MODEL REDUCTION VIA MATRIX PENCIL APPROACH

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Abstract

In this paper, we present a novel way of model reduction based on matrix pencil theory. Using only orthogonal transformations on state space models, we construct an approximation to the smallest perturbation to the coefficients that yields a lower order system. We derive some bounds on the stability of the resulting lower order system. We illustrate our method with an example arising from large flexible space structures.

1. Introduction

Model reduction refers to the approximation of a linear system model by a lower order model such that certain criteria are met. The motivations for model reduction are mainly economical since low order models can be analyzed, simulated or built at a much lower cost than high order models. Numerous papers have been written on model reduction. Reduction by modal expansion appears to be one of the oldest approaches. However, reduction by balanced realization [7] and by optimal Hankel norm approximation [9] as well as their weighted versions [8,10] have now found widespread acceptance among control engineers. In [15] we presented a novel constructive way of implementing model reduction based on some recent results in matrix pencil theory [1]. In this paper, we present the procedure for effecting model order reduction by truncating multiple states.

We now state some definitions and characterizations of matrix pencils [1,3,4,5]. Let A and B be two n x p matrices. The set of all matrices of the form $(A - \lambda B)$, where λ is any complex number is said to be a matrix pencil of dimension n x p. Two matrix pencils $(A_1 - \lambda B_1)$ and $(A_2 - \lambda B_2)$ of dimension n x p are said to be strictly equivalent when there exist constant invertible matrices P and Q of dimension n x n and p x p respectively such that $P(A_1 - \lambda B_1)Q = (A_2 - \lambda B_2)$. If $(A - \lambda B)$ is always full rank for any value of λ, then it is said to be non−deficient.

We shall consider the standard finite dimensional linear time invariant (FDLTI) continuous time system described by

$$\dot{x} = Fx + Gu\,; \quad y = Hx\,, \tag{1}$$

where $F \in R^{p \times p}$, $G \in R^{p \times m}$, and $H \in R^{r \times p}$ as the full order model. We have omitted the feedthrough matrix in the state space model description because it remains invariant in the model reduction scheme proposed in this paper. If the full order model is not strictly proper, the method and analysis outlined in this paper can still be applied to the strictly proper part of the model. We shall assume that the system described by (1) is stable, i.e., the real part of all the eigenvalues of F is strictly less than zero. We want to remark that this is a very mild assumption because the transfer function matrix $G(s)$ of any FDLTI system can be decomposed as $G(s) = G_S(s) + G_U(s)$, where $G_S(s)$ is the transfer function matrix of the stable part and $G_U(s)$ is the transfer function matrix of the unstable part. One can then proceed with the model order reduction of the stable part. We shall also assume that the state space realization described by (1) is minimal. According to the PBH rank test (see e.g., [4]), the dynamical system described by (1) is controllable if and only if the matrix pencil $[F-\lambda I|G] = [F|G] - \lambda[I|0]$, has full rank for any complex value λ.

In [1], the sensitivity of an algebraic (Kronecker) structure of rectangular matrix pencils to perturbations in the coefficients was examined and eigenvalue perturbation bounds in the spirit of Bauer-Fike was used to develop computational upper and lower bounds on the distance from a given pencil to one with a qualitatively different Kronecker structure. In this paper, we exploit the bounds derived in [1] for estimating the upper bound on the distance to a deficient pencil [1], to implement a model reduction scheme.

The rest of the paper is organized as follows. In Section 2, we review the background material on the computation of the nearest distance to a deficient pencil as espoused in [1]. In Section 3, we discuss how the ideas in Section 2 can be employed in a model reduction. In Section 4 we present an algorithm for model reduction and make some remarks on how it can be modified to suit the particular needs of the designer. We also make some remarks on the stability of the reduced order model. In Section 5 we derive an expression for the model reduction error in terms of the full order model and the perturbation of the matrix pencil that leads to a deficient pencil. For a special case of our method, the reduced order model is stable so long as the full order model is stable. For this special case, we derive an upper bound on the H_2 norm of the model reduction error. In Section 6, we present a worked example. Finally, in Section 7, we make some concluding remarks.

