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On the optimal approximation for the symmetric Procrustes problems of the matrix equation $AXB = C$

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Abstract

The explicit analytical expressions of the optimal approximation solutions for the symmetric Procrustes problems of the linear matrix equation $AXB = C$ are derived, with the projection theorem in Hilbert space , the quotient singular value decomposition (QSVD) and the canonical correlation decomposition (CCD) being used.

Key words: Linear matrix equation, least squares problem, optimal approximation , QSVD , CCD
MSC 2000: 65F15,65F20

1 Introduction

The least-squares problems of linear matrix equations are called Procrustes problems(cf. Higham,1988 and Andersson and Elfving, 1997). The unconstrained and constrained least squares problems have been of interest for many applications, including particle physics and geology , inverse Sturm-Liouville problem [11], inverse problems of vibration theory [6], control theory, digital image and signal processing, photogrammetry, finite elements, and multidimensional approximation [8]. Penrose(cf. [2], [13]) first considered the linear matrix equation

$$AX = B \tag{1.1}$$

and obtained its general solution and least-squares solution by making use of the Moore-Penrose generalized inverse, then Sun[14] obtained the least-squares solution and the related optimal approximation solution of Eq. (1.1) when X is a real matrix. When X

is constrained to be a real symmetric matrix , the least-squares solution of (1.1) was derived by Higham and Sun respectively in 1988([12] and [15]) , and Sun also obtained the related symmetric optimal approximation solution of Eq. (1.1) in [15].

In this paper , the following linear matrix equation

$$AXB = C \tag{1.2}$$

are considered . Fausett and Fulton[8] and Zha[18] considered the unconstrained least-squares problems of Eq. (1.2), Eric Chu[4] and Dai Hua[5] obtained the general expressions for the symmetric solution of Eq. (1.2) by using the general singular value decomposition of matrices (GSVD), and the symmetric and skew-symmetric least-squares solutions of Eq. (1.2) have been derived by Deng, Hu and Zhang[7]. But it remains unsolved about the optimal approximation solutions for the symmetric and skew-symmetric Procrustes problems of this equation. Therefore in the following, we will consider the optimal approximation solutions of the symmetric least squares problems of Eq. (1.2). We always suppose that $R^{m\times n}$ is the set of all $m \times n$ real matrices, $SR^{n\times n}$ and $OR^{n\times n}$ are the sets of all symmetric and orthogonal matrices in $R^{n\times n}$, respectively, $A * B$ represent the Hadamard product of A and B, and $\|Y\|_F$ denotes the Frobenius norm of a real matrix Y, defined as

$$\|Y\|_F^2 =< Y, Y >= \sum_{i,j} y_{ij}^2,$$

here the inner product is given by $< A, B >= trace(A^T B)$, and $R^{m\times n}$ become a Hilbert space with the inner product.

Problem I. Given matrices $A \in R^{m\times n}$,$B \in R^{n\times p}$,$C \in R^{m\times p}$ and $X_f \in R^{n\times n}$, let

$$S_E = \{X|X \in SR^{n\times n}, \|AXB - C\|_F = min\}. \tag{1.3}$$

Then find $X_e \in S_E$, such that

$$\|X_e - X_f\|_F = \min_{X\in S_E} \|X - X_f\|_F. \tag{1.4}$$

We first introduce some results about the quotient singular value decomposition (QSVD) and the canonical correlation decomposition (CCD)of matrices , as soon as the projection theorem on Hilbert space, which are essential tools for the Problem , see [3],[9], [10] and [16] for details.

The QSVD is a simple form of the GSVD. The QSVD of a pair of matrices (A, B^T) is as follows.

