ON BILINEAR FUNCTIONS

(SULLE FUNZIONI BILINEARI)

by

E. BELTRAMI

A Translation from the Original Italian of one of the Earliest Published Discussions of the Singular Value Decomposition.

Eugenio Beltrami *(Cremona, 1835 – Rome, 1900) was professor of Physics and Mathematics at the University of Pavia from 1876 and was named a senator in 1899. His writings were on the geometry of bilinear forms, on the foundations of geometry, on the theory of elasticity, electricity, and hydrodynamics, on the kinematics of fluids, on the attractions of ellipsoids, on potential functions, and even some in experimental physics.*

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E. D'OVIDIO, G. TORELLI e C. SARDI

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The theory of bilinear functions, already the subject of subtle and advanced research on the part of the eminent geometers KRONECKER and CHRISTOFFEL (Journal of BORCHARDT t. 68), gives occasion for elegant and simple problems, if one removes from it the restriction, almost always assumed until now, that the two series of variables be subjected to identical substitutions, or to inverse substitutions. I think it not entirely unuseful to treat briefly a few of these problems, in order to encourage the young readers of this Journal to become familiar ever more with these algebraic processes that form the fundamental subject matter of the new analytic geometry, and without which this most beautiful branch of mathematical science would remain confined within a symbolic geometry, which for a long time the perspicuity and the power of pure synthesis has dominated.

Let

$$f = \Sigma_{rs} c_{rs} x_r y_s$$

be a bilinear function formed with the two groups of independent variables

$$x_1, x_2, \ldots x_n;$$

$$y_1, y_2, \ldots y_n.$$

Transforming these variables simultaneously with two *distinct* linear substitutions

$$\text{(1)} \qquad x_r = \Sigma_r a_{rp} \xi_p, \quad y_s = \Sigma_s b_{sq} \eta_q,$$

(whose determinants one supposes to be always different from zero) one obtains a transformed form

$$\varphi = \Sigma_{pq} \gamma_{pq} \xi_p \eta_q,$$

whose coefficients γ_{pq} are related to the coefficients c_{rs} of the original function by the n^2 equations that have the following form:

$$\text{(2)} \qquad \gamma_{pq} = \Sigma_{rs} c_{rs} a_{rp} b_{sq}.$$

Setting for brevity

$$\Sigma_m c_{mr} a_{ms} = h_{rs}, \quad \Sigma_m c_{rm} b_{ms} = k_{rs},$$

this typical equation can be written in the two equivalent forms

$$\Sigma_s h_{sp} b_{sq} = \gamma_{pq}, \quad \Sigma_r k_{rq} a_{rp} = \gamma_{pq}. \tag{3}$$

Indicating with A, B, H, K, Γ the determinants formed respectively with the elements a, b, c, h, k, γ, one has, from these last equations, $\Gamma = \mathrm{HB} = \mathrm{KA}$. But, by the definition of the quantities h, k, one has as well $\mathrm{H} = \mathrm{CA}$, $\mathrm{K} = \mathrm{CB}$; hence

$$\Gamma = \mathrm{ABC},$$

that is, the determinant of the transformed function is equal to that of the original one multiplied by the products of the moduli of the two substitutions.

Let us suppose initially that the linear substitutions (1) are both orthogonal. In such a case, their $2n^2$ coefficients depend, as is known, on $n^2 - n$ independent parameters, and on the other hand, the transformed function φ can be, generally speaking, subjected to as many conditions. Now the coefficients γ_{pq} whose indices p, q are mutually unequal are exactly $n^2 - n$ in number: one can therefore seek if it is possible to annihilate all these coefficients, and to reduce the bilinear function f to the canonical form

$$\varphi = \Sigma_m \gamma_m \xi_m \eta_m.$$

To resolve this question, it suffices to observe that if, after having set in equations (3)

$$\gamma_{pq} = 0 \text{ for } p \gtrless q \text{ and } \gamma_{pp} = \gamma_p,$$

one multiplies the first by b_{rq} and one carries out on the result the summation Σ_q; then one multiplies the second by a_{sp} and one carries out the summation Σ_p, one obtains