2. Computation of Deficient Pencil

In this section we discuss two issues of importance to the reduction scheme as they were espoused in [1]. The first one concerns the problem of determining whether a given rectangular pencil is deficient or not. Specifically, given an n x p pencil $(A - \lambda B)$, with $n > p$, determine whether or not $(A - \lambda B)$ loses rank for any λ, including possibly λ infinite. By augmenting A and B matrices with arbitrary n x $(n - p)$ matrices C, D. We can examine the square n x n generalized eigenvalue problem

$$[A, C]v = \lambda[B, D]v\,. \tag{2}$$

In [1] two choices were proposed for the selection of C and D. One of these choices is to select C and D as orthonormal basis of the space orthogonal to the columns of A and B. This choice has the effect of limiting the increase to the condition numbers of $[A,C]$ and $[B,D]$ with respect to inversion. Thus in the case $B = [I_p,0]^T$, D is chosen as $D = [0,I_{n-p}]^T$ to turn the problem into an ordinary eigenvalue problem.

The second issue concerns the problem of computing the upper bound on the distance to a deficient pencil. Specifically, consider a non-deficient n x p pencil $(A - \lambda B)$. In this case, we know that B has full rank. We would like to estimate the size of the perturbation E to the matrix A that is needed to obtain a deficient pencil $(A+E-\lambda B)$. In [2], it was shown that the smallest perturbation E can be obtained by solving the minimization problem

$$\min_s \sigma_{\min}(A - sB)\ , \tag{3}$$

where $\sigma_{\min}(M)$ denotes the smallest singular value of the matrix M, and s varies over the entire complex plane. If we denote by σ^* and s^* the minimum in (3) and value of s achieving that minimum, respectively, then $\|E\| = \sigma^*$. In [1], it was shown that a simpler scheme which provides both a good estimate for $\|E\|$ and for that value of s that yields the minimum in (3) involves solving the eigenvalue problem (2). Suppose we partition the vector v as $v^T \equiv [x^T,y^T]$, where x is a p-vector, and y is a $(n-p)$-vector. Let λ_i, $v_i := [x_i^T,y_i^T]^T$, $i = 1,\cdots,n$ be the generalized eigenvalues and eigenvectors for (2). For each i we have the eigenvalue equation: $[A,C]v_i = \lambda_i[B,D]v_i$, substituting for v_i, the eigenvalue equation can be rewritten as $(\lambda_i B - A)x_i = (C - \lambda_i D)y_i$. For each i define the residual r_i as $r_i := (A - \lambda_i B)x_i$; and the perturbation E_i as

$$E_i := \frac{-r_i x_i^T}{\|x_i\|^2} \equiv (\lambda_i B - A)\frac{x_i x_i^T}{\|x_i\|^2} = (C - \lambda_i D)\frac{y_i}{\|x_i\|}\cdot\frac{x_i^T}{\|x_i\|}. \tag{4}$$

The norm on E_i can be simply computed as $\|E_i\| = \|r_i\|_2/\|x_i\|_2$. Note that $E_i x_i = -r_i$, and thus the expression for the residual becomes $(A+E_i-\lambda_i B)x_i = 0$. Now, $(A+E_i-\lambda B)$ is a deficient pencil, losing rank exactly at $\lambda = \lambda_i$, for each i. Let σ_i, u_i, w_i be, respectively, the smallest singular value and the corresponding left and right singular vectors of $(A - \lambda_i B)$ for each i. Then $E_i' := -\sigma_i u_i w_i^T$ is another smaller perturbation yielding a deficient pencil. By taking norms of (4), the following bounds are obtained for the perturbations: $\|E_i'\| \le \|E_i\| \le \|(C-\lambda_i D)y_i\|/\|x_i\|$. We also have the following error bound from [1]. If E denotes that perturbation with the smallest norm yielding a deficient pencil, then $\|E\| := \sigma^*$ satisfies

$$\sigma^* \le \min_i \|E_i'\| \equiv \min_i \sigma_{\min}(A - \lambda_i B) \le \min_i \|E_i\| \equiv \min_i \frac{\|(C-\lambda_i D)y_i\|}{\|x_i\|}. \tag{5}$$