QSVD Theorem. *Let $A \in R^{m\times n}, B \in R^{n\times p}$. Then there exist orthogonal matrices $U \in OR^{m\times m}, V \in OR^{p\times p}$ and a nonsingular matrix $Y \in R^{n\times n}$ such that*

$$A = U\Sigma_1 Y^{-1}, \quad B^T = V\Sigma_2 Y^{-1}, \tag{1.5}$$

where

$$\Sigma_1 = \begin{array}{cc} \left(\begin{array}{cccc} I_{r'} & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{array}{l} r' \\ s' \\ m-r'-s' \end{array} \\ \begin{array}{cccc} r' & s' & t' & n-k' \end{array} & \end{array}, \tag{1.6}$$

$$\Sigma_2 = \begin{array}{cc} \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & I_{s'} & 0 & 0 \\ 0 & 0 & I_{t'} & 0 \end{array}\right) & \begin{array}{l} p+r'-k' \\ s' \\ t' \end{array} \\ \begin{array}{cccc} r' & s' & t' & n-k' \end{array} & \end{array}, \tag{1.7}$$

$$k' = rank(A^T, B), r' = k' - rank(B),$$
$$s' = rank(A) + rank(B) - k', S = diag(\sigma_1, \cdots, \sigma_{s'}),$$
$$\sigma_i > 0 (i = 1, \cdots, s'), t' = k' - r' - s'.$$

When A and B^T are of full column rank, i.e. $r(B) = r(A) = n$, then $r' = 0, s' = n, k' = n$, and

$$\Sigma_1 = \begin{array}{cc} \left(\begin{array}{c} S \\ 0 \end{array}\right) & \begin{array}{c} n \\ m-n \end{array} \\ n & \end{array}, \qquad \Sigma_2 = \begin{array}{cc} \left(\begin{array}{c} 0 \\ I_{s'} \end{array}\right) & \begin{array}{c} p-n \\ n \end{array} \\ n & \end{array}. \tag{1.8}$$

The canonical correlations decomposition of the matrix pair (A^T, B) is given by the following theorem.

CCD Theorem. Let$A \in R^{m\times n}, B \in R^{n\times p}$, and assume that $g = rank(A), h = rank(B)$, $g \geq h$. Then there exist a orthogonal matrix $Q \in OR^{n\times n}$ and nonsingular matrices $X_A \in R^{m\times m}, X_B \in R^{p\times p}$ such that

$$A^T = Q[\Sigma_A, 0]X_A^{-1}, \qquad B = Q[\Sigma_B, 0]X_B^{-1}, \tag{1.9}$$

where $\Sigma_A \in R^{n\times g}$ and $\Sigma_B \in R^{n\times h}$ are of the forms:

$$\Sigma_A = \left(\begin{array}{ccc} I_i & 0 & 0 \\ 0 & \Lambda_j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Delta_j & 0 \\ 0 & 0 & I_t \end{array}\right), \qquad \Sigma_B = \left(\begin{array}{c} I_h \\ 0 \end{array}\right), \tag{1.10}$$

with the same row partitioning, and

$$\begin{array}{l} \Lambda_j = diag(\lambda_{i+1}, \ldots, \lambda_{i+j}), \quad 1 > \lambda_{i+1} \geq \ldots \geq \lambda_{i+j} > 0, \\ \Delta_j = diag(\delta_{i+1}, \ldots, \delta_{i+j}), \quad 0 < \delta_{i+1} \leq \ldots \leq \delta_{i+j} < 1, \\ \lambda_{i+1}^2 + \delta_{i+1}^2 = 1, \ldots, \lambda_{i+j}^2 + \delta_{i+j}^2 = 1, \quad i.e., \Lambda_j^2 + \Delta_j^2 = I, \end{array}$$

Here,

$$\begin{array}{l} i = rank(A) + rank(B) - rank[A^T, B], \\ j = rank[A^T, B] + rank(AB) - rank(A) - rank(B), \\ t = rank(A) - rank(AB), \quad g = i + j + t. \end{array}$$

Following is the projection theorem (cf. [16]).

