Vandermonde Factorization of a Hankel Matrix⋆

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Abstract. We show that an arbitrary Hankel matrix of a finite rank admits a Vandermonde decomposition: $H = V^T DV$, where V is a confluent Vandermonde matrix and D is a block diagonal matrix. This result was first derived by Vandevoorde; our contribution here is a presentation that uses only linear algebra, specifically, the Jordan canonical form. We discuss the choices for computing this decomposition in only $O(n^2)$ operations, and we illustrate how to employ the decomposition as a fast way to analyze a noisy signal.

1 Introduction

Let $\{h_k\}_{k=1}^{\infty}$ denote a complex-valued signal, and let H represent the associated infinite Hankel matrix whose (i, j)-element is defined by $H_{ij} = h_{i+j-1}$:

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & \cdot^{\cdot^{\cdot}} \\ h_2 & h_3 & h_4 & h_5 & \cdot^{\cdot^{\cdot}} \\ h_3 & h_4 & h_5 & h_6 & \cdot^{\cdot^{\cdot}} \\ h_4 & h_5 & h_6 & h_7 & \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \end{pmatrix} \tag{1}$$

This matrix is symmetric (not Hermitian if complex): $H^T = H$. Throughout this paper, the notation M^T denotes the transpose of M and not the conjugate transpose. Suppose that the underlying signal is a sum of r exponentials, i.e., for $k = 1, 2, \ldots,$

$$h_k = \sum_{i=1}^{r} \lambda_i^k d_i, \tag{2}$$

where the λ_i's are distinct complex numbers. Then the Hankel matrix H will have rank r. In this case, the Hankel matrix admits the factorization:

$$H = V^T DV,$$

where D is diagonal and V is Vandermonde:

$$D \triangleq \operatorname{diag}(d_1, d_2, \ldots, d_r)$$

and

$$V \triangleq \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \cdots \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \lambda_r & \lambda_r^2 & \lambda_r^3 & \cdots \end{pmatrix}.$$

We stress that a diagonal decomposition is possible only if the λ_j's are distinct.

In this paper, we consider the general case where the λ_j's are multiple. The factorization must be generalized so that the matrix D becomes block diagonal and the matrix V takes on a confluent Vandermonde structure. The theory was first developed in Vandevoorde's Ph.D. thesis [11]. The next section is devoted to a derivation of this generalized Vandermonde decomposition based entirely on concepts from linear algebra. In Section 3 we sketch how the decomposition can be computed quickly, viz., using $O(n^2)$ operations and $O(n)$ space. We conclude in Section 4 with an example illustrating to use the method to analyze a noisy signal.

2   Derivation via Jordan Canonical Form

Assume that the matrix H of (1) has rank r. By a theorem of Gantmacher [5, vol. 2, p. 207], the signal satisfies a recurrence relation of length r:

$$h_k = a_{r-1}h_{k-1} + a_{r-2}h_{k-2} + \cdots + a_0 h_{k-r}, \tag{3}$$

which generates the entire signal once the r initial values $\{h_1, h_2, \ldots, h_r\}$ are fixed. The recurrence (3) is a difference equation which can be used to solve for the a_i's after the next r values $\{h_{r+1}, h_{r+2}, \ldots, h_{2r}\}$ become known.

Let C denote the companion matrix corresponding to the polynomial:

$$p(\lambda) \triangleq \lambda^r - a_{r-1}\lambda^{r-1} - \cdots - a_1\lambda - a_0; \tag{4}$$

that is,

$$C \triangleq \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{r-2} & a_{r-1} \end{pmatrix}. \tag{5}$$

We show that the first r rows of H can be regarded as a Krylov sequence generated by C. Let

$$\mathbf{h}_k \triangleq \begin{pmatrix} h_k \\ h_{k+1} \\ \vdots \\ h_{k+r-1} \end{pmatrix} \tag{6}$$

denote the first r entries in the k-th column of H. The first r rows of H can be written as

$$H_{1:r,1:\infty} = (\,\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3 \quad \cdots\,) = (\,\mathbf{h}_1 \quad C\mathbf{h}_1 \quad C^2\mathbf{h}_1 \quad \cdots\,). \tag{7}$$

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_s$ denote the roots of the polynomial $p(\lambda)$ with respective multiplicities $m_1, m_2, \ldots, m_s$ so that

$$m_1 + m_2 + \cdots + m_s = r.$$

Denote a Jordan canonical decomposition of C by

$$C = PJP^{-1},$$

with J in the canonical form:

$$J = \begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{m_s}(\lambda_s) \end{pmatrix}_{r \times r},$$

where

$$J_{m_i}(\lambda) = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}_{m_i \times m_i}, \tag{8}$$

for $i = 1, 2, \ldots, s$. The companion matrix C is guaranteed to be nonderogatory, i.e., it has one Jordan block per distinct eigenvalue. The transformation P is not unique, but having fixed the order for the eigenvalues, any alternative transformation $\widetilde{P}$ also yielding the same Jordan canonical form must be related to P by

$$\widetilde{P}Q = P,$$

where

$$Q = \begin{pmatrix} Q_{m_1} & & & \\ & Q_{m_2} & & \\ & & \ddots & \\ & & & Q_{m_s} \end{pmatrix}_{r \times r}, \tag{9}$$

with each Q_{m_i} as an $m_i \times m_i$ nonsingular and upper triangular matrix whose diagonal entries are all the same. This is because the columns of any $\widetilde{P}$ are Jordan chains, and there are only limited kinds of transformations to the Jordan chains that will yield the same Jordan form [5, vol. 1, p. 172]. Note that Q is block diagonal with blocks conforming to the Jordan blocks.