The importance of (5) was noted and stated as follows.

Proposition 1 [1]: Let $(A - \lambda B)$ be an n x p pencil, with $n > p$. Let C, D be two arbitrary full-rank n x $(n - p)$ matrices. Then the smallest perturbation E such that $A+E-\lambda B$ is a deficient pencil satisfies the bound (5), where λ_i, v_i, $i = 1,\cdots,n$ are the eigenpairs of the generalized eigenproblem (2) and y_i is defined by $v_i := [x_i^T,y_i^T]^T$.

3. Application to Model Reduction

Our goal is to use the results stated in the previous section to implement a model reduction scheme. This is done by constructing a matrix pencil based on the PBH test for controllability or observability. The matrix pencil so constructed is augmented with "C" and "D" matrices as described in Section 2. One then proceeds to solve the eigenvalue problem which is (2). Next, the norm of the perturbations $\|E_i\|$, $i,\cdots,n$, are computed, and sorted in increasing order. In [14,15] we showed how to effect model order reduction when E_i is formed from a real eigenpair and only one state is truncated at a time. In general, solving the eigenvalue equation (2) may yield real eigenvalues and complex eigenvalues which occur as complex conjugates. When the perturbation matrix E_i corresponds to a complex eigenvalue then the reduction scheme in [15] needs be modified so that the two states corresponding to a pair of complex eigenvalues are simultaneously truncated. Also, for a large order system, it will be more economical to reduce the order of the system by multiple states instead of a single state each time the matrix pencil is perturbed. In this section, we show how to extend the method outlined in [15] to handle both situations. Specifically, suppose we want to truncate N modes from the full order model. Then, using the eigenpairs (λ_i, v_i), $i = 1,\cdots,N$ corresponding to the first N smallest values of $\|E_i\|$, one constructs a new perturbation matrix E_N such that $(A + E_N - \lambda B)$ is a deficient pencil, losing rank exactly at $\lambda = \lambda_i$, for each $i = 1,\cdots,N$. The resulting uncontrollable or unobservable states are then truncated to yield a reduced order model.

3.1 Model Order Reduction

Here, we shall assume that the A and B matrices of the matrix pencil $(A - \lambda B)$ are defined by: $A = [F,G]^T$, $B = [I_p,0]^T$, where the underlying matrices F and G are given by (1), i.e., we form the pencil using controllability criteria. We shall assume that we want to truncate N modes which correspond to the N smallest $\|E_i\|$ values. From the eigenvalues $\lambda_i = \alpha_i + j\beta_i$, with the corresponding eigenvectors $v_i = [x_i^H,y_i^H]^H$, $i = 1,\cdots,N$, we form the matrices T, $V := [W^T,U^T]^T$ as follows:

$$\begin{aligned} T &:= diag\{T_1,T_2,\cdots,T_N\}, \quad \text{with} \quad T_i := \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}; \\ W &:= [Re(x_1),Im(x_1),Re(x_2),Im(x_2),\cdots,Re(x_N),Im(x_N)]\ ; \\ U &:= [Re(y_1),Im(y_1),Re(y_2),Im(y_2),\cdots,Re(y_N),Im(y_N)]\ . \end{aligned} \tag{6}$$