$$h_{rp} = \gamma_p b_{rp}, \quad k_{sq} = \gamma_q a_{sq}.$$

These two typical equations are mutually equivalent to the corresponding ones of equations (3), and, as a consequence of these latter ones, one could in this way recover the equations (3) in the process. In them are contained the entire resolution of the problem posed, that one obtains in this way: Writing out the last two equations in the following way

$$(4) \qquad \begin{cases} c_{1r}a_{1s} + c_{2r}a_{2s} + ... + c_{nr}a_{ns} = \gamma_s b_{rs}, \\ c_{r1}b_{1s} + c_{r2}b_{2s} + ... + c_{rn}b_{ns} = \gamma_s a_{rs}, \end{cases}$$

then setting, for brevity,

$$\begin{aligned} c_{r1}c_{s1} + c_{r2}c_{s2} + ... + c_{rn}c_{sn} = \mu_{rs}, \\ c_{1r}c_{1s} + c_{2r}c_{2s} + ... + c_{nr}c_{ns} = \nu_{rs}, \end{aligned}$$

(so that $\mu_{rs} = \mu_{sr}$, $\nu_{rs} = \nu_{sr}$), the substitution into the second equation (4) of the values of the quantities b obtained from the first one yields

$$(5)_1 \qquad \mu_{rq}a_{1s} + \mu_{r2}a_{2s} + ... + \mu_{rn}a_{ns} = \gamma^2 a_{rs};$$

likewise, the substitution onto the first equation (4) of the values of the quantities a recovered from the second one yields

$$(5)_2 \qquad \nu_{r1}b_{1s} + \nu_{r2}b_{2s} + ... + \nu_{rn}b_{ns} = \gamma^2 b_{rs}.$$

The elimination of the quantities a from the n equations that one deduces from equations $(5)_1$ setting $r = 1, 2, ...\ n$ in succession, leads one to the equation

$$\Delta_1 = \begin{vmatrix} \mu_{11} - \gamma^2 & \mu_{12} & ... & \mu_{1n} \\ \mu_{21} & \mu_{22} - \gamma^2 & ... & \mu_{2n} \\ ...\ ...\ ... & ...\ ...\ ... & ... & ...\ ...\ ... \\ \mu_{n1} & \mu_{n2} & ... & \mu_{nn} - \gamma^2 \end{vmatrix} = 0$$

which the n values γ_1^2, γ_2^2, ... γ_n^2 of γ^2 must satisfy. Likewise, the elimination of the quantities b from the n equations that one deduces from equation $(5)_2$ setting $r = 1, 2, ...n$ in succession, leads one to the equation

$$\Delta_2 = \begin{vmatrix} \nu_{11} - \gamma^2 & \nu_{12} & ... & \nu_{1n} \\ \nu_{21} & \nu_{22} - \gamma^2 & ... & \nu_{2n} \\ ...\ ...\ ... & ...\ ...\ ... & ... & ...\ ...\ ... \\ \nu_{n1} & \nu_{n2} & ... & \nu_{nn} - \gamma^2 \end{vmatrix} = 0,$$

possessing the same properties as the preceding equation. It follows that the two determinants Δ_1, and Δ_2 are mutually identical for any value of gamma (*). In fact, they are entire functions of degree n with respect to γ^2,

(*) This theorem one finds demonstrated in a different way in §VII of *Determinants* by BRIOSCHI which become identical for $n+1$ values of γ^2, that is for the values γ_1^2, γ_2^2, ... γ_n^2 that simultaneously make both determinants zero, and for the value $\gamma = 0$ that makes them both equal to C^2.

