STRONGLY SUBGAUSSIAN PROBABILITY DISTRIBUTIONS

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Abstract. We explore the class of probability distributions on the real line whose Laplace transform admits a strong upper bound of subgaussian type. Using Hadamard's factorization theorem, sufficient conditions for this property are given in terms of location of zeros of the associated characteristic functions in the complex plane.

1. **Introduction**

A random variable X is called subgaussian, if its distribution has subgaussian tails. More precisely, the subgaussianity refers to the bound

$$\mathbb{P}\{|X| \geq x\} \leq c_1 e^{-x^2/c_0^2}, \quad x \in \mathbb{R},$$

where the constants $c_0, c_1 > 0$ are independent of x. Without loss of generality, this inequality may be stated with $c_1 = 2$, by choosing a larger value of c_0 if necessary. An equivalent definition is that

$$\mathbb{E}\, e^{X^2/c^2} \leq 2$$

for some constant $c \geq 0$. In terms of the Orlicz norm

$$\|X\|_{\psi_2} = \inf\{c > 0 : \mathbb{E}\, \psi_2(X/c) \leq 1\},$$

generated by the Young function $\psi_2(r) = \exp\{r^2\} - 1$, this bound may be stated as $\|X\|_{\psi_2} \leq c$.

If X has mean zero, the subgaussian property is equivalent as well to the following bound for the moment generating function (or the two-sided Laplace transform) of X, namely

$$\mathbb{E}\, e^{tX} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}, \tag{1.1}$$

with some constant $\sigma \geq 0$. It is easy to verify that the optimal value of σ in (1.1) satisfies

$$C_0 \, \|X\|_{\psi_2} \leq \sigma \leq C_1 \, \|X\|_{\psi_2}$$

up to some absolute constants $C_1 > C_0 > 0$.

Immediate consequences of inequality (1.1) are the finiteness of moments of all orders of X and in particular the relations

$$\mathbb{E}X = 0 \quad \text{and} \quad \mathbb{E}X^2 \leq \sigma^2,$$

which follow by an expansion of both sides of (1.1) around $t = 0$. Here the case $\sigma^2 = \mathbb{E}X^2 = \mathrm{Var}(X)$ is of particular interest in terms of applications like the Rényi divergence of the infinite order extending the results in Rényi divergence of finite order investigated

2010 *Mathematics Subject Classification.* Primary 60E, 60F.

Key words and phrases. Subgaussian distributions, zeros, entire functions.

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in [1]. To distinguish this class of distributions, we introduce the following sharpening of the subgaussian property.

Definition. We say that the random variable X is *strongly subgaussian*, or the distribution of X is *strongly subgaussian*, if (1.1) holds with the optimal constant $\sigma^2 = \mathrm{Var}(X)$.

Thus, the word "strong" reflects the requirement that the variance of X is exactly σ^2 in contrast with the usual subgaussianity, when (1.1) is required to hold for all t with some constant σ^2.

This class of probability distributions seems rather rich. After a short discussion of their basic properties (Section 2), we list a number of interesting examples/subclasses in Section 3.

Equivalently, one could describe the class of all strongly subgaussian distributions, for example, in terms of the characteristic function

$$f(z) = \mathbb{E}\, e^{izX}, \quad z \in \mathbb{R}. \tag{1.2}$$

The subgaussian property (1.1) ensures that f has an analytic extension to the whole complex plane $\mathbb{C}$ as an entire function of order at most 2, extending the definition (1.2) to arbitrary complex values of z. Note that if the characteristic function $f(z)$ of a subgaussian distribution does not have any real or complex zeros, a well-known theorem due to Marcinkiewicz implies that the distribution of X is already Gaussian, cf. [7]. Thus, richer classes of subgaussion distributions like the strong subgaussian distributions need to have zeros. Interesting questions in this context studied in sections 8 and 9 are "what locations of a single zero of $f(z)$ would be compatible with the strong subgaussian property and the assumption that $f(z)$ is a characteristic function" and in section 5 "to what extent does the Hadamard product representation of $f(z)$ in terms of zeros correspond to a stochastic decomposition of X as a sum of independent random variables?"

In particular, applying Goldberg-Ostrovskiĭ's refinement of Hadamard's factorization theorem, we have the following simple sufficient condition for strong subgausssian distributions.

Theorem 1.1. *Let X be a subgaussian random variable with mean zero. If all zeros of $f(z)$ are real, then X is strongly subgaussian.*

This condition can easily be verified for many interesting classes including, for example, arbitrary Bernoulli sums and (finite or infinite) convolutions of uniform distributions on bounded symmetric intervals. It is however far from being necessary, as illustrated by the next generalization of Theorem 1.1.

Theorem 1.2. *Let X be a subgaussian random variable with a symmetric distribution. If all zeros of $f(z)$ with $\mathrm{Re}(z) \geq 0$ lie in the cone centered on the real axis defined by*

$$|\mathrm{Arg}(z)| \leq \frac{\pi}{8}, \tag{1.3}$$

then X is strongly subgaussian.

On the other hand, it seems that (1.3) is close to a necessary condition for the strong subgaussianity. At least, this is true for the following subclass of probability distributions.

Theorem 1.3. *Let X be a random variable with a symmetric subgaussian distribution. Suppose that f has exactly one zero $z = x + iy$ in the positive quadrant $x, y \geq 0$. Then X is strongly subgaussian, if and only if* (1.3) *holds true.*

In order to clarify this particular case, we shall study the hidden assumption that a given entire function with exactly one non-trivial zero is positive definite on the real axis. These issues are discussed in Sections 7-9 which include the proof of Theorem 1.3. As a consequence, one can partially address the following question from the theory of entire characteristic functions (which is one of the central problems in this area): What can one say about the possible location of zeros of such functions?

Theorem 1.4. *Let (z_n) be a finite or infinite sequence of non-zero complex numbers in the angle $|\mathrm{Arg}(z_n)| \leq \frac{\pi}{8}$ such that*

$$\sum_n \frac{1}{|z_n|^2} < \infty.$$

Then there exists a symmetric strongly subgaussian distribution whose characteristic function has zeros exactly at the points $\pm z_n$, $\pm \bar{z}_n$.

It will be shown that a random variable X with such distribution may be constructed as the sum $X = \sum_n X_n$ of independent strongly subgaussian random variables X_n whose characteristic functions have zeros at the points $\pm z_n$, $\pm \bar{z}_n$ for every n (and only at these points). Moreover, one may require that

$$\mathrm{Var}(X) = \Lambda \sum_n \frac{1}{|z_n|^2}$$

with any prescribed value $\Lambda \geq \Lambda_0$ where Λ_0 is a universal constant ($\Lambda_0 \sim 5.83$).

Returning to the setting of Theorems 1.1-1.2, it will also be shown that, if a strongly subgaussian random variable X is not normal, the inequality (1.1) may further be sharpened as follows: For any $t_0 > 0$, there exists $c = c(t_0)$, $0 < c < \sigma^2 = \mathrm{Var}(X)$, such that

$$\mathbb{E}\, e^{tX} \leq e^{ct^2/2}, \quad |t| \geq t_0. \tag{1.4}$$

Such a refinement is important in the study of local limit theorems (such as CLT for the Rényi divergence, cf. [1]). We prefer to discuss these applications in detail in a separate paper, and here let us only mention one result in this direction. Let $(X_n)_{n\geq 1}$ be independent copies of a random variable X with standard deviation σ, and suppose that the normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}}$$

have densities p_n for some and hence for all n large enough. If (1.4) holds true, then

$$\mathrm{ess\,sup}_{x\in\mathbb{R}}\, \frac{p_n(x) - \varphi(x)}{\varphi(x)} \to 0 \quad \text{as } n \to \infty, \tag{1.5}$$

where φ is the standard normal density. Here, in general a two-sided bound does not hold any more, in particular in the case of p_n being compactly supported.

The non-uniform local limit theorem such as (1.5) can easily be shown to imply the strong subgaussianity (1.1). On the other hand, we do not know whether or not (1.1) implies the stronger property (1.4) in the entire class of non-normal strongly subgaussian distributions, or whether it implies even the weaker property

$$L(t) < e^{\sigma^2 t^2/2}, \quad t \neq 0, \tag{1.6}$$

for the Laplace transform $L(t) = \mathbb{E}\exp\{tX\}$. Given a subgaussian random variable X with mean zero, its cumulant generating function (or the log-Laplace transform) $K(z) = \log L(z)$ is analytic for complex z in a domain of $\mathbb{C}$ not containing points z with $L(z) = 0$ and thus contains at least a neighborhood of the real axis. Then its derivative $K'(z)$ is a meromorphic function defined on $\mathbb{C}$ with poles at the zeros of $L(z)$. Obviously we get by local expansion $K(0) = K'(0) = 0$, $K''(0) = \mathrm{Var}(X)$, as well as $K^{(3)}(0) = 0$ for symmetric distributions around zero. (The derivatives of K at zero are the cumulants of X).