That is the columns of V form a real basis for the invariant subspace corresponding to the eigenvalues in T. The eigenspace equation becomes: $[A,C]V = VT$. We want to remark that actually any basis for the invariant subspace corresponding to the N modes to truncate will do, whether or not they are complex. In such a case, T may be full but will have the same eigenvalues. The following development carries through almost unchanged. The eigenspace equation simplifies to: $AW - [(WT)^T,0]^T = [0,(WT)^T]^T - CU$. Define the residual R as $R := [0,(WT)^T]^T - CU$; and from $E_N W + R = 0$, define E_N as $E_N := -R(W^TW)^{-1}W^T$. The expression for the residual R becomes: $[A + E_N]W = AW - [(WT)^T,0]^T$. Suppose the perturbation matrix E_N is partitioned as $E_N^T = [E_1,E_2]$, where E_1 and E_2 have the dimensions of F and G respectively. Substituting for the A and E_N matrices, we obtain

$$\begin{bmatrix} F^T + E_1^T \\ G^T + E_2^T \end{bmatrix} \cdot W = \begin{bmatrix} WT \\ 0 \end{bmatrix}. \tag{7}$$

The top part of (7) yields $(F + E_1)^T W = WT$. Now choose an orthonormal Q such that $QW = [S^T,0]^T$, where S is a square upper triangular matrix. Thus when the top part of (7) is premultiplied by Q, we have $Q(F+E_1)^T Q^T QW = QWT$; upon substituting for QW the equation becomes: $Q(F+E_1)Q^T[S^T,0]^T = [(ST)^T,0]^T$. Similarly, the bottom part of (7) yields $(G + E_2)^T Q^T QW = 0$; upon substituting for QW and transposing, we obtain $[S^T,0]Q(G + E_2) = 0$. The perturbed system can be written as:

$$\dot{z} = Q(F+E_1)Q^T z + Q(G+E_2)u\ ;\ y = HQ^T z\,. \tag{8}$$

For notational simplicity denote $Q(F+E_1)Q^T$, $Q(G+E_2)$, HQ^T by F, G, and H respectively. The perturbed system described by (8) can be written in partitioned form as

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}\tilde{F}_{11} & 0\\ \tilde{F}_{21} & \tilde{F}_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ \tilde{G}_2\end{bmatrix}u\ ;\ y = \begin{bmatrix}\tilde{H}_1 & \tilde{H}_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}, \tag{9}$$

where z_1 represents the uncontrollable states (if the N eigenpairs used to construct E_N are all complex, then clearly z_1 represents $2N$ states). If the uncontrollable states z_1 are stable, they can be directly truncated to yield a reduced order model:

$$\dot{z}_2 = \tilde{F}_{22}z_2 + \tilde{G}_2 u\ ;\ \ y = \tilde{H}_2 z_2\ . \tag{10}$$

One can also use observability criteria by doing exactly the same reduction on the dual system: $x = F^T x + H^T u$; $y = G^T x$. The trace of the controllability and observability grammians give a measure of how "controllable" or how "observable" the model is in its current state space realization. Denote by κ_C the norm of the perturbation to the input matrix G that will render the realization uncontrollable, and κ_O the norm of the perturbation to the output matrix H that will render the same realization unobservable. We shall also denote the controllability and observability grammians by W_{contr} and W_{obs} respectively. It is desirable to effect the model reduction by applying the smallest possible perturbation. If trace(W_{contr}) < trace(W_{obs}) then we normally expect to have $\kappa_C < \kappa_O$, in that case the controllability method should be used. On the other hand if trace(W_{contr}) > trace(W_{obs}) then we normally expect to have $\kappa_C > \kappa_O$, and in that case the observability method should be used.

4. Matrix Pencil Algorithm for Model Reduction

Given a FDLTI system as described by (1), a reduced order model can be computed using the following algorithm.