The n roots γ_1^2, γ_2^2, ... γ_n^2 of the equation $\Delta = 0$ (likewise indicating indifferently $\Delta_1 = 0$ or $\Delta_2 = 0$) are all *real* , by virtue of a very well known theorem; to convince oneself that they are also *positive* , it suffices to observe that the coefficients of

$$\gamma^0, -\gamma^2, \gamma^4, -\gamma^6, \text{ etc.}$$

are sums of squares. But one can also consider that, by the elementary theory of ordinary quadratic forms and by virtue of the preceding equations, one has

$$\mathrm{F} = \Sigma_{rs}\mu_{rs}x_r x_s = \gamma_1^2\xi_1^2 + \gamma_2^2\xi_2^2 + ... + \gamma_n^2\xi_n^2,$$
$$\mathrm{G} = \Sigma_{rs}\nu_{rs}y_r y_s = \gamma_1^2\eta_1^2 + \gamma_2^2\eta_2^2 + ... + \gamma_n^2\eta_n^2;$$

on the other hand, one has as well

$$\mathrm{F} = \Sigma_m(c_{1m}x_1 + c_{2m}x_2 + ... + c_{nm}x_n)^2,$$
$$\mathrm{G} = \Sigma_m(c_{m1}y_1 + c_{m2}y_2 + ... + c_{mn}y_n)^2;$$

hence the two quadratic functions F, and G are essentially positive, and the coefficients γ_1^2, γ_2^2, ... γ_n^2 of the transformed expressions, i.e. the roots of the equation $\Delta = 0$, are necessarily all positive.

The proposed problem is therefore susceptible of a real solution, and here is the procedure: Find first the roots γ_1^2, γ_2^2, ... γ_n^2 of the equation $\Delta = 0$, (which is equivalent to reducing one or the other of the quadratic functions F, G to the canonical form); then with the help of the equations of the form $(5)_1$ and of those of the form

$$a_{1s}^2 + a_{2s}^2 + ... + a_{ns}^2 = 1,$$

one determines the coefficients a of the first substitution (coefficients that admit an ambiguity of sign common to all those items in the same column). This done, the equations that have the form (4) supply the values of the coefficients b of the second substitution (coefficients that also admit an ambiguity of sign common to all those items in the same column, so that each of the quantities γ_s are determined only by its square γ_s^2). Having done all these operations, one has two orthogonal substitutions that yield, exactly as desired by the problem, the identity

$$\Sigma_{rs}c_{rs}x_r y_s = \Sigma_m\gamma_m\xi_m\eta_m,$$

in which everyone of the coefficients γ_m must be taken with the same sign that it is assigned in the calculation of the coefficients b.

It is worth observing that the quadratic functions denominated F and G can be derived from the bilinear function f setting in the latter on the one hand

$$y_s = c_{1s}x_1 + c_{2s}x_2 + ... + c_{ns}x_n,$$

and on the other

$$x_r = c_{r1}y_1 + c_{r2}y_2 + ... + c_{rn}y_n.$$

Now if in these two relations one applies the substitutions (1), one then sees immediately that they are respectively converted into the following relations in the new variables ξ and η:

$$\eta_m = \gamma_m\xi_m, \quad \xi_m = \gamma_m\eta_m,$$

which transform the canonical bilinear function

$$\gamma_1\xi_1\eta_1 + \gamma_2\xi_2\eta_2 + ... + \gamma_n\xi_n\eta_n$$

into the respective quadratic functions

$$\gamma_1^2\xi_1^2 + \gamma_2^2\xi_2^2 + ... + \gamma_n^2\xi_n^2,$$
$$\gamma_1^2\eta_1^2 + \gamma_2^2\eta_2^2 + ... + \gamma_n^2\eta_n^2.$$

And, in fact, we have already noted that these two last functions are equivalent to the quadratics F, G.

We ask of what form must be the bilinear function f so that the two orthogonal substitutions that reduce it to the canonical form turn out substantially mutually identical. To this end, we observe that setting

$$b_{1s} = \pm a_{1s}, \quad b_{2s} = \pm a_{2s}, \quad ... \quad b_{ns} = \pm a_{ns}$$

the equations (4) are converted to the following:

$$c_{1r}a_{1s} + c_{2r}a_{2s} + ... + c_{nr}a_{ns} = \pm\gamma_s a_{rs},$$
$$c_{r1}a_{1s} + c_{r2}a_{2s} + ... + c_{rn}a_{ns} = \pm\gamma_s a_{rs},$$

which, since they must hold for every value of r and s, give $c_{rs} = c_{sr}$. Reciprocally this hypothesis implies the equivalence of the two linear substitutions. So every bilinear form of the desired form is *associated harmonically* with an ordinary quadratic form: that is to say that designating this quadratic form with