If we define a stronger notion of subgaussian distribution via

$$K'(t) \leq \sigma^2 t, \quad t > 0$$

(in the symmetric case), then we get of course (1.6). In fact, the assumptions of Theorem 1.2 ensure the stronger property that $K(\sqrt{t})$ is concave in $t > 0$, or equivalently $tK''(t) \leq K'(t)$. In this case we may derive as well the sharpened form (1.4) in the non-normal case, compare Proposition 2.5 in the next section.

The proof of Theorem 1.4 is given in Section 11. Theorems 1.1 and 1.2 are proved in Section 5 respectively Section 10. In Section 4 we recall basic results related to the Hadamard factorization theorem, and include some remarks on the growth of moments of strongly subgaussian random variables (Section 6). Thus, our plan is the following:

1. Introduction
2. Basic properties of strongly subgaussian distributions
3. Basic examples
4. Hadamard's and Goldberg-Ostrovskiĭ's theorems
5. Characteristic functions with real zeros
6. Growth of moments
7. More examples of strongly subgaussian distributions
8. Characterization of characteristic functions
9. Strongly subgaussian symmetric distributions with characteristic functions having exactly one non-trivial zero
10. General case of zeros in the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$
11. Proof of Theorem 1.4

2. **Basic properties of strong subgaussian distributions**

In addition to the properties $\mathbb{E}X = 0$ and $\mathbb{E}X^2 \leq \sigma^2$, the Taylor expansion of the exponential function in (1.1) around zero implies as well that necessarily

$$\mathbb{E}X^3 = 0, \quad \mathbb{E}X^4 \leq 3\sigma^4. \tag{2.1}$$

Here an equality is attained for symmetric normal distributions (but not exclusively so).

Turning to other properties and some examples, first let us emphasize the following two immediate consequences of (1.1).

Proposition 2.1. *If the random variables $X_1, \dots, X_n$ are independent and strongly subgaussian, then their sum $X = X_1 + \dots + X_n$ is strongly subgaussian, as well.*

Proposition 2.2. *If a sequence of strongly subgaussian random variables $(X_n)_{n \geq 1}$ converges weakly in distribution to a random variable X with finite second moment, and $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$ as $n \to \infty$, then X is strongly subgaussian.*

Proof. Putting $\sigma_n^2 = \mathrm{Var}(X_n)$, we have

$$\mathbb{E}\, e^{tX_n} \leq e^{\sigma_n^2 t^2/2}, \quad t \in \mathbb{R}. \tag{2.2}$$

By the assumption of weak convergence,

$$\lim_{n \to \infty} \mathbb{E}\, u(X_n) = \mathbb{E}\, u(X)$$

for any bounded, continuous function u on the real line. In particular, for any $c \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{E}\, e^{t \min(X_n, c)} = \mathbb{E}\, e^{t \min(X, c)}.$$

Hence, by (2.2), for any $t \in \mathbb{R}$,

$$\mathbb{E}\, e^{t \min(X,c)} \leq \liminf_{n \to \infty} \mathbb{E}\, e^{tX_n} \leq e^{\sigma^2 t^2/2}.$$

Letting $c \to \infty$, we get (1.1). $\square$

Combining Proposition 2.1 with Proposition 2.2, we obtain:

Corollary 2.3. *Suppose that independent, strongly subgaussian random variables $(X_n)_{n \geq 1}$ have variances satisfying $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$. Then the series*

$$X = \sum_{n=1}^{\infty} X_n$$

represents a strongly subgaussian random variable.

Here, the assumption that $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$ ensures that the series $\sum_{n=1}^{\infty} X_n$ is convergent with probability one (by the Kolmogorov theorem), so that the partial sums of the series are weakly convergent to the distribution of X.

Thus, the class of strongly subgaussian distributions is closed in the weak topology under infinite convolutions. Obviously, it is also closed when taking convex mixtures.

Proposition 2.4. *If X_n are strongly subgaussian random variables with $\mathrm{Var}(X_n) = \sigma^2$, and μ_n are distributions of X_n, then for any sequence $p_n \geq 0$ such that $\sum_{n=1}^{\infty} p_n = 1$, the random variable with distribution*

$$\mu = \sum_{n=1}^{\infty} p_n \mu_n$$

is strongly subgaussian as well and has variance $\mathrm{Var}(X) = \sigma^2$.

One should also mention that, if X is strongly subgaussian, then λX is strongly subgaussian as well, for any $\lambda \in \mathbb{R}$.

Finally, let us give a simple sufficient condition for the property (1.4). Recall the notation

$$K(t) = \log \mathbb{E}\, e^{tX}, \quad t \in \mathbb{R}.$$

Proposition 2.5. *Let X be a non-normal strongly subgaussian random variable. If the function $t \rightarrow K(\sqrt{|t|})$ is concave on the half-axis $t > 0$ and concave on the half-axis $t < 0$, then* (1.4) *holds true.*

Proof. Let $\mathrm{Var}(X) = \sigma^2$. For $t \geq 0$, write

$$\mathbb{E}\, e^{tX} = e^{\frac{1}{2}\,\sigma^2 t^2 - W(t^2)}.$$

By the assumption, $W(s)$ is non-negative and convex in $s \geq 0$, with $W(0) = 0$. In addition, it is C^∞-smooth on $(0, \infty)$. Since X is not normal, necessarily $W(s) > 0$ and $W'(s) > 0$ for all $s > 0$. Using that $W'(s) \uparrow r$ as $s \rightarrow \infty$ for some $r \in (0, \infty]$, it follows that

$$r(s) \,\equiv\, \frac{1}{s} W(s) \,=\, \int_0^1 W'(sv)\, dv \,\uparrow\, r \quad \text{as } s \rightarrow \infty.$$

In particular, given $s_0 > 0$, we have $\frac{1}{s} W(s) \geq r(s_0) > 0$ for all $s \geq s_0$, or equivalently

$$K(t) \leq \Big(\frac{1}{2}\sigma^2 - r(s_0)\Big) t^2, \quad t \geq \sqrt{s_0},$$

which is the desired conclusion. A similar argument works for $t < 0$ as well. $\square$

3. Basic examples

An application of Corollary 2.3 allows to construct a rather rich family of strongly subgaussian probability distributions.

Examples

3.1. First of all, if a random variable X has a normal distribution with mean zero and variance σ^2, that is, $X \sim N(0, \sigma^2)$, then it is strongly subgaussian. In this case,

$$\mathbb{E}\, e^{tX} = e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R},$$

so that the inequality in (1.1) becomes an equality.

3.2. If X has a symmetric Bernoulli distribution, supported on two points, say a and $-a$, then it is strongly subgaussian. If, for definiteness, $a = 1$, then $\mathrm{Var}(X) = 1$, and the Laplace transform of the distribution of X is given by

$$\mathbb{E}\, e^{tX} = \cosh(t) = \frac{e^t + e^{-t}}{2}, \quad t \in \mathbb{R}.$$

3.3. If X is an infinite Bernoulli sum, that is,

$$X = \sum_{n=1}^\infty a_n X_n, \quad \mathbb{P}\{X_n = \pm 1\} = \frac{1}{2}, \quad \sum_{n=1}^\infty a_n^2 < \infty,$$

with X_n independent symmetric Bernoulli random variables, then it is strongly subgaussian with variance $\sigma^2 = \mathrm{Var}(X) = \sum_{n=1}^{\infty} a_n^2$. The corresponding Laplace transform and characteristic function f of X are given by

$$\mathbb{E}\, e^{tX} = \prod_{n=1}^{\infty} \cosh(a_n t), \quad f(t) = \prod_{n=1}^{\infty} \cos(a_n t).$$

3.4. If the random variable X is uniformly distributed on a finite interval $[-a, a]$, $a > 0$, then it is strongly subgaussian. In this case it may be represented (in the sense of distributions) as the sum

$$X = \sum_{n=1}^{\infty} \frac{a}{2^n}\, X_n, \quad \mathbb{P}\{X_n = \pm 1\} = \frac{1}{2},$$

with X_n independent symmetric Bernoulli random variables. Hence, this case is covered by the previous example. The corresponding Laplace transform is given by

$$\mathbb{E}\, e^{tX} = \frac{\sinh(at)}{at}.$$

3.5. If the random variables X_n are independent and uniformly distributed on the interval $[-1, 1]$, then the infinite sum

$$X = \sum_{n=1}^{\infty} a_n X_n \quad \text{with} \quad \sum_{n=1}^{\infty} a_n^2 < \infty$$

represents a strongly subgaussian random variable. The corresponding Laplace transform is given by

$$\mathbb{E}\, e^{tX} = \prod_{n=1}^{\infty} \frac{\sinh(a_n t)}{a_n t}.$$

3.6. Suppose that X has density $p(x) = x^2 \varphi(x)$, where $\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ is the standard normal density. Then $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 = 3$, and the Laplace transform satisfies

$$\mathbb{E}\, e^{tX} = (1 + t^2)\, e^{t^2/2} \le e^{3t^2/2}.$$

Hence, X is strongly subgaussian.