Step 1: Compute trace(W_{contr}) and trace(W_{obs}). If trace(W_{contr}) < trace(W_{obs}) then use the PBH controllability test to form the matrix pencil $(A - \lambda B)$ where A and B are defined by: $A = [F,G]^T$, $B = [I_p,0]^T$. Otherwise use the PBH observability test to form the matrix pencil $(A - \lambda B)$. Augment the pencil with C and D matrices as described in Section 2.

Step 2: Solve the corresponding eigenvalue problem which is (2). Next, compute $\|E_i\|$, $i = 1,\cdots,n$, and sort them in increasing order. Make a decision on the number of states to be truncated. Using the eigenpairs (λ_i, v_i), $i = 1,\cdots,N$ corresponding to the first N smallest values of $\|E_i\|$, construct a new perturbation matrix E_N as defined in Section 3.

Step 3: Extract the perturbed system from the matrix $A + E_N$, and then apply the appropriate orthonormal matrix Q to the perturbed system to reveal the uncontrollable or unobservable states. Verify the stability of the perturbed system. A reduced order model can be obtained by direct truncation if the resulting uncontrollable or unobservable states are stable.

Remarks

1. The algorithm outlined above may be modified to suit particular needs of the designer. For example, if trace(W_{contr}) < trace(W_{obs}) but close in value, one may use the controllability method to truncate a few states and then implement the observability method to truncate other states.

2. The algorithm proposed above could be extended to include frequency dependent weightings. The modes of the full order model can be weighted differently and a different criterion can be used for truncation. For example, one could use a weighted average of the existing measure and the maximum singular values of the transfer function matrix. For simplicity, we have not done that here.

3. The method presented in this paper involves using an orthonormal matrix Q to effect a similarity transformation. Model reduction by optimal Hankel norm reduction [9] and by truncation by balanced realization [7] also involve using a matrix T to effect a similarity transformation, however, the T used in [7,9] is not in general an orthonormal matrix. As is well known, use of non-orthonormal (or non-unitary) transformation can lead to worsening of the condition numbers and loss of accuracy of the system matrices.

4. A question of great importance in any model reduction scheme is whether the reduced order model obtained from a stable full order model is stable or not. In our scheme, this amounts to asking whether the real part of all the eigenvalues of $Q(F+E_1)Q^T$ are strictly less than zero. Since $(F+E_1)$ and $Q(F+E_1)Q^T$ are similar matrices, it will suffice to consider the eigenvalues of $(F+E_1)$. We shall focus on a reduced order model obtained by applying the controllability criteria, however the discussions concerning the stability of $(F+E_1)$ are equally valid for a reduced order model obtained from the observability criteria. In order to form the perturbation matrix E_N, we first solve the eigenvalue equation $[A,C]v_i = \lambda_i[B,D]v_i$, where A is formed as $A = [F,G]^T$, $B = [I_p,0]^T$ and D set to $D = [0,I_{n-p}]^T$ to turn the problem into an ordinary eigenvalue problem. We also select C as orthonormal basis of the space orthogonal to the columns of A. Suppose, we partition the matrix C as $C^T = [C_1^T,C_2^T]$, where C_1 and C_2 are of dimensions p x $(n - p)$ and $(n - p)$ x $(n - p)$ respectively; and also introduce the parameter α by writing the matrix C as $C^T = [\alpha C_1^T,C_2^T]$, with $\alpha \in [0,1]$. When $\alpha = 0$, there is no perturbation of the F matrix. In that case the reduced order model remains stable so long as the original model is stable. As α increases, the perturbation E_1 to F is no longer zero but varies continuously with α. Therefore stability of $(F + E_1)$ is maintained for small values of α. Application of the Bauer-Fike theorem [6] gives us the following result. Let K be the condition number of the matrix of eigenvectors of F, and d the distance to the imaginary axis of the eigenvalue of F which is closest to the imaginary axis. Then stability of $(F + E_1)$ is guaranteed as long as $K\|E_1\| < d$. The reader is referred to [14] for details. In practice we found it just sufficed to verify the stability of the perturbed system.