Lemma 1.1 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ be a subspace of $\mathcal{H}$, and $\mathcal{M}^\perp$ be the orthogonal complement subspace of $\mathcal{M}$. For a given $H \in \mathcal{H}$, if there exists an $M_0 \in \mathcal{M}$ such that $\|H - M_0\| \leq \|H - M\|$ holds for any $M \in \mathcal{M}$, then M_0 is unique and $M_0 \in \mathcal{M}$ is the unique minimization vector in $\mathcal{M}$ if and only if $(H - M_0) \perp \mathcal{M}$, i.e.,$(H - M_0) \in \mathcal{M}^\perp$.*

2 The main results

In this section, the explicit expression for the solution of Problem I is derived. Without loss of generality, we suppose that $rank(A) \geq rank(B)$.

Instead of considering the solution of Problem I, we will find a matrix C_0, and then transform Problem I to the following equivalent problem.

Problem I_0. Given matrices $A \in R^{m\times n}$,$B \in R^{n\times p}$,$C_0 \in R^{m\times p}$ and $X_f \in R^{n\times n}$, let

$$S_{E_0} = \{X|X \in SR^{n\times n}, AXB = C_0\}. \tag{2.11}$$

Then find $X_e \in S_{E_0}$, such that

$$\|X_e - X_f\|_F = \min_{X\in S_{E_0}} \|X - X_f\|_F. \tag{2.12}$$

First we use the projection theorem on $R^{m\times p}$.

Theorem 2.1 *Given $A \in R^{m\times n}, B \in R^{n\times p}, C \in R^{m\times p}$, let X_0 be one of the symmetric least-squares solutions of the matrix equation (1.2) and define*

$$C_0 = AX_0B, \tag{2.13}$$

then the matrix equation

$$AXB = C_0, \tag{2.14}$$

is consistent in $SR^{n\times n}$, and the symmetric solution set S_{E_0} of the matrix equation (2.13) is the same as the symmetric least-squares solution set S_E of the matrix equation (1.2).

Proof. Let

$$\mathcal{L} = \{Z|Z = AXB, X \in SR^{n\times n}\}. \tag{2.15}$$

Then $\mathcal{L}$ is obviously a linear subspace of $R^{m\times p}$.Because X_0 is the symmetric least-squares solutions of the matrix equation (1.2), from (2.13) we see that $C_0 \in \mathcal{L}$ and

$$\|C_0 - C\|_F = \|AX_0B - C\|_F$$

$$= \min_{X\in SR^{n\times n}} \|AXB - C\|_F$$

$$= \min_{Z\in\mathcal{L}} \|Z - C\|_F.$$

Then by Lemma 1.1 we have

$$(C_0 - C)\perp\mathcal{L} \;\; or \;\; (C_0 - C) \in \mathcal{L}^{\perp}.$$

Next for all $X \in SR^{n\times n}$, $AXB - C_0 \in \mathcal{L}$, it then follows that

$$\begin{aligned}
&\|AXB - C\|_F^2 \\
&= \|(AXB - C_0) + (C_0 - C)\|_F^2 \\
&= \|AXB - C_0\|_F^2 + \|C_0 - C\|_F^2.
\end{aligned}$$

Hence, $S_E = S_{E_0}$, and the conclusion of the theorem is true. $\square$

Now suppose $A \in R^{m\times n}, B \in R^{n\times p}$ and the matrix pair (A, B^T) has the QSVD (1.5), and partition U^TCV into the following blocks matrix.

$$U^TCV = \begin{array}{cl} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} & \begin{array}{l} r' \\ s' \\ m-r'-s' \end{array} \\ \begin{array}{ccc} p+r'-k' & s' & t' \end{array} & \end{array}, \tag{2.16}$$

then the expression of C_0 will be shown in the following theorem.

Theorem 2.2 *Let A, B, C be given in Problem I, the matrix pair (A, B^T) have the QSVD (1.5), and U^TCV be partitioned by (2.16), then for any symmetric least-squares solution X_0 of the matrix equation (1.2) the matrix C_0 defined by (2.13) can be determined by the following form.*

$$C_0 = UC^*V^T, \quad C^* = \begin{array}{cl} \begin{pmatrix} 0 & C_{12} & C_{13} \\ 0 & S\hat{X}_{22} & C_{23} \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} r' \\ s' \\ m-r'-s' \end{array} \\ \begin{array}{ccc} p+r'-k' & s' & t' \end{array} & \end{array}, \tag{2.17}$$

where

$$\begin{aligned}
&\hat{X}_{22} = \phi * (C_{22}^T S + SC_{22}), \\
&\phi = (\varphi_{kl}) \in SR^{s'\times s'}, \varphi_{kl} = \tfrac{1}{\sigma_k^2+\sigma_l^2}, 1 \le k, l \le s'.
\end{aligned} \tag{2.18}$$