$$\psi = \Sigma_{rs}c_{rs}x_r y_s \qquad (c_{rs} = c_{sr})$$

the bilinear function is

$$f = \Sigma_s \frac{1}{2}\frac{d\psi}{dx_s} y_s.$$

In this case, the equation $\Delta = 0$ can be decomposed in this way:

$$\Delta = \begin{vmatrix} c_{11}-\gamma & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22}-\gamma & \ldots & c_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ c_{n1} & c_{n2} & \ldots & c_{nn}-\gamma \end{vmatrix} \begin{vmatrix} c_{11}+\gamma & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22}+\gamma & \ldots & c_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ c_{n1} & c_{n2} & \ldots & c_{nn}+\gamma \end{vmatrix}.$$

The first factor of the second member, set to zero, gives the well known equation that serves to reduce the function ψ to its canonical form. If the two substitutions are absolutely mutually identical, then coefficients γ are the same for the quadratic function and for the bilinear one. But if, as we have already supposed, one concedes the possibility of an opposition of sign between the coefficients a and b belonging to two columns of equal index, the coefficient γ_s of the corresponding index in the bilinear function can have sign opposite to that in the quadratic. From this, the presence in the equation $\Delta = 0$ of a factor having for roots the quantities γ taken negatively.

In the particular case just now considered, the quadratic function denoted F is

$$\mathrm{F} = \Sigma_r \left(\frac{1}{2}\frac{d\psi}{dx_r}\right)^2,$$

and, on the other hand, one finds that there always exists an orthogonal substitution which makes simultaneously identical the two equations

$$\psi = \Sigma\gamma\xi^2, \quad \Sigma\left(\frac{1}{2}\frac{d\psi}{dx}\right)^2 = \Sigma\gamma^2\xi^2.$$

This is a consequence of the fact that, as is well known,

$$\Sigma_r \left(\frac{1}{2}\frac{d}{dx_r}\right)^2$$

is a symbol invariant with respect to any orthogonal substitution.

We translate into geometrical language the results of the preceding analysis, assuming (as is generally useful to do) that the signs of the coefficients γ are chosen to make $\mathrm{AB} = 1$ and hence $\Gamma = \mathrm{C}$.

Let S_n, S'_n be two spaces of n dimensions with null curvature, referred to, respectively, by the two systems of orthogonal linear coordinates x and y, for which we will call O and O′ the origins. To a straight line S_1 drawn through the origin O in the space S_n, there corresponds a specific set of ratios $x_1 : x_2 : ... : x_n$; and on the other hand, the equation $f = 0$, homogeneous and of first degree in the x_1, x_2, ..., x_n and in the y_1, y_2, ..., y_n defines a correlation of figures in which to each line through the point O in the space S_n corresponds a locus of first order in $n-1$ dimensions that we call S'_{n-1} within the space S'_n; and vice-versa.

By virtue of the demonstrated theorem, it is always possible to substitute for the original coordinate axes in the x's and y's new axes in the ξ's and η's, respectively, with the same origins O and O$'$, so that the correlation rule assumes the simpler form

$$\gamma_1\xi_1\eta_1 + \gamma_2\xi_2\eta_2 + ... + \gamma_n\xi_n\eta_n = 0.$$

Said this, one may think of the axis system η, together with its figure, moved so that its origin O$'$ falls on O, and that each axis η_r falls on its homologous axis ξ_r (*). In such a hypothesis, the last equation expresses evidently that the two figures are found to be in *polar* or *involutory* correlation with respect to the quadric cone (in $n-1$ dimensions)

$$\gamma_1\xi_1^2 + \gamma_2\xi_2^2 + ... + \gamma_n\xi_n^2 = 0$$

that has its vertex on O. Hence, one can always convert a correlation of first degree of the above type, through a motion of one of the figures, into a polar or involutory correlation with respect to a quadric cone (in $n-1$ dimensions) having its vertex on the common center of the two figures overlaid.