5. Another issue of concern is that of the $n - p$ spurious eigenvalues introduced when the eigenvalue equation is solved. These spurious eigenvalues arise because the underlying dynamical system of the matrix pencil is of order p whereas the eigenvalue equation yields n eigenvalues. Note that when $\alpha = 0$, the spurious eigenvalues pose no problem because the vector $x_i = 0$, thus $\|E_i\| = \infty$ for the spurious eigenvalues and hence can be ignored in the model reduction process. However, when $\alpha \neq 0$, the value of $\|E_i\|$ for the spurious roots is finite; but generally, $\|E_i\|$ for the spurious eigenvalues will be much larger than for the non-spurious eigenvalues. Space does not permit an extensive discussion here, but the reader is referred to [14] for a detailed explanation.

5. Model Reduction Error

In this section we derive expressions for the model reduction error. We consider two cases which are 1) $\alpha = 0$, and 2) $\alpha \neq 0$. When $\alpha = 0$, then E_1 is a zero matrix, and it is only the G matrix which is perturbed by E_2. If we apply the orthonormal matrix Q from Section 3 to the state space representation of the full order model $G(s)$ (i.e., (1)), we have the following: $\dot{z} = QFQ^T z + QGu$; $y = HQ^T z$. For notational simplicity, denote QFQ^T, QG, and HQ^T by F, Γ, and $\hat{H}$ respectively; we shall also denote $(-QE_2)$ by E_C. Then it can be shown that the model reduction error $K(s) = G(s) - G_r(s)$ is given by: $\dot{z} = Fz + E_C u$; $y = \hat{H}z$. We now seek to compute an upper bound on the H_2 norm of the model error $K(s)$ when $\alpha = 0$. Using equation (5.35) from [11] to compute the H_2 norm of $G(s)$ and $K(s)$ we have: $\|G\|_2^2 = \mathrm{trace}(\Gamma^T W_{obs} \Gamma)$; $\|K\|_2^2 = \mathrm{trace}(E_C^T W_{obs} E_C)$ where W_{obs} is the observability grammian of both $G(s)$ and $K(s)$. Using cholesky factorization of W_{obs}, it can be shown that $\mathrm{trace}(E_C^T W_{obs} E_C) \leq \mathrm{trace}(W_{obs})\|E_C\|_F^2$. Now $\|E_C\|_F$ is small since $E_C = -QE_2$, and E_2 is just the perturbation we are committing to arrive at the uncontrollable system.

Roughly, the term $\mathrm{trace}(W_{obs})$ measures the energy that can be retrieved at the output from the system states, and $\|G\|_2^2$ measures the energy that can be retrieved at the output from a unit impulse input. We may define the ratios: $\rho_{input} := \sqrt{(\mathrm{trace}(W_{obs}))}/\|G\|$; $\rho_{output} := \sqrt{(\mathrm{trace}(W_{contr}))}/\|G\|$, where ρ_{input} roughly measures the *input transmission ratio* of energy, and ρ_{output} roughly measures the *output transmission ratio* of energy. These ratios are properties inherent to a given state-space realization, regardless of whatever model reduction method is used. Using the ratio ρ_{input}, we may bound the norm of the error transfer function $K(s)$ as $\|K\| \leq \rho_{input}\|G\|\|E_C\|_F$. We could also apply an observability perturbation to the outputs, obtaining a reduced order model $\hat{G}_r(s)$ and error transfer function $K_{obs}(s) = G(s) - \hat{G}_r(s)$. Working through the same development, we could obtain the bound $\|K_{obs}\| \leq \rho_{output}\|G\|\|E_O\|_F$; where $E_O = -E_2 Q^T$, and E_2 is the perturbation to the output matrix H.