Proof. From Theorem 2.1 in [7] we know that the symmetric least-squares solution of the matrix equation (1.2) can be obtained using of the QSVD of matrix pair (A, B^T) and the general form of the solution is

$$X_0 = Y \begin{bmatrix} X'_{11} & C_{12} & C_{13} & X'_{14} \\ C_{12}^T & \hat{X}_{22} & S^{-1}C_{23} & X'_{24} \\ C_{13}^T & (S^{-1}C_{23})^T & X'_{33} & X'_{34} \\ X'^T_{14} & X'^T_{24} & X'^T_{34} & X'_{44} \end{bmatrix} Y^T, \tag{2.19}$$

where $\hat{X}_{22}$ is given by (2.18) and $X'_{11} \in SR^{r'\times r'}, X'_{33} \in SR^{t'\times t'}, X'_{44} \in SR^{(n-k')\times(n-k')}$,

$X'_{14} \in R^{r' \times (n-k')}$, $X'_{24} \in R^{s' \times (n-k')}$, $X'_{34} \in R^{t' \times (n-k')}$ are arbitrary matrix blocks. Substituting (1.5),(2.19) into (2.13), we can easily obtain (2.17). $\square$

Evidently, (2.17) shows that the matrix C_0 in theorem 2.2 is dependent only on the matrices A, B and C, but is independent on the symmetric least-squares solution X of the matrix equation (1.2). Since C_0 is known, from Theorem 2.1 we know that Problem I is equivalent to Problem I_0. In Problem I_0, since $S_{E_0} \neq \emptyset$, we can derive the general expression of of the elements of S_{E_0} in the following theorem. In this theorem, given $A \in R^{m \times n}, B \in R^{n \times p}$,while C_0 is given by (2.17),and assume that $g = rank(A), h = rank(B)$, the matrix pair (A^T, B) has CCD (1.9).Notice that we only state the result with $g = h$, because in the case $g > h$, the results of the theorem and process of the proof are similar, only the partitions of the related matrices are more complex.

Suppose $X \in S_{E_0}$, then partition the symmetric matrix $X^* \equiv Q^T X Q$ into blocks matrix,

$$X^* = (X_{kl})_{6 \times 6}, \tag{2.20}$$

with the row numbers (and the related column numbers) of blocks are $i, j, t, n - g - j - t, j, t$ respectively, and $X_{kl} = X_{lk}^T, k, l = 1, 2, \ldots, 6$. Let $E = X_A^T C_0 X_B$ and also partition E into blocks matrix,

$$E = (E_{kl})_{4 \times 4}, \tag{2.21}$$

with the row numbers of blocks are $i, j, t, m - g$ and the column numbers of blocks are $i, j, t, p - g$ respectively.

Theorem 2.3 *In Problem I_0, the general form of the elements of S_{E_0} can be expressed as $X = QX^*Q^T$, where X^* has the form*

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} & X_{14} & X_{51}^{*T} & E_{31}^T \\ E_{12}^T & X_{22} & X_{23} & X_{24} & X_{52}^{*T} & E_{32}^T \\ E_{13}^T & X_{23}^T & X_{33} & X_{34} & X_{53}^{*T} & E_{33}^T \\ X_{14}^T & X_{24}^T & X_{34}^T & X_{44} & X_{45} & X_{46} \\ X_{51}^* & X_{52}^* & X_{53}^* & X_{45}^T & X_{55} & X_{56} \\ E_{31} & E_{32} & E_{33} & X_{46}^T & X_{56}^T & X_{66} \end{pmatrix} \tag{2.22}$$

where $X_{51}^* = \Delta_j^{-1}(E_{21} - \Lambda_j E_{12}^T), X_{52}^* = \Delta_j^{-1}(E_{22} - \Lambda_j X_{22}), X_{53}^* = \Delta_j^{-1}(E_{23} - \Lambda_j X_{23})$,while $X_{kk} = X_{kk}^T, 2 \leq k \leq 6$, $X_{14}, X_{23}, X_{24}, X_{34}, X_{45}, X_{46}$ and X_{56} are arbitrary matrices with the associated sizes.