In the case of $n = 2$, this general proposition yields the very well known theorem that two homographic bundles of rays can always be overlaid in such a way that they constitute a quadratic involution of rays.

In the case that $n = 3$, one has the theorem, also known, that two correlative stars (i.e., such that to every ray in one corresponds a plane in the other, and vice-versa) can always be overlaid in such a way that they become reciprocal polar with respect to a quadric cone having its vertex on the common center.

One can interpret the analytic theorem in another way, and recover other geometric properties in the cases $n = 2$, and $n = 3$. If with

$$y_1Y_1 + y_2Y_2 + ... + y_nY_n + 1 = 0$$

(*) Something that is possible by having AB = 1.

one represents a locus of first order (S'_{n-1}) lying in the space S'_n, to every orthogonal substitution of the form

$$y_s = \Sigma_q b_{sq} \eta_q$$

applied on the local coordinates y, corresponds an *identical* orthogonal substitution

$$\mathrm{Y}_s = \Sigma_q b_{sq} \mathrm{E}_q$$

applied to the tangential coordinates Y. Given this, suppose that between the x's and y's one institutes the $n-1$ relations that result from setting equal the n ratios

$$\frac{y_r}{c_{1r}x_1 + c_{2r}x_2 + ... + c_{nr}x_n} \qquad (r = 1, 2, ...n).$$

This is equivalent to considering two *homographic* stars with centers on the points O and O$'$. With such hypotheses, the equation

$$y_1\mathrm{Y}_1 + y_2\mathrm{Y}_2 + ... + y_n\mathrm{Y}_n = 0,$$

which corresponds to this other equation in the η-axis system

$$\eta_1\mathrm{E}_1 + \eta_2\mathrm{E}_2 + ... + \eta_n\mathrm{E}_n = 0,$$

is equivalent to the following relation between the x's and the y's

$$\Sigma_{rs} c_{rs} x_r \mathrm{Y}_s = 0,$$

and this is in turn reducible, with two simultaneous orthogonal substitutions, to the canonical form

$$\gamma_1\xi_1\mathrm{E}_1 + \gamma_2\xi_2\mathrm{E}_2 + ... + \gamma_n\xi_n\mathrm{E}_n = 0.$$

Now the relation that this establishes between the new tangential coordinates E cannot differ from that contained in the third to last equation; hence it must be

$$\frac{\eta_1}{\gamma_1\xi_1} = \frac{\eta_2}{\gamma_2\xi_2} = ... = \frac{\eta_n}{\gamma_n\xi_n}.$$

From these equations, which are nothing else than the relations of homography, expressed in the new coordinates ξ and η, it emerges evidently that the axis ξ_1 and η_1, ξ_2 and η_2, ... ξ_n and η_n are pairs of corresponding straight lines in the two stars. Since it is thus possible to move one of the stars in such a way that the ξ axes and the η axes coincide one for one, one concludes that, given two homographic stars in an n dimensional space, one can always overlay one upon the other so that the n double rays acted upon by the overlaying, constitute a system of n orthogonal cartesian rays.

From this, setting $n = 2$ and $n = 3$, one deduces that two homographic groups of rays can always be overlaid so that the two double rays are orthogonal; and that the two homographic stars can always be overlaid so that the three double rays form an orthogonal cartesian triple.

Returning now to the hypothesis of two *arbitrary* linear substitutions, but always acting to give to the transformed function φ the canonical form $\varphi = \Sigma_m \gamma_m \xi_m \eta_m$, one solve the equations (3) with respect to the quantities h and k, respectively, with which one finds the two typical equations, mutually equivalent,

$$\text{(6)} \qquad \mathrm{B} h_{rs} = \mathrm{B}_{rs} \gamma_s, \qquad \mathrm{A} k_{rs} = \mathrm{A}_{rs} \gamma_s,$$

in which A_{rs}, B_{rs} are the algebraic complements of the elements a_{rs}, b_{rs} in the respective determinants A, B. The equations (4) are nothing else than particularizations of these.

Let f' be a second bilinear function, given by the expression

$$f' = \Sigma_{rs} c'_{rs} x_r y_s,$$

and suppose we want to transform simultaneously, with the very same linear substitutions (1), the function f into the canonical form φ and the function f' into the canonical form $\varphi' = \Sigma_m \gamma'_m \xi_m \eta_m$.