3.7. More generally, if X has a density of the form

$$p(x) = \frac{1}{(2d-1)!!}\, x^{2d} \varphi(x), \quad x \in \mathbb{R},\ d = 1, 2, \dots,$$

then $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2 = 2d + 1$, and the Laplace transform satisfies

$$\mathbb{E}\, e^{tX} = \frac{1}{(2d-1)!!}\, H_{2d}(it)\, e^{t^2/2} \le e^{(2d+1)\, t^2/2}, \quad t \in \mathbb{R}.$$

Hence, X is strongly subgaussian. The last inequality follows from Theorem 1.1, since the Chebyshev-Hermite polynomials have real zeros, only. Note that the characteristic function of X is given by

$$\mathbb{E}\, e^{itX} = \frac{1}{(2d-1)!!}\, H_{2d}(t)\, e^{-t^2/2}.$$

4. Hadamard's and Goldberg-Ostrovskiĭ's theorems

All the previous examples may be included as partial cases of a more general setup. First, let us recall some basic definitions and notations related to the Hadamard theorem from the theory of complex variables.

Given an entire function $f(z)$, introduce

$$M_f(r) = \max_{|z| \leq r} |f(z)| = \max_{|z|=r} |f(z)|, \quad r \geq 0,$$

which characterizes the growth of f at infinity. The order of f is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

Thus, $\rho \geq 0$ is an optimal value such that, for any $\varepsilon > 0$, we have $M_f(r) < e^{r^{\rho+\varepsilon}}$ for all sufficiently large r.

If f is a polynomial, then $\rho = 0$. If ρ is finite, then the type of f is defined by

$$\tau = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho}.$$

Thus, $\tau \geq 0$ is an optimal value such that, for any $\varepsilon > 0$, we have $M_f(r) < e^{(\tau+\varepsilon)\, r^\rho}$ for all sufficiently large r. If $0 < \tau < \infty$, the function f is said to be of normal type.

For integers $p \geq 0$, introduce the functions

$$G_p(u) = (1-u) \exp\Big\{u + \frac{u^2}{2} + \dots + \frac{u^p}{p}\Big\}, \quad u \in \mathbb{C},$$

called the primary factors, with the convention that $G_0(u) = 1 - u$. Given a sequence of complex numbers $z_n \neq 0$ such that $|z_n| \uparrow \infty$, one considers a function of the form

$$\Pi(z) = \prod_{n=1}^\infty G_p(z/z_n) \tag{4.1}$$

called a canonical product. An integer $p \geq 0$ is called the genus of this product, if it is the smallest integer such that

$$\sum_{n=1}^\infty \frac{1}{|z_n|^{p+1}} < \infty. \tag{4.2}$$

There is a simple estimate $\log |G_p(u)| \leq A_p\, |u|^{p+1}$ where the constant A_p depends on p only. Therefore, the product in (4.1) is uniformly convergent as long as (4.2) is fulfilled.

See e.g. Levin [5] for the next classical theorem.

Theorem 4.1 (Hadamard). *Any entire function f of a finite order ρ can be represented in the form*

$$f(z) = z^m\, e^{P(z)} \prod_{n \geq 1} G_p(z/z_n), \quad z \in \mathbb{C}. \tag{4.3}$$

Here z_n are the non zero roots of $f(z)$, the genus of the canonical product satisfies $p \leq \rho$, $P(z)$ is a polynomial of degree $\leq \rho$, and $m \geq 0$ is the multiplicity of the zero at the origin.

In order to describe the convergence of the canonical product, assume that $f(z)$ has an infinite sequence of non zero roots z_n arranged in increasing order of their moduli so that

$$0 < |z_1| \le |z_2| \le \cdots \le |z_n| \to \infty \quad \text{as } n \to \infty.$$

Define the convergence exponent of the sequence a_n by

$$\rho_1 = \inf\Big\{\lambda > 0 : \sum_{n=1}^{\infty} \frac{1}{|z_n|^\lambda} < \infty\Big\}.$$

A theorem due to Borel asserts that the order ρ of the canonical product $\Pi(z)$ satisfies $\rho \le \rho_1$. Moreover, Theorem 6 from [5], p.16, states that the convergence exponent of the zeros of any entire function $f(z)$ does not exceed its order: $\rho_1 \le \rho$. Thus, for canonical products the convergence exponent of the zeros is equal to the order of the function: $\rho_1 = \rho$ (Theorem 7).

There is also the following elementary relation between the convergence exponent and the genus of the canonical product:

$$p \le \rho_1 \le p + 1.$$

Assuming that ρ_1 is an integer, we have that $\sum_{n=1}^{\infty} |z_n|^{-\rho_1} = \infty \Rightarrow p = \rho_1$, while $p = \rho_1 + 1$ means that the latter series is convergent.

The next theorem due to Goldberg and Ostrovskiĭ [2] refines Theorem 4.1 for the class of ridge entire functions whose all zeros are real. Recall that f is a ridge function, if it satisfies

$$|f(x + iy)| \le |f(iy)|, \quad x, y \in \mathbb{R}.$$

Theorem 4.2 (Goldberg-Ostrovskiĭ). *Suppose that an entire ridge function f of a finite order has only real roots. Then it can be represented in the form*

$$f(z) = c\, e^{i\beta z - \gamma z^2/2} \prod_{n \ge 1} \Big(1 - \frac{z^2}{z_n^2}\Big), \quad z \in \mathbb{C}, \tag{4.4}$$

for some $c \in \mathbb{C}$, $\beta \in \mathbb{R}$, $\gamma \ge 0$, and $z_n > 0$ such that $\sum_{n\ge1} z_n^{-2} < \infty$.

We refer to [2]. See also Kamynin [4] for generalizations of Theorem 4.2 to the case where the zeros of f are not necessarily real.

5. **Characteristic functions with real zeros**

We are now prepared to prove Theorem 1.1, including the relation (1.4) in the non-Gaussian case which is stronger than (1.1).

Thus, let X be a subgaussian random variable with mean zero and variance $\sigma^2 = \mathrm{Var}(X)$. Then the inequality (1.1) may be extended to the complex plane in the form

$$|\mathbb{E}\, e^{zX}| \le e^{b\,\mathrm{Re}(z)^2/2}, \quad z \in \mathbb{C},$$

for some constant $b \ge \sigma^2$. Equivalently, the characteristic function f of X admits the bound

$$|f(z)| \le e^{b\,\mathrm{Im}(z)^2/2}.$$

Hence, f is a ridge entire function of order $\rho \leq 2$. We are therefore in position to apply Theorem 4.2 which yields the representation

$$f(z) = c\, e^{i\beta z - \gamma z^2/2} \prod_{n \geq 1} \Big(1 - \frac{z^2}{z_n^2}\Big)$$

for some $c \in \mathbb{C}$, $\gamma \geq 0$, $\beta \in \mathbb{R}$, and for some finite or infinite sequence $z_n > 0$ such that $\sum_{n \geq 1} z_n^{-2} < \infty$. Note that $f(z_n) = f(-z_n) = 0$, so that $\{z_n, -z_n\}$ are all zero of f (this set may be empty). Since $f(0) = 1$ and $f'(0) = 0$, we necessarily have $c = 1$ and $\beta = 0$. Hence, this representation is simplified to

$$f(z) = e^{-\gamma z^2/2} \prod_{n \geq 1} \Big(1 - \frac{z^2}{z_n^2}\Big). \tag{5.1}$$

Since $f''(0) = -\sigma^2$, we also have

$$\frac{1}{2}\sigma^2 = \frac{1}{2}\gamma + \sum_{n \geq 1} \frac{1}{z_n^2}, \tag{5.2}$$

so that $\gamma \leq \sigma^2$. Applying (5.1) with $z = -it$, $t \in \mathbb{R}$, we get a similar representation for the Laplace transform

$$\mathbb{E}\, e^{tX} = e^{\gamma t^2/2} \prod_{n \geq 1} \Big(1 + \frac{t^2}{z_n^2}\Big). \tag{5.3}$$

Using $1 + x \leq e^x$ $(x \in \mathbb{R})$, we see that the right-hand side above does not exceed

$$e^{\gamma t^2/2} \prod_{n \geq 1} e^{t^2/z_n^2} = e^{\sigma^2 t^2/2},$$

where we used (5.2). Hence (5.3) leads to the desired bound (1.1), and Theorem 1.1 is proved.

Let us also verify the property (1.4) in the case where the random variable X is not normal. Then the product in (5.3) is not empty and therefore $\gamma < \sigma^2$. Let us rewrite (5.3) as

$$\mathbb{E}\, e^{tX} = e^{V(t^2)}, \quad V(s) = \gamma s + \sum_{n \geq 1} \log\Big(1 + \frac{s}{z_n^2}\Big).$$

Since the function V is concave, it remains to refer to Proposition 2.5. $\square$

Remark. Using (5.2), let us rewrite (5.1) with $z = t \in \mathbb{R}$ in the form

$$\begin{aligned} f(t) &= \exp\Big\{-\frac{1}{2}\Big(\gamma - \sum_{n \geq 1} \frac{1}{z_n^2}\Big)\, t^2\Big\} \prod_{n \geq 1} \Big(1 - \frac{t^2}{z_n^2}\Big)\, e^{-\frac{t^2}{2z_n^2}} \\ &= e^{-(3\gamma - \sigma^2)\, t^2/4} \prod_{n \geq 1} \Big(1 - \frac{t^2}{z_n^2}\Big)\, e^{-\frac{t^2}{2z_n^2}}. \end{aligned} \tag{5.4}$$

The terms in the product represent characteristic funtions of random variables $\frac{1}{z_n} X_n$ such that all X_n have density $p(x) = x^2 \varphi(x)$ which we discussed in Example 3.6. Hence, if

$$\gamma \geq \sum_{n \geq 1} \frac{1}{z_n^2} \quad \text{or equivalently} \quad \frac{1}{3}\sigma^2 \leq \gamma \leq \sigma^2,$$

the function $f(t)$ in (5.4) represents the characteristic function of the random variable

$$X = cZ + \sum_{n \geq 1} \frac{1}{z_n} X_n,$$

assuming that X_n are independent, and Z is a standard normal random variable independent of all X_n. Necessarily, $c^2 = \frac{3}{2}\gamma - \frac{1}{2}\sigma^2$.