Proof. Suppose $X \in S_{E_0}$, then

$$AXB = C_0. \tag{2.23}$$

Substitute (1.9) into (2.23),we have

$$\begin{pmatrix} \Sigma_A^T \\ 0 \end{pmatrix} X^* (\Sigma_B, 0) = E, \tag{2.24}$$

then substitute (1.10),(2.20) and (2.21) into (2.24), it holds

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & 0 \\ \Lambda_j X_{21}+\Delta_j X_{51} & \Lambda_j X_{22}+\Delta_j X_{52} & \Lambda_j X_{23}+\Delta_j X_{53} & 0 \\ X_{61} & X_{62} & X_{63} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix} \tag{2.25}$$

Because the matrix equation (2.23) is consistent, therefore we can obtain some X_{ij} from (2.25) directly. Comparing with both sides of (2.25) , the expression (2.22) of X^* can be derived according to the symmetric property of X^*. $\square$

The following lemmas are needed for the main results.

Lemma 2.1 [17] *For given J_1, J_2, J_3 and $J_4 \in R^{m\times n}$,*

$$S_a = diag(a_1,\dots,a_m) > 0,\quad S_b = diag(b_1,\dots,b_m) > 0,$$
$$S_c = diag(c_1,\dots,c_m) > 0,\quad S_d = diag(d_1,\dots,d_m) > 0,$$

there exists a unique $W \in R^{m\times n}$, such that

$$\|S_aW-J_1\|_F^2+\|S_bW-J_2\|_F^2+\|S_cW-J_3\|_F^2+\|S_dW-J_4\|_F^2 = min$$

and W can be expressed as

$$W = P*(S_aJ_1+S_bJ_2+S_cJ_3+S_dJ_4),$$

where

$$P=(p_{kl})\in R^{m\times n}, p_{kl}=1/(a_k^2+b_k^2+c_k^2+d_k^2), 1\le k\le m, 1\le l\le n.$$

Lemma 2.2 *For given J_1, J_2 and $J_3 \in R^{s\times s}$, $S_a = diag(a_1,\dots,a_s) > 0$,$S_b = diag(b_1,\dots,b_s) > 0$, $S_c = diag(c_1,\dots,c_s) > 0$, there exists a unique symmetric matrix $W \in SR^{s\times s}$, such that*

$$\mu \equiv \|S_aW-J_1\|_F^2+\|S_bW-J_2\|_F^2+\|S_cW-J_3\|_F^2 = min,$$

and W can be expressed as

$$W = \Phi*(S_aJ_1+J_1^TS_a+S_bJ_2+J_2^TS_b+S_cJ_3+J_3^TS_c), \tag{2.26}$$

where

$$\Phi=(\phi_{kl})\in R^{s\times s}, \phi_{kl}=1/(a_k^2+a_l^2+b_k^2+b_l^2+c_k^2+c_l^2), 1\le k,l\le s.$$

Proof. For $W\in SR^{s\times s}$, it holds $w_{kl}=w_{lk}$ $(1\le k,l\le s)$, and

$$\begin{aligned}\mu &= \textstyle\sum_{k=1}^s[(a_kw_{kk}-J_{1kk})^2+(b_kw_{kk}-J_{2kk})^2+(c_kw_{kk}-J_{3kk})^2] \\ &\quad +\textstyle\sum_{1\le k<l\le s}[(a_kw_{kl}-J_{1kl})^2+(a_lw_{kl}-J_{1lk})^2+(b_kw_{kl}-J_{2kl})^2 \\ &\qquad +(b_lw_{kl}-J_{2lk})^2+(c_kw_{kl}-J_{3kl})^2+(c_lw_{kl}-J_{3lk})^2].\end{aligned}$$