Indicating with h' and k' quantities analogous to the h and k for the second function, together with the equations (6), these further equations must hold

$$\text{(6)}' \qquad \mathrm{B} h'_{rs} = \mathrm{B}_{rs} \gamma'_s, \qquad \mathrm{A} k'_{rs} = \mathrm{A}_{rs} \gamma'_s.$$

Dividing the first two equations of each of the pairs (6) and (6)$'$ one by the other, one obtains

$$h_{rs} - \lambda_s h'_{rs} = 0 \qquad \text{where} \qquad \lambda_s = \frac{\gamma_s}{\gamma'_s},$$

or

$$\text{(7)} \qquad (c_{1r} - \lambda_s c'_{1r}) a_{1s} + (c_{2r} - \lambda_s c'_{2r}) a_{2s} + \ldots + (c_{nr} - \lambda_s c'_{nr}) a_{ns} = 0.$$

Setting $r = 1, 2, ...n$ in this last equation and eliminating the quantities a_{1s}, a_{2s}, ... a_{ns} from the n equations obtained in this way, one arrives at the equation

$$\Theta = \begin{vmatrix} c_{11} - \lambda c'_{11} & c_{12} - \lambda c'_{12} & ... & c_{1n} - \lambda c'_{1n} \\ c_{21} - \lambda c'_{21} & c_{22} - \lambda c'_{22} & ... & c_{2n} - \lambda c'_{2n} \\ ... ... & ... ... & ... & ... ... \\ c_{n1} - \lambda c'_{n1} & c_{n2} - \lambda c'_{n2} & ... & c_{nn} - \lambda c'_{nn} \end{vmatrix} = 0$$

which must be satisfied by the n values λ_1, λ_2, ... λ_n of λ. One can convince oneself of this by another way, by observing that Θ is the determinant of the bilinear function $f - \lambda f'$, and that hence one has by virtue of the general theorem on the transformation of this determinant (*)

$$\Theta = \mathrm{AB}(\gamma_1 - \lambda\gamma'_1)(\gamma_2 - \lambda\gamma'_2)...(\gamma_n - \lambda\gamma'_n),$$

whence emerges just as expected, for A and B to be different from zero, that the equation $\Theta = 0$ has for its roots the n ratios

$$\frac{\gamma_1}{\gamma'_1}, \quad \frac{\gamma_2}{\gamma'_2}, \quad ...\frac{\gamma_n}{\gamma'_n}.$$

Of the rest, since the two series of quantities γ and γ' are not determined except by these ratios, it is useful to assume for more simplicity that $\gamma'_1 = \gamma'_2 = ... = \gamma'_n = 1$, and hence $\gamma_s = \lambda_s$.

Here then is the procedure that leads to the solution of the problem: Find the n roots (real or imaginary) λ_1, λ_2, ..., λ_n of the equation $\Theta = 0$; then substitute them successively into the equations of the form (7). One obtains in this way, for each value of s, n linear and homogeneous equations, one of which is a consequence of the other $n - 1$, so that one can recover only the values of the ratios $a_{1s} : a_{2s} : ... : a_{ns}$. Having chosen arbitrarily the values of the quantities $a_{1s}, a_{2s}, ..., a_{ns}$ so that they have these mutual ratios, the equations that are of the form of the second part of $(6)'$ yield

$$b_{rs} = \frac{\mathrm{C}'_{1r}\mathrm{A}_{1s} + \mathrm{C}'_{2r}\mathrm{A}_{2s} + ... + \mathrm{C}'_{nr}\mathrm{A}_{ns}}{\mathrm{C}'\mathrm{A}}; \tag{8}$$

and in this way all the unknown quantities can be determined.

(*) from this one sees that the determinant of a bilinear function is zero only when the function itself can be reduced to contain two fewer variables, a property that one can easily show directly.