On the other hand, the formula (5.1) does not always define a characteristic function. For example, when there is only one term in the product, we obtain $f(t) = e^{-\gamma t^2/2}(1 - \frac{t^2}{z_1^2})$. It is a characteristic function, if and only if $\gamma \geq \frac{1}{z_1^2}$ (cf. e.g. [6], p. 34). We will return to this question in Section 8.

6. **Growth of moments**

Let X be a mean zero random variable with characteristic function $f(t) = \mathbb{E}\, e^{itX}$. Note that X is subgaussian, when for some $b \geq 0$ and $t_0 \geq 0$,

$$\mathbb{E}\, e^{tX} \leq e^{bt^2/2}, \quad |t| \geq t_0. \tag{6.1}$$

In terms of the characteristic function, this is equivalent to the statement that f can be extended to the complex plane as an entire function of order $\rho \leq 2$ and some finite type τ (in the case $\rho = 2$). Assuming this, one can reformulate the refinement of Theorem 1.1 in the form (1.4) as follows: If f has only real zeros, and X is not normal with $\rho = 2$, then

$$\tau < \frac{1}{2}\,\sigma^2, \quad \sigma^2 = \mathrm{Var}(X).$$

On the other hand, both the order ρ and the type τ of f may be related to the coefficients c_n in the Taylor expansion

$$f(z) = \sum_{n=1}^{\infty} c_n z^n.$$

It is well-known (see e.g. [5], p.4) that

$$\rho = \limsup_{n \to \infty} \frac{n \log n}{\log \frac{1}{|c_n|}}, \quad (\tau e \rho)^{1/\rho} = \limsup_{n \to \infty} \big(n^{1/\rho} |c_n|^{1/n}\big).$$

Introduce the moments and norms

$$\alpha_n = \mathbb{E} X^n, \quad \beta_n = \mathbb{E}\, |X|^n, \quad \|X\|_n = \beta_n^{1/n},$$

so that $c_n = \frac{i^n}{n!}\, \alpha_n$, $|c_n| \leq \frac{1}{n!}\, \beta_n$. Starting from (6.1), we have

$$|\mathbb{E}\, e^{zX}| \leq e^{b\,|z|^2/2}, \quad z \in \mathbb{C},\ |z| \geq t_0.$$

Hence $M_f(r) \leq e^{br^2/2}$ for all $r \geq t_0$, and, by Cauchy's theorem,

$$|c_{2n}| \leq \frac{M_f(r)}{r^{2n}} \leq \frac{e^{br^2/2}}{r^{2n}}.$$

Optimizing over all $r \geq t_0$ (attained when $br^2 = 2n$ when n is large enough), we get

$$\beta_{2n} \leq (2n)!\, \frac{(be)^n}{(2n)^n}, \quad n \geq n_0,$$

so that, by Stirling's formula,

$$\|X\|_{2n} \leq d_n \sqrt{\frac{2bn}{e}}, \quad d_n \rightarrow 1 \text{ as } n \rightarrow \infty. \tag{6.2}$$

Conversely, starting from (6.2) and using $\|X\|_{2n-1} \leq \|X\|_{2n}$, we have a similar upper bound on the moments, and as a result, we arrive at (6.1) with any $b' > b$ in pace of b by choosing a suitable value of t_0 (see for details the proof of Theorem 2 in [5], pp. 4-5).

In order to compare with moments of Gaussian random variables, note that, if $Z \sim N(0,1)$, then

$$\mathbb{E}Z^{2n} = (2n-1)!! = \frac{(2n)!}{2^n n!} \sim \sqrt{2}\,\Big(\frac{2n}{e}\Big)^{2n},$$

so that

$$\|Z\|_{2n} = d_n \sqrt{\frac{2n}{e}}, \quad d_n \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Therefore, we have the following characterization of property (6.1) in terms of moments.

Proposition 6.1. *Let X be a mean zero random variable. If it satisfies* (6.1) *with parameter $b \geq 0$, then, for any $b' > b$, the inequality*

$$\|X\|_{2n} \leq \sqrt{b'}\,\|Z\|_{2n} \tag{6.3}$$

holds true for all n large enough. Conversely, the latter implies (6.1) *for any prescribed value $b > b'$ with some $t_0 > 0$.*

Thus, by (1.4), if the characteristic function $f(z)$ of a subgaussian, non-normal random variable X such that $\mathbb{E}X = 0$, $\mathrm{Var}(X) = \sigma^2$ has only real zeros in the complex plane, then (6.3) is fulfilled with some $b' < \sigma^2$ for all n large enough.

7. More examples of strongly subgaussian distributions

In connection with the problem of location of zeros, we now examine probability distributions with characteristic functions of the form

$$f(t) = e^{-t^2/2}\,(1 - \alpha t^2 + \beta t^4), \tag{7.1}$$

where $\alpha, \beta \in \mathbb{R}$ are parameters. It was already mentioned that when $\beta = 0$, we obtain a characteristic function

$$f(t) = e^{-t^2/2}\,(1 - \alpha t^2),$$

if and only if $0 \leq \alpha \leq 1$. As we will see, in the general case, it is necessary that $\beta \geq 0$ for $f(t)$ to be a characteristic function (although negative values of α are possible for small β). Before deriving a full characterization, first let us emphasize the following.

Proposition 7.1. *Given $\beta \geq 0$, a random variable X with characteristic function of the form* (7.1) *is strongly subgaussian, if and only if α satisfies $\alpha \geq \sqrt{2\beta}$.*

Proof. Recall that X is strongly subgaussian, if and only if, for all $t \in \mathbb{R}$,

$$\mathbb{E}\,e^{tX} \leq e^{\sigma^2 t^2/2}, \quad \sigma^2 = -f''(0). \tag{7.2}$$

Near zero, the characteristic function $f(t)$ as in (7.1) behaves like a quadratic polynomial

$$f(t) = 1 - \frac{1}{2}\,t^2 - \alpha t^2 + O(t^4),$$

so that $\sigma^2 = 1 + 2\alpha$ (in particular, $\alpha \geq -\frac{1}{2}$). Hence, applying (7.1) to the values $-it$ instead of t, one may rewrite (7.2) equivalently (multiplying both sides by $\exp(-t^2/2)$) as

$$1 + \alpha t^2 + \beta t^4 \leq e^{\alpha t^2} = 1 + \alpha t^2 + \frac{1}{2}\,\alpha^2 t^4 + \frac{1}{6}\,\alpha^3 t^6 + \dots$$

If $\alpha \geq 0$, this inequality holds for all $t \in \mathbb{R}$, if and only if $\alpha^2 \geq 2\beta$. As for the case $\alpha < 0$, this is impossible, since then $e^{\alpha t^2} \to 0$ as $t \to \infty$ exponentially fast. $\square$

As already emphasized, if a random variable X is subgaussian (even if it is not strongly subgaussian), its characteristic function $f(t)$ may be extended to the complex plane as an entire function

$$f(z) = \mathbb{E}\, e^{izX}, \quad z \in \mathbb{C},$$

of order $\rho \leq 2$ and of finite type, like in the strongly subgaussian case (7.2). If $z = x + iy$, $x, y \in \mathbb{R}$, is a zero of f, that is,

$$\mathbb{E}\, \cos(xX)\, e^{-yX} = 0, \quad \mathbb{E}\, \sin(xX)\, e^{-yX} = 0,$$

then so is $-x + iy$,

$$\begin{aligned} f(-x+iy) &= \mathbb{E}\, e^{i(-x+iy)X} \\ &= \mathbb{E}\, \cos(xX)\, e^{-yX} + i\, \mathbb{E}\, \sin(xX)\, e^{-yX} = 0. \end{aligned}$$

Thus, $-\bar{z}$ is a zero of f as well. If in addition the distribution of X is symmetric about zero, then $-z$ and $\bar{z}$ will also be zeros of f. Thus, in this case with every non-real zero z, the characteristic function has 3 more distinct zeros, and hence we have 4 distinct zeros $\pm x \pm iy$, $x, y > 0$. One can now apply Proposition 7.1 to prove Theorem 1.3.