Since the function μ is a continuous and differentiable function of $\frac{1}{2}s(s+1)$ variables w_{kl}, hence μ obtains its minimum value at $\{w_{kl}\}$ when $\frac{\partial \mu}{\partial w_{kl}} = 0$, *i.e.*,

$$w_{kl} = \frac{a_k J_{1kl} + a_l J_{1lk} + b_k J_{2kl} + b_l J_{2lk} + c_k J_{3kl} + c_l J_{3lk}}{a_k^2 + a_l^2 + b_k^2 + b_l^2 + c_k^2 + c_l^2}, \quad 1 \leq k \leq l \leq s.$$

Therefore W can be expressed by (2.26). $\square$

Finally we give the the optimal approximation solutions for the symmetric least-squares problems of the linear matrix equation $AXB = C$, and we still suppose that $rank(A) = rank(B)$.

Theorem 2.4 *Let matrices A, B, C and X_f be given in Problem I, suppose $rank(A) = rank(B)$, partition the matrix $Q^T X_f Q$ into blocks matrix*

$$Q^T X_f Q = (X_{kl}^{(f)})_{6\times 6}, \tag{2.27}$$

with the same row and column numbers as X^* of (2.20). Then the unique solution X_e of Problem I can be expressed as $X_e = Q X_* Q^T$, and X_* is equal to

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} & \{X_{14}^{(f)}\} & \bar{X}_{51}^T & E_{31}^T \\ E_{12}^T & \bar{X}_{22} & \bar{X}_{23} & \{X_{24}^{(f)}\} & \bar{X}_{52}^T & E_{32}^T \\ E_{13}^T & \bar{X}_{23}^T & \{X_{33}^{(f)}\} & \{X_{34}^{(f)}\} & \bar{X}_{53}^T & E_{33}^T \\ \{X_{41}^{(f)}\} & \{X_{42}^{(f)}\} & \{X_{43}^{(f)}\} & \{X_{44}^{(f)}\} & \{X_{45}^{(f)}\} & \{X_{46}^{(f)}\} \\ \bar{X}_{51} & \bar{X}_{52} & \bar{X}_{53} & \{X_{54}^{(f)}\} & \{X_{55}^{(f)}\} & \{X_{56}^{(f)}\} \\ E_{31} & E_{32} & E_{33} & \{X_{64}^{(f)}\} & \{X_{65}^{(f)}\} & \{X_{66}^{(f)}\} \end{pmatrix} \tag{2.28}$$

where $\bar{X}_{51} = \Delta_j^{-1}(E_{21} - \Lambda_j E_{12}^T), \bar{X}_{52} = \Delta_j^{-1}(E_{22} - \Lambda_j \bar{X}_{22}), \bar{X}_{53} = \Delta_j^{-1}(E_{23} - \Lambda_j \bar{X}_{23}),$

$$\{X_{kl}^{(f)}\} = \frac{1}{2}(X_{kl}^{(f)} + X_{lk}^{(f)T}) = \{X_{lk}^{(f)}\}^T,$$

$$\bar{X}_{22} = \Psi * [X_{22}^{(f)} + X_{22}^{(f)T} + \Delta_j^{-1}\Lambda_j(\Delta_j^{-1}E_{22} - X_{25}^{(f)T}) + (\Delta_j^{-1}E_{22} - X_{25}^{(f)T})^T \Lambda_j \Delta_j^{-1}$$

$$+\Delta_j^{-1}\Lambda_j(\Delta_j^{-1}E_{22} - X_{52}^{(f)}) + (\Delta_j^{-1}E_{22} - X_{52}^{(f)})^T \Lambda_j \Delta_j^{-1}],$$