When $\alpha \neq 0$ then in general the eigenvalues of the full order model are different from the eigenvalues of the reduced order model and we do not have any simple way of expressing the model error dynamics. However we can augment the states x of the full order model with the states z_2 of the reduced order model to form a composite system with states (x,z_2). From (1) and (10) we form the state space representation for the model reduction error $K(s)$ as

$$\begin{bmatrix} \dot{x} \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & \tilde{F}_{22} \end{bmatrix} \begin{bmatrix} x \\ z_2 \end{bmatrix} + \begin{bmatrix} G \\ \tilde{G}_2 \end{bmatrix} u \; ; \; y_e = \begin{bmatrix} H & -\tilde{H}_2 \end{bmatrix} \begin{bmatrix} x \\ z_2 \end{bmatrix}. \qquad (11)$$

For both $\alpha = 0$ and $\alpha \neq 0$, the state space expression for the model reduction error is in terms of the full order model, the perturbation E_N and an orthonormal matrix Q. The sup-norm of the model reduction error can be computed using an iteration method such as in [12].

6. Numerical Results

We applied the matrix pencil reduction algorithm outlined in Section 4 with the parameter $\alpha = 1$. The example is a MIMO model of the "CSI evolutionary" structure [13]; the model which was used in this paper was furnished by Prof. Gary Balas. The full order model consists of 26 states, 8 inputs and 10 outputs.

Table 1 shows the norm of the perturbations and the corresponding eigenvalues of $[A,C]$. Note the gap in the norm of the perturbations corresponding to the 26^{th} and 27^{th} eigenvalues in the table. We thus identify all the eigenvalues whose $\|E_i\|$ is greater than or equal to 1.4207 as spurious eigenvalues. The eigenpair (λ_1,v_1) was used to generate E_N for reduced order model of 24 states, and the pairs $\{(\lambda_1,v_1),(\lambda_3,v_3)\}$ were used for reduced order model of 22 states. For reduced order models of order 24 and 22 there is no noticeable difference between singular values plot of the original and the reduced order models. The eigenpairs $\{(\lambda_1,v_1), (\lambda_3,v_3), (\lambda_5,v_5)\}$ were used to generate E_N for reduced order 20 model. Figure 1 is a plot of the maximum and minimum singular values versus frequency of the reduced

i	$\|E_i\|$	$\lambda_i[A,C]$
1,2	0.0103	$-0.0215 \pm j21.4750$
3,4	0.0759	$-0.0159 \pm j15.8949$
5,6	0.1200	$-0.0142 \pm j14.1052$
⋮	⋮	⋮
25,26	0.4860	$-0.1369 \pm j1.0555$
27	1.4207	0.9980
34	4.1977	1.0000

Table 1: $\|E_i\|$ and eigenvalues of [A,C] sorted by $\|E_i\|$.

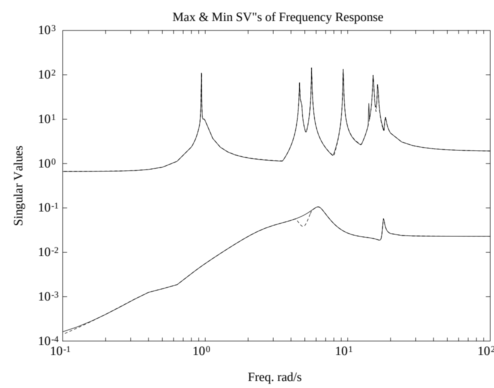

Figure 1 Max & Min SV’s: Matrix Pencil Approach (dashed); Full order model (solid).