Proof of Theorem 1.3. Given a random variable X with a symmetric subgaussian distribution, suppose that its characteristic function has exactly one zero $z = x + iy$ in the positive quadrant $x, y \geq 0$. We need to show that X is strongly subgaussian, if and only if

$$0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{8}. \tag{7.3}$$

The case where $z = x$ is real is covered by Theorem 1.1. The argument below also works in this case, but for definiteness let us assume that z is complex, so that $x, y > 0$ (the case $x = 0$ and $y > 0$ is impossible, since then $f(z) = f(iy) \geq 1$).

Thus, let $f(z)$ have four distinct roots $z_1 = z$, $z_2 = -z = -x - iy$, $z_3 = \bar{z} = x - iy$, $z_4 = -\bar{z} = -x + iy$. Applying Hadamard's theorem, we get a representation

$$f(z) = e^{P(z)} \Big(1 - \frac{z}{z_1}\Big)\Big(1 - \frac{z}{z_2}\Big)\Big(1 - \frac{z}{z_3}\Big)\Big(1 - \frac{z}{z_4}\Big),$$

where $P(z)$ is a quadratic polynomial. Since $f(0) = 1$, necessarily $P(0) = 0$. Also, by the symmetry of the distribution of X, we have $f(z) = f(-z)$, which implies $P(z) = P(-z)$ for all $z \in \mathbb{C}$. It follows that $P(z)$ has no linear term, so that $P(z) = -\frac{1}{2}\gamma z^2$ for some $\gamma \in \mathbb{C}$. Thus, putting $w = a + bi = \frac{1}{x+iy}$, we have

$$\begin{aligned} f(t) &= e^{-\gamma t^2/2}\,(1-wt)(1+wt)(1-\bar{w}t)(t+\bar{w}t) \\ &= e^{-\gamma t^2/2}\left(1-(w^2+\bar{w}^2)\,t^2+|w|^4t^4\right) \\ &= e^{-\gamma t^2/2}\left(1-2(a^2-b^2)\,t^2+(a^2+b^2)^2\,t^4\right). \end{aligned} \tag{7.4}$$

Comparing both sides of (7.4) near zero according to Taylor's expansion, we get that

$$\gamma + 4(a^2 - b^2) = \sigma^2. \tag{7.5}$$

In particular, γ must be a real number, necessarily positive (since otherwise $f(t)$ would not be bounded on the real axis). Moreover, the case $a = |b|$ is impossible, since then

$$f(t) = e^{-\sigma^2 t^2/2}\,(1 + 2b^4t^4).$$

Rescaling the variable and applying Proposition 7.1 with $\alpha = 0$, we would conclude that the random variable X is not strongly subgaussian.

Thus, let $a \neq |b|$ (as we will see, necessarily $\gamma > \sigma^2$). Again rescaling of the t-variable, one may assume that $\gamma = 1$ in which case the representation (7.4) becomes

$$f(t) = e^{-t^2/2}\left(1-2(a^2-b^2)\,t^2+(a^2+b^2)^2\,t^4\right).$$

One can now apply Proposition 7.1 with parameters

$$\alpha = 2(A-B),\ \beta = (A+B)^2, \quad \text{where}\ \ A = a^2,\ B = b^2.$$

Since the condition $\alpha \geq 0$ is necessary for $f(t)$ to be a characteristic function of a strongly subgaussian distribution, we may assume that $A \geq B$ (in fact, we have $A > B$, since $a \neq |b|$). The condition $\beta \leq \frac{1}{2}\,\alpha^2$, that is, $2(A-B)^2 \geq (A+B)^2$ is equivalent to

$$(A+B)^2 \geq 8AB \iff (a^2+b^2)^2 \geq 8a^2b^2.$$

To express this in polar coordinates, put $a = r\cos\theta$, $b = r\sin\theta$ with $r^2 = a^2 + b^2$ and $|\theta| \leq \frac{\pi}{2}$. Since $A \geq B$, that is $a \geq |b|$, necessarily $|\theta| \leq \frac{\pi}{4}$, and the above turns out to be the same as

$$\cos^2(\theta)\,\sin^2(\theta) \leq \frac{1}{8} \iff \sin^2(2\theta) \leq \frac{1}{2} \iff |\theta| \leq \frac{\pi}{8}.$$

Since $\theta = \mathrm{Arg}(a + bi) = -\mathrm{Arg}(z)$, the desired characterization (7.3) follows. $\square$

8. **Characterization of characteristic functions**

It remains to decide whether or not the characteristic functions in Proposition 7.1 with non-real zeros do exist. Therefore, we now turn to the characterization of the property that the functions of the form

$$f(t) = e^{-t^2/2}\,(1 - \alpha t^2 + \beta t^4) \tag{8.1}$$

are positive definite (that is, they represent characteristic functions). Note that the more general class of functions

$$f(t) = e^{-\gamma t^2/2}\,(1 - \alpha t^2 + \beta t^4), \quad \gamma > 0,$$

is reduced to (8.1) by rescaling the t-variable.

Proposition 8.1. *The equality* (8.1) *defines a characteristic function, if and only if the point* (α, β) *belongs to one of the following two regions:*

$$4\beta - 2\sqrt{\beta(1-2\beta)} \le \alpha \le 3\beta + 1, \qquad 0 \le \beta \le \frac{1}{3}, \tag{8.2}$$

or

$$4\beta - 2\sqrt{\beta(1-2\beta)} \le \alpha \le 4\beta + 2\sqrt{\beta(1-2\beta)}, \quad \frac{1}{3} \le \beta \le \frac{1}{2}. \tag{8.3}$$

The expression on the left-hand sides in (8.2)-(8.3) is negative, if and only if $\beta < \frac{1}{6}$. Hence, for such values of β, the parameter α may be negative.

Combining Propositions 7.1 and 8.1, we obtain a full characterization of strongly subgaussian distributions with characteristic functions of the form (8.1). To this aim, one should complement (8.2)-(8.3) with the bound $\alpha \ge \sqrt{2\beta}$. To describe the full region, we need to solve the corresponding inequalities. First, it should be clear that $\sqrt{2\beta}$ is smaller than the right-hand sides of (8.2)-(8.3) for all $0 \le \beta \le \frac{1}{2}$. In this β-interval, we also have

$$\begin{aligned} 4\beta - 2\sqrt{\beta(1-2\beta)} \le \sqrt{2\beta} \quad &\Longleftrightarrow \quad 16\beta^2 \le 2\beta + 4\beta(1-2\beta) + 4\beta\sqrt{2(1-2\beta)} \\ &\Longleftrightarrow \quad 12\beta - 3 \le 2\sqrt{2(1-2\beta)}. \end{aligned}$$

The latter is fulfilled automatically for $\beta \le \frac{1}{4}$. For $\frac{1}{4} \le \beta \le \frac{1}{2}$, squaring the above inequality, we arrive at the quadratic inequality

$$144\beta^2 - 56\beta + 1 \le 0.$$

The corresponding quadratic equation has two real roots, one of which 0.0188... is out of our interval, while the other one

$$\beta_0 = \frac{1}{36}\,(7 + 2\sqrt{10}) \sim 0.3701...$$

belongs to the interval $(\frac{1}{3}, \frac{1}{2})$. Therefore, the left-hand side in (8.2) should be replaced with $\sqrt{2\beta}$ on the whole interval $0 \le \beta \le \frac{1}{3}$, while the lower bounds in (8.3) should be properly changed for $\beta \le \beta_0$ and $\beta \ge \beta_0$. That is, we obtain:

Proposition 8.2. *The equality* (8.1) *defines a characteristic function of a strongly subgaussian distribution, if and only if*

$$\sqrt{2\beta} \le \alpha \le 3\beta + 1, \qquad 0 \le \beta \le \frac{1}{3}, \tag{8.4}$$

$$\sqrt{2\beta} \le \alpha \le 4\beta + 2\sqrt{\beta(1-2\beta)}, \quad \frac{1}{3} \le \beta \le \beta_0, \tag{8.5}$$

$$4\beta - 2\sqrt{\beta(1-2\beta)} \le \alpha \le 4\beta + 2\sqrt{\beta(1-2\beta)}, \quad \beta_0 \le \beta \le \frac{1}{2}. \tag{8.6}$$

Proof of Proposition 8.1. Recall that the Chebyshev-Hermite polynomial $H_k(x)$ of degree $k = 0, 1, 2, \dots$ is defined via the identity

$$\varphi^{(k)}(x) = (-1)^k H_k(x) \varphi(x).$$

In particular, $H_0(x) = 1$, $H_2(x) = x^2 - 1$, $H_4(x) = x^4 - 6x^2 + 3$.