$$\Psi = (\psi_{kl}) \in R^{j\times j}, \quad \psi_{kl} = \frac{1}{2(1 + (\frac{\delta_{i+k}}{\lambda_{i+k}})^2) + (\frac{\delta_{i+l}}{\lambda_{i+l}})^2)}, 1 \leq k, l \leq j.$$

and

$$\bar{X}_{23} = G * [X_{23}^{(f)} + X_{32}^{(f)T} + \Delta_j^{-1}\Lambda_j(\Delta_j^{-1}E_{23} - X_{35}^{(f)T}) + \Delta_j^{-1}\Lambda_j(\Delta_j^{-1}E_{23} - X_{53}^{(f)}),$$

$$G = (g_{kl}) \in R^{i\times t}, \quad g_{kl} = \frac{1}{2}\lambda_{i+k}, 1 \leq k, \leq i, 1 \leq l \leq t.$$

*P*roof. Suppose $X \in S_E = S_{E_0}$, by using (2.22) and (2.27), we have

$$
\begin{aligned}
&\|X - X_f\|_F^2 = \|X^* - Q^T X_f Q\|_F^2 \\
&= (\|X_{33} - X_{33}^{(f)}\|_F^2) + (\|X_{44} - X_{44}^{(f)}\|_F^2) + (\|X_{55} - X_{55}^{(f)}\|_F^2) + (\|X_{66} - X_{66}^{(f)}\|_F^2) \\
&+(\|X_{14} - X_{14}^{(f)}\|_F^2 + \|X_{14}^T - X_{41}^{(f)}\|_F^2) + (\|X_{24} - X_{24}^{(f)}\|_F^2 + \|X_{24}^T - X_{42}^{(f)}\|_F^2) \\
&+(\|X_{34} - X_{34}^{(f)}\|_F^2 + \|X_{34}^T - X_{43}^{(f)}\|_F^2) + (\|X_{45} - X_{45}^{(f)}\|_F^2 + \|X_{45}^T - X_{54}^{(f)}\|_F^2) \\
&+(\|X_{46} - X_{46}^{(f)}\|_F^2 + \|X_{46}^T - X_{64}^{(f)}\|_F^2) + (\|X_{56} - X_{56}^{(f)}\|_F^2 + \|X_{56}^T - X_{65}^{(f)}\|_F^2) \\
&+(\|X_{22} - X_{22}^{(f)}\|_F^2 + \|(\Delta_j^{-1}(E_{22} - \Lambda_j X_{22}))^T - X_{25}^{(f)}\|_F^2 + \\
&\|\Delta_j^{-1}(E_{22} - \Lambda_j X_{22}) - X_{52}^{(f)}\|_F^2) + (\|X_{23} - X_{23}^{(f)}\|_F^2 + \|X_{23}^T - X_{32}^{(f)}\|_F^2 + \\
&\|(\Delta_j^{-1}(E_{23} - \Lambda_j X_{23}))^T - X_{35}^{(f)}\|_F^2 + \|\Delta_j^{-1}(E_{23} - \Lambda_j X_{23}) - X_{53}^{(f)}\|_F^2) + \alpha_0,
\end{aligned}
\tag{2.29}
$$

where α_0 is a constant.

According to (2.29), $\|X - X_f\|_F^2 = min$ if and only if each of the brackets in (2.29) takes minimum. Notice that $X_{kk} = X_{kk}^T, k = 3, 4, 5, 6$ and by making use of Lemma 2.1 and Lemma 2.2, the results of this theorem can be derived easily. □

Conclusions. Using the projection theorem in Hilbert space , the quotient singular value decomposition and the canonical correlation decomposition , we have obtained the explicit analytical expressions of the optimal approximation solutions for the symmetric least-squares problems of the linear matrix equation $AXB = C$. In fact, we have also obtained the explicit analytical expressions of the optimal approximation solutions for the skew-symmetric least-squares problems of the linear matrix equation $AXB = C$, because of the limitation of the pages, we omit the content here, and we can design new algorithms to solve the large scale least-square problems of linear matrix equation $AXB = C$. These new results have generalized the work of Eric Chu [4], Dai Hua [5], Higham [12] and Sun [15] in some aspects.

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