1} + \lambda + 1$ is minimized at $\lambda = 1 + \frac{1}{\sqrt{n-1}}$, which leads to

$$\int |\nabla f|^2\,d\sigma_{n-1} \le \frac{c_n}{n-1}\int \|f''\|_{\rm HS}^2\,d\sigma_{n-1}, \qquad c_n = 1 + \Big(1 + \frac{1}{\sqrt{n-1}}\Big)^2. \qquad (6.9)$$

Clearly, $c_n \le 5$, thus proving (6.1). $\square$

Note that $c_n \to 2$ as $n \to \infty$. So, the constant 5 in (6.1) may be improved for large values of n.

Combining (6.1) with the Poincaré inequality (1.1), we get a second order Poincaré-type inequality in the Euclidean setup,

$$\int (f-m)^2 \, d\sigma_{n-1} \leq \frac{5}{(n-1)^2} \int \|f''\|_{\rm HS}^2 \, d\sigma_{n-1},$$

assuming that f is orthogonal to all linear functions, and where m is the mean of f with respect to σ_{n-1}. Here the left integral will not change when it is applied to $f_a(x) = f(x) - \frac{a}{2}\,|x|^2$ in place of f, while the right integral will depend on a. More precisely, we get

$$\int (f-m)^2 \, d\sigma_{n-1} \leq \frac{5}{(n-1)^2} \int \|f'' - aI_n\|_{\rm HS}^2 \, d\sigma_{n-1},$$

Hence, we arrive at:

Corollary 6.2. *If f is orthogonal to all affine functions with respect to σ_{n-1}, then for any $a \in \mathbf{R}$,*

$$\int f^2 \, d\sigma_{n-1} \leq \frac{5}{(n-1)^2} \int \|f'' - aI_n\|_{\rm HS}^2 \, d\sigma_{n-1}.$$

7. Proof of Theorems 1.2-1.3

Having proved Proposition 6.1, the proof of Theorem 1.2 is almost identical to the proof of Theorem 1.1.

Proof of Theorem 1.2. Let f be orthogonal to all linear functions with mean m. Applying (6.1) to the function $f - m$ in the bound (3.3) of Corollary 3.2, we get

$$\log \int \exp\Big\{\frac{n-1}{2}\,(f-m)\Big\} \, d\sigma_{n-1} \ \leq\ \frac{5}{2} \int \|f''\|_{\rm HS}^2 \, d\sigma_{n-1}.$$

Applying it to $f_a(x) = f(x) - \frac{a}{2}\,|x|^2$ in place of f, we get

$$\log \int \exp\Big\{\frac{n-1}{2}\,(f-m)\Big\} \, d\sigma_{n-1} \ \leq\ \frac{5}{2} \int \|f'' - aI_n\|_{\rm HS}^2 \, d\sigma_{n-1} \ \leq\ \frac{5}{2}\, b^2.$$

Assuming that $m = 0$ and applying a similar inequality to the function $-f$, we obtain

$$\int e^{\frac{n-1}{2}\,|f|} \, d\sigma_{n-1} \ \leq\ 2e^{5b^2/2}.$$

Hence, for any $\lambda \geq 1$,

$$\int e^{\frac{n-1}{2}\,|f|/\lambda} \, d\sigma_{n-1} \leq \Big(\int e^{\frac{n-1}{2}\,|f|} \, d\sigma_{n-1}\Big)^{1/\lambda} \ \leq\ (2e^{5b^2/2})^{1/\lambda}.$$

It remains to note that $(2e^{5b^2/2})^{1/\lambda} = 2$ for $\lambda = 1 + \frac{5b^2}{\log 4} \leq 1 + 3.7\,b^2$. $\square$

Proof of Theorem 1.3. Let $l(\theta) = \langle v, \theta\rangle$ be the linear part of f, and recall that

$$|v|^2 = n^2 I, \qquad I = \iint \langle x, y\rangle\, f(x) f(y)\, d\sigma_{n-1}(x) d\sigma_{n-1}(y).$$

To control Gaussian tails of l under σ_{n-1}, we apply an exponential bound

$$\int e^{tl(\theta)}\, d\sigma_{n-1}(\theta) \,\leq\, e^{\frac{t^2}{2(n-1)}\,|v|^2}, \qquad t \in \mathbf{R},$$

which is implied by the logarithmic Sobolev inequality on the sphere, (3.1). Choosing $t = n-1$ and using the assumption $I \leq \frac{b_0}{n^3}$, we get $\int e^{(n-1)|l|}\, d\sigma_{n-1} \leq 2e^{b_0^2/2}$ and hence

$$\int \exp\Big\{\frac{n-1}{1+b_0^2}\,|l|\Big\}\, d\sigma_{n-1} \leq 2.$$

On the other hand, by Theorem 1.2 with the same assumption on the second derivative of f, we have

$$\int \exp\Big\{\frac{n-1}{2(1+4b^2)}\,|Tf|\Big\}\, d\sigma_{n-1} \leq 2.$$

Using $|f| \leq |Tf| + |l|$ and applying the Cauchy-Schwarz inequality, we conclude that

$$\int e^{(n-1)\,|f|/2\lambda}\, d\sigma_{n-1} \,\leq\, \Big(\int e^{(n-1)\,|Tf|/\lambda}\, d\sigma_{n-1}\Big)^{1/2} \Big(\int e^{(n-1)\,|l|/\lambda}\, d\sigma_{n-1}\Big)^{1/2} \,\leq\, 2,$$

provided that $\lambda \geq 2(1+4b^2)$ and $\lambda \geq 1 + b_0^2$. $\square$

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