Differentiating the equality

$$e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx} \varphi(x)\,dx$$

k times, we therefore get the identity

$$(-1)^k H_k(t)\, e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx}\,(ix)^k \varphi(x)\,dx.$$

Hence, by the Fourier inversion formula,

$$(ix)^k \varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\,(-1)^k H_k(t)\, e^{-t^2/2}\,dt.$$

Applying this equality with even orders $2k$ and changing notations, we arrive at

$$t^{2k}\, e^{-t^2/2} = \int_{-\infty}^{\infty} (-1)^k H_{2k}(x) \varphi(x)\, e^{itx}\,dx.$$

Therefore, the function in (8.1) represents the Fourier transform of the (continuous) function

$$\begin{aligned} p(x) &= \big(1 + \alpha H_2(x) + \beta H_4(x)\big)\, \varphi(x) \\ &= \big(1 + \alpha(x^2 - 1) + \beta(x^4 - 6x^2 + 3)\big)\, \varphi(x) \\ &= \big((1 - \alpha + 3\beta) + (\alpha - 6\beta)x^2 + \beta x^4\big)\, \varphi(x), \end{aligned}$$

whose total integral is $f(0) = 1$. As a consequence, $p(x)$ represents a probability density, if and only if

$$\psi(y) \equiv (1 - \alpha + 3\beta) + (\alpha - 6\beta)y + \beta y^2 \geq 0 \quad \text{for all } y \geq 0.$$

Choosing $y = 0$ and $y \to \infty$, we obtain necessary conditions

$$\alpha \leq 3\beta + 1, \quad \beta \geq 0. \tag{8.7}$$

Assuming this, a sufficient condition for the inequality $\psi(y) \geq 0$ to hold for all $y \geq 0$ is $\alpha \geq 6\beta$. As a result, we obtain a natural region for the parameters, namely

$$6\beta \leq \alpha \leq 3\beta + 1, \quad 0 \leq \beta \leq \frac{1}{3}, \tag{8.8}$$

for which $f(t)$ in (8.1) is a characteristic function.

In the case $\alpha < 6\beta$, we obtain a second region. Note that the quadratic function $\psi(y) = c_0 + 2c_1 y + c_2 y^2$ with $c_0, c_2 \geq 0$ and $c_1 < 0$ is non-negative in $y \geq 0$, if and only if $c_1^2 \leq c_0 c_2$. For the coefficients $c_2 = \beta > 0$ and $2c_1 = \alpha - 6\beta < 0$, the condition $c_1^2 \leq c_0 c_2$ means that

$$\begin{aligned} \Big(\frac{\alpha - 6\beta}{2}\Big)^2 \leq (1 - \alpha + 3\beta)\beta \quad &\Longleftrightarrow \quad \alpha^2 - 8\alpha\beta + 24\beta^2 \leq 4\beta \\ &\Longleftrightarrow \quad (\alpha - 4\beta)^2 \leq 4\beta(1 - 2\beta). \end{aligned}$$

Thus, necessarily $\beta \leq \frac{1}{2}$, and then admissible values of α are described by the relations

$$4\beta - 2\sqrt{\beta(1 - 2\beta)} \leq \alpha \leq 4\beta + 2\sqrt{\beta(1 - 2\beta)} \tag{8.9}$$

in addition to the assumption $\alpha < 6\beta$ and the necessary conditions in (8.7).

If $\frac{1}{3} \leq \beta \leq \frac{1}{2}$, we arrive at the desired relations in (8.3), since

$$4\beta + 2\sqrt{\beta(1 - 2\beta)} \leq 3\beta + 1 \leq 6\beta.$$

If $\beta \le \frac{1}{3}$, then $6\beta \le 3\beta + 1$. In the case $\alpha < 6\beta$, the upper bound in (8.9) will hold automatically, since

$$6\beta \le 4\beta + 2\sqrt{\beta(1-2\beta)} \quad \text{for all } 0 \le \beta \le \frac{1}{3}.$$

So, for the values $\alpha < 6\beta$ and $\beta \le \frac{1}{3}$, (8.9) simplifies to

$$4\beta - 2\sqrt{\beta(1-2\beta)} \le \alpha \le 6\beta, \quad 0 < \beta \le \frac{1}{3}. \tag{8.10}$$

It remains to take the union of the two regions described by (8.10) with (8.8), and then we arrive at (8.2). □

9. Strongly subgaussian symmetric distributions with characteristic functions having exactly one non-trivial zero

One may illustrate Proposition 8.2 by the following simple example. For $\beta = \frac{1}{3}$, admissible values of α cover the interval $\sqrt{2/3} \le \alpha \le 2$, following both (8.4) and (8.5). Choosing $\alpha = \sqrt{2/3}$, we obtain the characteristic function

$$f(t) = e^{-t^2/2}\left(1 - \sqrt{\frac{2}{3}}\,t^2 + \frac{1}{3}t^4\right)$$

of a strongly subgaussian random variable. It has four distinct complex zeros z_k defined by $z^2 = r^2\,(1 \pm i)$ with $r^2 = \frac{1}{3}\sqrt{2/3}$, so

$$z_1 = (2r)^{1/4}\,e^{i\pi/8}, \quad z_2 = (2r)^{1/4}\,e^{-i\pi/8}, \quad z_3 = (2r)^{1/4}\,e^{7i\pi/8}, \quad z_4 = (2r)^{1/4}\,e^{-7i\pi/8}.$$

Note that $|\mathrm{Arg}(z_{1,2})| = \frac{\pi}{8}$. As already mentioned, it was necessary that $|\mathrm{Arg}(z)| \le \frac{\pi}{8}$ for all zeros with $\mathrm{Re}(z) > 0$ in the class of all strongly subgaussian probability distributions with characteristic functions of the form (8.1).

In order to describe the possible location of zeros, let us see what Proposition 8.2 is telling us about the class of functions

$$f(t) = e^{-t^2/2}\,(1 - wt)(1 + wt)(1 - \bar{w}t)(t + \bar{w}t), \quad t \in \mathbb{R}, \tag{9.1}$$

with $w = a + bi$. Thus, in the complex plane $f(z)$ has two or four distinct zeros $z = \pm 1/w$, $z = \pm 1/\bar{w}$ depending on whether $b = 0$ or $b \neq 0$. Note that

$$|\mathrm{Arg}(z)| = |\mathrm{Arg}(w)|$$

when z and w are taken from the half-plane $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$.

Proposition 9.1. *Let $w = a + bi$ with $a > 0$. The function $f(t)$ in (9.1) represents a characteristic function of a strongly subgaussian random variable, if and only if*

$$a \le 2^{-1/4} \sim 0.8409,$$

while $|b|$ is sufficiently small. More precisely, this is the case whenever $|b| \le b(a)$ with a certain function $b(a) \ge 0$ such that $b(2^{-1/4}) = 0$ and $b(a) > 0$ for $0 < a < 2^{-1/4}$.

Moreover, there exists a universal constant $0 < a_0 < 2^{-1/4}$, $a_0 \sim 0.7391$, such that for $0 \le a \le a_0$ and only for these a-values, the property $|b| \le b(a)$ is equivalent to the angle requirement $\mathrm{Arg}(w) \le \frac{\pi}{8}$. As for the values $a_0 < a \le 2^{-1/4}$, this angle must be smaller.

Proof. We may assume that $b \geq 0$. The function in (9.1) may be expressed in the form

$$f(t) = e^{-t^2/2}\,(1 - \alpha t^2 + \beta t^4) \tag{9.2}$$

with parameters

$$\alpha = 2(A - B),\ \beta = (A + B)^2, \quad \text{where} \ \ A = a^2,\ B = b^2.$$

Since the condition $\alpha \geq 0$ is necessary for $f(t)$ to be a characteristic function of a strongly subgaussian distribution, we may require that $a \geq b$, that is, $A \geq B$. Recall that

$$\mathrm{Arg}(w) \leq \frac{\pi}{8} \iff \alpha \geq \sqrt{2\beta} \iff b \leq \frac{1}{\sqrt{2}+1}\,a. \tag{9.3}$$

In fact, as easy to check, if $w = re^{i\theta}$, then

$$\alpha^2 - 2\beta = 2\beta\cos(4\theta).$$

In order to apply Proposition 8.2, first note that the above parameters satisfy $\alpha \leq 2\sqrt{\beta}$. In this case, the upper bounds in (8.4)-(8.6) are fulfilled automatically. Therefore, we only need to take into account the lower bounds in (8.4)-(8.6). Thus, $f(t)$ in (9.2) represents the characteristic function of a strongly subgaussian distribution, if and only if

$$\frac{1}{\sqrt{2}}\,(A + B) \leq A - B \qquad \text{for} \ \ 0 < A + B < \sqrt{\beta_0} \tag{9.4}$$

or

$$2(A + B)^2 - (A + B)\sqrt{1 - 2(A + B)^2} \leq A - B \qquad \text{for} \ \ \sqrt{\beta_0} \leq A + B \leq \frac{1}{\sqrt{2}}, \tag{9.5}$$

where $\beta_0 = \frac{1}{36}\,(7 + 2\sqrt{10}) \sim 0.3701...$ Since the condition $A + B \leq 2^{-1/2}$ is necessary, we should require that $a \leq 2^{-1/4}$. Moreover, for $a = 2^{-1/4}$, there is only one admissible value $b = 0$, when w is a real number, $w = 2^{-1/4}$.

Let us recall that

$$\sqrt{2\beta} < 4\beta - 2\sqrt{\beta(1 - 2\beta)} \quad \text{for} \ \ \beta_0 < \beta \leq \frac{1}{2},$$

in which case there is a strict inequality $\alpha > \sqrt{2\beta}$ for admissible values of α in (8.6). Hence, $\mathrm{Arg}(w) < \frac{\pi}{8}$ according to (8.3). Thus, $\mathrm{Arg}(w) < \frac{\pi}{8}$ for the region described in (9.5).