One can observe that if for the coefficients a_{rs} one substitutes the products $\rho_s a_{rs}$, something that is legitimate by a previous observation, the coefficients b_{rs}, determined by the last equation, are converted into $\frac{b_{rs}}{\rho_s}$. This is the same as saying that if for the variables ξ_s one substitutes $\rho_s \xi_s$, the variables η_s are converted to $\frac{\eta_s}{\rho_s}$; but this change leaves unaltered the transformed functions φ and φ'.

If one denotes by Θ_{rs} the algebraic complement of the element $c_{rs} - \lambda c'_{rs}$ in the determinant Θ, and by $\Theta_{rs}(\lambda_s)$ what results by setting $\lambda = \lambda_s$ in this complement, it is easy to see that the equations (7) are satisfied by setting

$$a_{rs} = \alpha_s \Theta_{rs}(\lambda_s).$$

In this way, if one forms the equations analogous to (7) and containing the coefficients b in place of the coefficients a, they can be satisfied by setting

$$b_{rs} = \beta_s \Theta_{sr}(\lambda_s).$$

The quantities α_s and β_s are not determined completely: but the equations analogous to (2) determine the product $\alpha_s \beta_s$. In any case, these factors are not essential, since by writing ξ_s in place of $\alpha_s \xi_s$ and η_s in place of $\beta_s \eta_s$, they can be removed. From the expressions

$$a_{rs} = \Theta_{rs}(\lambda_s), \qquad b_{rs} = \Theta_{sr}(\lambda_s),$$

that result from this supposition, it is evident that this important property emerges, that when $c_{rs} = c_{sr}$, $c'_{rs} = c'_{sr}$, i.e., when the two bilinear functions are associated harmonically to two quadratic functions, the substitutions that reduce them both simultaneously to canonical form *are substantially mutually identical.*

If we suppose that the second bilinear function f' has the form

$$f' = x_1 y_1 + x_2 y_2 + ... + x_n y_n,$$

the general problem just treated assumes the following form: Reduce the bilinear function f to the canonical form

$$\varphi = \lambda_1 \xi_1 \eta_1 + \lambda_2 \xi_2 \eta_2 + ... + \lambda_n \xi_n \eta_n$$

with two simultaneous linear substitutions, so that the function

$$x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

is transformed into itself. In this case, the general formula (8) becomes

$$\mathrm{A}b_{rs} = \mathrm{A}_{rs},$$

whence

$$\mathrm{A}y_r = \mathrm{A}_{r1}\eta_1 + \mathrm{A}_{r2}\eta_2 + ... + \mathrm{A}_{rn}\eta_n,$$

and hence

$$\eta_s = a_{1s}y_1 + a_{2s}y_2 + ... + a_{ns}y_n.$$

This last formula shows that the two substitutions (a) and (b) are *inverses* of each other, something that follows necessarily from the nature of the function that is transformed into itself. The relations among the coefficients a and b that follow from this

$$\begin{gathered}\Sigma_r a_{rs}b_{rs} = 1, \qquad \Sigma_s a_{rs}b_{rs} = 1\\ \Sigma_r a_{rs}b_{rs'} = 0, \qquad \Sigma_s a_{rs}b_{r's} = 0,\\ \mathrm{AB} = 1\end{gathered}$$

make a perfect contrast to those that hold among the coefficients of *one* orthogonal substitution, and they reduce to them when $a_{rs} = b_{rs}$, such a case arising (by what we have recently seen) when the form f is *associated* to a quadratic form. In the special problem which we have mentioned (and which has already been treated by Mr. Christoffel at the end of his Memoirs), the equation $\Theta = 0$ takes on the form

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22}-\lambda & \dots & c_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ c_{n1} & c_{n2} & \dots & c_{nn}-\lambda \end{vmatrix} = 0,$$

and under the hypothesis $c_{rs} = c_{sr}$ (that we have just alluded to), it is identified, as is natural, with what the analogous problem leads to via a true quadratic form.

We will not add any word on the geometric interpretation of the preceding results, since their intimate connection with the whole theory of homogeneous coordinates is evident.

Likewise, we will not discuss for now the transformations of bilinear functions into themselves, an important argument but less easy to handle than the preceding ones, for which Mr. Christoffel has already presented the treatment in the case of a single function and a single substitution.