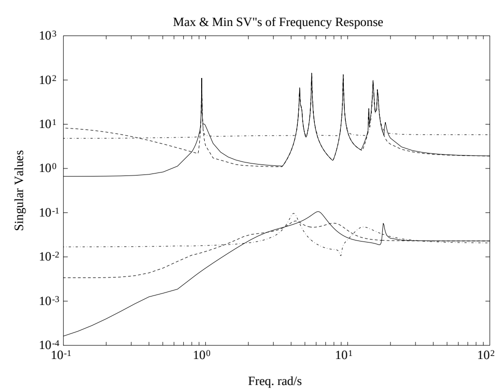

Figure 2 Max & Min SV’s: Full order model (solid); Balanced Realization (dashed); Hankel Norm (dashdot).

order 20 model obtained using the matrix pencil approach. As can be seen from Figure 1, for the reduced order 20 model, the singular values plots are almost identical with those of the full order model of 26 states.

We next compared the performance of the matrix pencil reduction scheme with 1) truncation by balanced realization [7], and 2) optimal Hankel norm reduction [9] methods. Figure 2 is a plot of the maximum and minimum singular values versus frequency of the reduced order models of 20 states using the optimal Hankel norm and balanced realization methods. The model from balanced realization gives a good fit at intermediate and high frequencies but a very poor fit at low frequencies. However, we see that the optimal Hankel norm method gives the worst fit of the singular values.

As is well known, the SISO notion of phase is not easily generalized for the MIMO system. However, in some applications, phase information of the individual transfer functions are important. We thus decided to verify how closely the Bode plots of the individual transfer functions of the reduced order models follow that of the full order model. We arbitrarily picked the transfer function between the first output and the first input, i.e., the (1,1) entry for this purpose. Figure 3 is the Bode magnitude and phase plots of the (1,1) transfer function for reduced order 20 model. From Figure 3 we see that the matrix pencil approach approximates both the phase and magnitude of the full order model much better than the balanced truncation method.

7. Conclusion

In this paper we have presented a novel way of model reduction based on matrix pencil theory. We have given some initial results on the stability of the reduced order model in terms of a perturbation applied to the system matrix F. However, for the special case of $\alpha = 0$, the reduced order model is always stable so long as the full order model is. We have also derived an upper bound on the H_2 norm of the model reduction error when $\alpha = 0$. The computational cost of the matrix pencil method is comparable to other methods which require balanced realization. Some of the advantages of the matrix pencil method over the balanced realization and optimal Hankel norm methods are: 1) there is no need to transform the full order model into balanced realization, and 2) the matrix pencil method uses only orthonormal transformations.

We have given an example to illustrate features of the method. The result indicates that the matrix pencil may yield models that are much better approximations than those from the optimal Hankel norm method. The models from the matrix pencil method tend to follow the phase of the full order model better than models from the balanced realization method, and are otherwise comparable.

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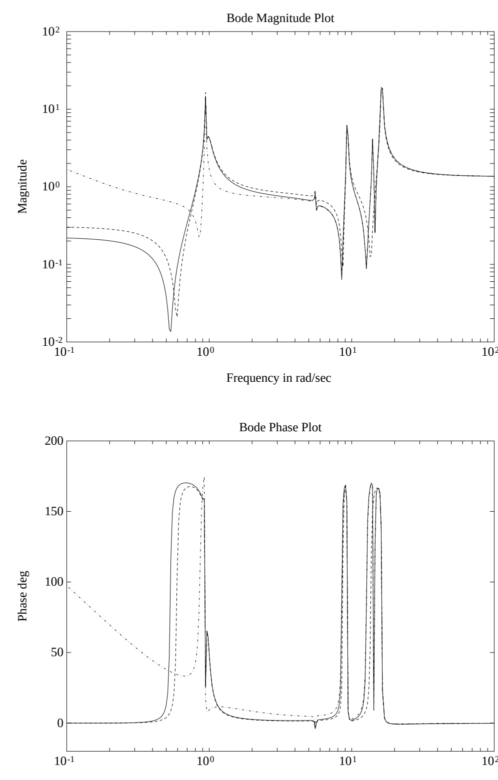

Figure 3 Bode Plots of (1,1) Transfer Function: Full order model (solid); Matrix Pencil (dashed); Balanced Realization (dash-dot).