Turning to the region of couples (A, B) as in (9.4), let us fix a value $0 < A < \sqrt{\beta_0}$. The first inequality in (9.4) is equivalent to

$$B \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}\,A = \frac{1}{(\sqrt{2}+1)^2}\,A,$$

which is the same as (9.3). The value $B = \frac{1}{(\sqrt{2}+1)^2}\,A$ does satisfy the second constraint, if and only if

$$\Big(1 + \frac{1}{(\sqrt{2}+1)^2}\Big)\,A \leq \sqrt{\beta_0},$$

which is equivalent to $0 < a \leq a_0$ with

$$a_0 = \beta_0^{1/4}\,\frac{\sqrt{2}+1}{\sqrt{4 + 2\sqrt{2}}} \sim 0.7391. \tag{9.6}$$

Therefore, in this a-interval Proposition 9.1 holds true with $b(a) = \frac{1}{\sqrt{2}+1}\,a$.

Now, let $a_0 < a < 2^{-1/4}$. Since $A < \frac{1}{2}$, both (9.4) and (9.5) are fulfilled for all B small enough. Indeed, if $A \geq \sqrt{\beta_0}$ and $B = 0$, (9.5) becomes

$$2A^2 - A\sqrt{1-2A^2} \leq A,$$

which holds with a strict inequality sign. To show that (9.5) is solved as $B \leq B(A)$ for a certain positive function $B(A)$, it is sufficient to verify that the left-hand side of (9.5) is increasing in B (since the right-hand side is decreasing in B). Consider the function

$$u(x) = 2x^2 - x\sqrt{1-2x^2}, \quad \sqrt{\beta_0} \leq x < \frac{1}{\sqrt{2}}.$$

We have

$$u'(x) = 4x - \sqrt{1-2x^2} + \frac{2x^2}{\sqrt{1-2x^2}} \geq 0$$

for $x \geq \frac{1}{2}$, hence for $x \geq \sqrt{\beta_0}$. Thus, $u(A+B)$ is increasing in B, proving the claim. □

10. **General case of zeros in the angle** $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$

We are now prepared to prove Theorem 1.2, which covers the case where the zeros of the characteristic function

$$f(z) = \mathbb{E}\, e^{izX}, \quad z \in \mathbb{C},$$

of the subgaussian random variable X are not necessarily real, but belong to the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$. Let us state it once more together with the stronger property (1.4).

Theorem 10.1. *Let X be a subgaussian random variable with a symmetric distribution. If all zeros of $f(z)$ with $\mathrm{Re}(z) \geq 0$ lie in the angle $|\mathrm{Arg}(z)| \leq \frac{\pi}{8}$, then X is strongly subgaussian. Moreover, if X is not normal, then for any $t_0 > 0$, there exists $c = c(t_0)$, $0 < c < \sigma^2 = \mathrm{Var}(X)$, such that*

$$\mathbb{E}\, e^{tX} \leq e^{ct^2/2}, \quad |t| \geq t_0. \tag{10.1}$$

In the proof of (10.1) we employ Proposition 2.5, which asserts that (10.1) would follow from the property that the function $t \to \log \mathbb{E}\, e^{\sqrt{t}X}$ is concave on the positive half-axis $t \geq 0$ (in the symmetric case). In this connection let us remind Proposition 7.1: A random variable ξ with characteristic function

$$f_\xi(t) = e^{-t^2/2}\,(1 - \alpha t^2 + \beta t^4)$$

is strongly subgaussian, if and only if $\beta \geq 0$ and $\alpha \geq \sqrt{2\beta}$. In fact, the latter description is also equivalent to the concavity of the function

$$t \to \log \mathbb{E}\, e^{\sqrt{t}\xi} = -\frac{1}{2}\,t + \log(1 + \alpha t + \beta t^2), \quad t \geq 0.$$

That is, we have:

Lemma 10.2. *Given $\alpha, \beta \geq 0$, the function $Q(t) = \log(1+\alpha t+\beta t^2)$ is concave in $t \geq 0$, if and only if $\alpha \geq \sqrt{2\beta}$, and then the function $R(t) = \alpha t - Q(t)$ is convex and non-decreasing.*

Indeed, by the direct differentiation,

$$R'(t) = \frac{(\alpha^2 - 2\beta)t + \alpha\beta t}{1 + \alpha t + \beta t^2}$$

and

$$Q''(t) = -\frac{(\alpha^2 - 2\beta) + 2\alpha\beta t + 2\beta^2 t^2}{(1 + \alpha t + \beta t^2)^2},$$

from which the claim readily follows.

Proof of Proposition 10.1. We may assume that X is not normal. By the symmetry assumption, with every zero $z = x + iy$, we have more zeros $\pm x \pm iy$. So, one may arrange all zeros in increasing order of their moduli and by coupling $\pm z_1, \pm \bar{z}_1, \dots$. Let us enumerate only the zeros $z_n = x_n + iy_n$ lying in the quadrant $x_n \geq 0$, $y_n \leq 0$ and deal with $-z_n, \bar{z}_n, -\bar{z}_n$ as associated zeros. If z_n is real, then we have only one associated zero $-z_n$. For simplicity of notations, let us assume that all zeros are complex.

Since X is subgaussian, the characteristic function $f(t)$ may be extended from the real line to the complex plane as an entire function satisfying

$$|f(z)| \leq e^{b\, \mathrm{Im}(z)^2/2}, \quad z \in \mathbb{C},$$

for some constant $b \geq 0$. Therefore, f is a ridge entire function of order $\rho \leq 2$ and of a finite type like in the strongly subgaussian case. Thus, Hadamard's theorem is applicable, with parameters $\rho \leq 2$ and $p \leq 2$. In this case, the representation (4.1) takes the form

$$f(z) = e^{P(z)} \prod_{n \geq 1} G_p(z/z_n)\, G_p(z/\bar{z}_n) G_p(-z/z_n)\, G_p(-z/\bar{z}_n).$$

Here, the genus of the canonical product satisfies $p \leq 2$, and $P(z)$ is a polynomial of degree at most 2 such that $P(0) = 0$. Thus, putting in the sequel

$$w_n = \frac{1}{z_n} = a_n + b_n i,$$

we have

$$f(z) = e^{i\beta z - \gamma z^2/2} \prod_{n \geq 1} \pi_{p,n}(z) \tag{10.2}$$

for some $\beta, \gamma \in \mathbb{C}$, where

$$\pi_{p,n}(z) = G_p(w_n z)\, G_p(-w_n z) G_p(\bar{w}_n z)\, G_p(-\bar{w}_n z).$$

By the symmetry assumption, $f(-z) = f(z)$ for all $z \in \mathbb{C}$. Since also $\pi_{p,n}(-z) = \pi_{p,n}(z)$, we conclude that $\beta = 0$.

Put

$$\alpha_n = w_n^2 + \bar{w}_n^2 = 2(a_n^2 - b_n^2), \quad \beta_n = |w_n|^4 = (a_n^2 + b_n^2)^2.$$

There are three cases for the values of the genus, $p = 0$, $p = 1$, and $p = 2$, for which

$$G_0(u) = 1 - u, \quad G_1(u) = (1 - u)\, e^u, \quad G_2(u) = (1 - u)\, e^{u + \frac{u^2}{2}}.$$

Since

$$G_1(u) G_1(-u) = 1 - u^2 = G_0(u) G_0(-u)$$

and

$$G_2(-u) G_2(u) = (1 - u^2)\, e^{u^2},$$

(10.2) is simplified to
$$f(z) = e^{-\gamma z^2/2} \prod_{n \geq 1} Q_{p,n}(z), \tag{10.3}$$
where
$$\begin{aligned} Q_{0,n}(z) \ = \ Q_{1,n}(z) \ &= \ (1 - w_n^2 z^2)(1 - \bar{w}_n^2 z^2) \\ &= \ 1 - (w_n^2 + \bar{w}_n^2) z^2 + |w|^4 z^4 \ = \ 1 - \alpha_n z^2 + \beta_n z^4 \end{aligned}$$
and
$$\begin{aligned} Q_{2,n}(z) \ &= \ (1 - w_n^2 z^2)(1 - \bar{w}_n^2 z^2)\, e^{(w_n^2 + \bar{w}_n^2) z^2} \\ &= \ (1 - \alpha_n z^2 + \beta_n z^4)\, e^{\alpha_n z^2}. \end{aligned}$$
These functions are real-valued for $z = t \in \mathbb{R}$, as well as $f(t)$, by the symmetry assumption on the distribution of X. Hence, necessarily $\gamma \in \mathbb{R}$. Moreover, we have $\gamma \geq 0$, since otherwise $f(t)$ would not be bounded on the real axis $t \in \mathbb{R}$.

Since $\mathrm{Arg}(z_n) = -\mathrm{Arg}(w_n)$, we have $\mathrm{Arg}(w_n) \leq \frac{\pi}{8}$, by the main angle hypothesis. In particular, $a_n > b_n > 0$ so that $\alpha_n > 0$ (since we have agreed that $x_n > 0$, $y_n < 0$). As already noticed in the proof of Theorem 1.3, the angle hypothesis is equivalent to the relation
$$\alpha_n^2 \geq 2\beta_n.$$

Applying (10.3) with $z = it$, $t \in \mathbb{R}$, we get that
$$\mathbb{E}\, e^{tX} = e^{\gamma t^2/2} \prod_{n \geq 1} Q_{p,n}(it) \tag{10.4}$$
with positive factors given by
$$Q_{0,n}(it) = Q_{1,n}(it) = 1 + \alpha_n t^2 + \beta_n t^4, \qquad Q_{2,n}(it) = (1 + \alpha_n t^2 + \beta_n t^4)\, e^{-\alpha_n t^2}.$$
We have already observed in the proof of Proposition 7.1 that, by the angle hypothesis,
$$1 + \alpha_n t^2 + \beta_n t^4 < e^{\alpha_n t^2}, \quad t > 0, \tag{10.5}$$
so that $Q_{2,n}(it) < 1$. Moreover, this inequality was strengthened by improving the constant α_n in the exponent, provided that t is bounded away from zero. We will thus repeat some steps from the proof of Proposition 7.1. However, formally, we need to consider the three cases separately according to the three possible values of p.

Genus $p = 2$. By the very definition of the genus, the following sum converges
$$\sum_{n \geq 1} |w_n|^3 = \sum_{n \geq 1} (a_n^2 + b_n^2)^{3/2} = \sum_{n \geq 1} \beta_n^{3/4} < \infty.$$
Since
$$Q_{2,n}(it) = 1 + O(|w_n|^4 t^4) = 1 + O(\beta_n t^4) \quad \text{ as } t \to 0,$$
the product in (10.4) is absolutely convergent. Moreover, the right-hand side of (10.4) near zero is $1 + \gamma t^2 + O(t^3)$. Hence, necessarily $\gamma = \sigma^2$, and (10.4) becomes
$$\mathbb{E}\, e^{tX} = e^{\sigma^2 t^2/2} \prod_{n \geq 1} Q_{2,n}(it). \tag{10.6}$$
Recalling the bound $Q_{2,n}(it) \leq 1$, we conclude that
$$\mathbb{E}\, e^{tX} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}, \tag{10.7}$$

which means that X is strongly subgaussian.

For the second claim of the theorem, write

$$\mathbb{E}\, e^{tX} = e^{V(t^2)}, \tag{10.8}$$

where

$$\begin{aligned} V(s) &= \frac{1}{2}\gamma s + \sum_{n\geq 1} \log Q_n(it) \\ &= \frac{1}{2}\sigma^2 s + \sum_{n\geq 1} \Big[\log(1+\alpha_n s+\beta_n s^2) - \alpha_n s\Big], \quad s \geq 0, \end{aligned}$$

and define

$$W(s) \, = \, \frac{1}{2}\sigma^2 s - V(s) \, = \, \sum_{n\geq 1} R_n(s), \quad R_n(s) = \alpha_n s - \log(1+\alpha_n s+\beta_n s^2).$$

By Lemma 10.2, and using the assumption $\alpha_n^2 \geq 2\beta_n$, all $R_n(s) > 0$ for $s > 0$, representing convex increasing functions. Hence, W is a convex increasing function with $W(0) = 0$. It remains to apply Proposition 2.5, and we obtain the property (10.1).

Genus $p = 1$. By definition, the following sum converges

$$\sum_{n\geq 1} |w_n|^2 = \sum_{n\geq 1} (a_n^2 + b_n^2) = \sum_{n\geq 1} \beta_n^{1/2} < \infty.$$

Since

$$\alpha_n = 2(a_n^2 - b_n^2) \leq 2(a_n^2 + b_n^2) = 2\beta_n^{1/2},$$

the product in (10.4) is convergent. Moreover, the right-hand side of (10.4) near zero is

$$1 + \frac{1}{2}\gamma t^2 + t^2 \sum_{n\geq 1} \alpha_n + O(t^3).$$

Hence, necessarily

$$\frac{1}{2}\sigma^2 = \frac{1}{2}\gamma + \sum_{n\geq 1} \alpha_n,$$

so that the characteristic function and the Laplace transform admit the same representation (10.6). As a result, since the summation property defining the genus became stronger, we immediately obtain (10.7) and its improvement (10.1) using the previous step.

Genus $p = 0$. By definition, the following sum converges

$$\sum_{n\geq 1} |w_n| = \sum_{n\geq 1} (a_n^2 + b_n^2)^{1/2} = \sum_{n\geq 1} \beta_n^{1/4} < \infty.$$

Since this assumption is stronger than the one of the previous step, while $Q_{0,n} = Q_{1,n}$, we are reduced to the previous step. $\square$

11. **Proof of Theorem 1.4**

As in the proof of Proposition 10.1, let us enumerate the points $z_n = x_n + iy_n$ lying in the quadrant $x_n \geq 0$, $y_n \leq 0$ and deal with $-z_n, \bar{z}_n, -\bar{z}_n$ as associated zeros. For simplicity of notations, we assume that all these numbers are complex. Put

$$w_n = \frac{1}{z_n} = a_n + b_n i$$

and define

$$\begin{aligned} f_n(z) &= e^{-\gamma_n z^2/2}\,(1 - w_n z)(1 + w_n z)(1 - \bar{w}_n z)(1 + \bar{w}_n z) \\ &= e^{-\gamma_n z^2/2}\,(1 - \alpha_n z^2 + \beta_n z^4), \quad z \in \mathbb{C}, \end{aligned}$$

for a given sequence $\gamma_n > 0$ (to be precised later on). Here as before

$$\alpha_n = 2(a_n^2 - b_n^2), \quad \beta_n = (a_n^2 + b_n^2)^2.$$

By the assumption, $a_n, b_n > 0$. Moreover, the angle assumption $|\mathrm{Arg}(z_n)| = \mathrm{Arg}(w_n) \leq \frac{\pi}{8}$ equivalent to $\alpha_n^2 \geq 2\beta_n$, which may also be written as

$$b_n \leq \frac{1}{\sqrt{2}+1}\, a_n. \tag{11.1}$$

Now, if γ_n is sufficiently large, $f_n(t)$, $t \in \mathbb{R}$, will be the characteristic function of a strongly subgaussian distribution. A full description of the minimal possible value of γ_n is provided in Proposition 9.1. More precisely, consider the function

$$\begin{aligned} g_n(t) \,=\, f_n\Big(\frac{t}{\sqrt{\gamma_n}}\Big) &= e^{-t^2/2}\,(1 - w_n' t)(1 + w_n' t)(1 - \bar{w}_n' t)(1 + \bar{w}_n' t) \\ &= e^{-t^2/2}\,(1 - \alpha_n' t^2 + \beta_n' t^4) \end{aligned}$$

with

$$w_n' = a_n' + b_n' i, \quad a_n' = \frac{a_n}{\sqrt{\gamma_n}}, \quad b_n' = \frac{b_n}{\sqrt{\gamma_n}}$$

and

$$\alpha_n' = \frac{2(a_n^2 - b_n^2)}{\gamma_n}, \quad \beta_n' = \frac{(a_n^2 + b_n^2)^2}{\gamma_n^2}.$$

As we know, $g_n(t)$ represents the characteristic function of a strongly subgaussian random variable X_n', as long as

$$b_n' \leq \frac{1}{\sqrt{2}+1}\, a_n', \quad a_n' \leq a_0,$$

where the universal constant a_0 was explicitly identified in (9.6), $a_0 \sim 0.7391$. Here, the first condition is satisfied in view of (11.1), while the second one is equivalent to

$$\gamma_n \geq \frac{a_n^2}{a_0^2}. \tag{11.2}$$

Moreover, X_n' has variance

$$\mathrm{Var}(X_n') = -g_n''(0) = 2\alpha_n' + 1 = \frac{4(a_n^2 - b_n^2)}{\gamma_n} + 1.$$

Thus, subject to (11.2), $f_n(t)$ will be the characteristic function of the strongly subgaussian random variable $X_n = \sqrt{\gamma_n}\, X_n'$, whose variance is given by

$$\mathrm{Var}(X_n) = 4(a_n^2 - b_n^2) + \gamma_n. \tag{11.3}$$

Now, assuming that $\Lambda \geq 4 + \frac{1}{a_0^3} \sim 5.83$, let us choose

$$\gamma_n = (\Lambda - 4)a_n^2 + (\Lambda + 4)b_n^2,$$

so that the expression in (11.3) would be equal to $\Lambda(a_n^2 + b_n^2) = \Lambda |w_n|^2$. Then the condition (11.2) is satisfied, and also

$$\sum_n \gamma_n < \infty.$$

As a result, the series $\sum_n X_n$ is convergent with probability one, and the sum of the series, call it X, represents a strongly subgaussian random variable with characteristic function

$$f(z) = \prod_n f_n(z)$$

(cf. Proposition 2.2). By the construction, all $f_n(z)$ have exactly prescribed zeros, and

$$\mathrm{Var}(X) = \sum_n \mathrm{Var}(X_n) = \Lambda \sum_n |w_n|^2.$$

$\square$

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