For the purposes of computation, or for fixing ideas, it is often convenient to choose a specific P. One such choice is that of a confluent Vandermonde matrix [6, p. 188]:

$$P \triangleq \begin{pmatrix} \mathbf{p}^T \\ \mathbf{p}^T J \\ \vdots \\ \mathbf{p}^T J^{r-1} \end{pmatrix}_{r \times r}, \tag{10}$$

where

$$\mathbf{p}^T = \begin{pmatrix} \mathbf{e}_1^{[m_1]T} & \mathbf{e}_1^{[m_2]T} & \dots & \mathbf{e}_1^{[m_s]T} \end{pmatrix}_{1 \times r};$$

so $\mathbf{p}$ is partitioned conformally with the Jordan canonical form J and each partition $\mathbf{e}_1^{[m_i]T}$ represents a first unit-coordinate vector:

$$\mathbf{e}_1^{[m_i]T} = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}_{1 \times m_i}.$$

It is easy to verify (for any choice of a starting vector $\mathbf{p}$) that

$$CP = \begin{pmatrix} \mathbf{p}^T J \\ \mathbf{p}^T J^2 \\ \vdots \\ \mathbf{p}^T J^{r-1} \\ \mathbf{p}^T \cdot (a_{r-1}J^{r-1} + \dots + a_1 J + a_0 I) \end{pmatrix} = \begin{pmatrix} \mathbf{p}^T J \\ \mathbf{p}^T J^2 \\ \vdots \\ \mathbf{p}^T J^{r-1} \\ \mathbf{p}^T J^r \end{pmatrix} = PJ,$$

where we have used the the characteristic equation for J:

$$p(J) = J^r - a_{r-1}J^{r-1} - \dots - a_1 J - a_0 I = 0.$$

Hence $C = PJP^{-1}$, supporting our choice of the special form of P in (10).

Now, define a generalized Vandermonde matrix V by

$$V \triangleq \begin{pmatrix} \mathbf{v} & J\mathbf{v} & J^2\mathbf{v} & \cdots \end{pmatrix},$$

where

$$\mathbf{v} \triangleq P^{-1}\mathbf{h}_1.$$

The relation (7) can be written as

$$H_{1:r,1:\infty} = PV. \tag{11}$$

Let V_r denote the first r columns of V. We will express the Jordan decomposition of C in terms of V_r. Forming the product $V_r C^T$ columnwise, we get

$$\begin{aligned} V_r C^T &= \begin{pmatrix} \mathbf{v} & J\mathbf{v} & \cdots & J^{r-2}\mathbf{v} & J^{r-1}\mathbf{v} \end{pmatrix} C^T \\ &= \begin{pmatrix} J\mathbf{v} & J^2\mathbf{v} & \cdots & J^{r-1}\mathbf{v} & (a_0 + a_1 J + \dots + a_{r-1}J^{r-1})\mathbf{v} \end{pmatrix} \\ &= \begin{pmatrix} J\mathbf{v} & J^2\mathbf{v} & \cdots & J^{r-1}\mathbf{v} & J^r\mathbf{v} \end{pmatrix} \\ &= JV_r. \end{aligned} \tag{12}$$

Since the first r columns of (11) are independent by assumption, the matrix V_r must be nonsingular. So we obtain the decomposition:

$$C^T = V_r^{-1}JV_r,$$

or equivalently,

$$C = V_r^T J^T V_r^{-T}.$$

We will use this result to express the transformation P in terms of V_r. Define the block diagonal "flip" matrix as follows:

$$F \triangleq \begin{pmatrix} F_{m_1} & & & \\ & F_{m_2} & & \\ & & \ddots & \\ & & & F_{m_s} \end{pmatrix}_{r \times r}, \tag{13}$$

where

$$F_{m_i} \triangleq \begin{pmatrix} & & & 1 \\ & 0 & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix}_{m_i \times m_i},$$

for $i = 1, 2, \ldots, s$, partitioned conformally with the Jordan block $J_{m_i}(\lambda_i)$. This matrix is involutory ($F^2 = I$) and symmetric ($F^T = F$). When applied to the Jordan matrix J, the matrix F has the effect of transposing it:

$$FJF = J^T.$$

We can thereby write the Jordan decomposition of C as

$$C = V_r^T J^T V_r^{-T} = V_r^T F J F V_r^{-T}. \tag{14}$$

Setting

$$P = V_r^T F Q$$

for some matrix Q of the form (9), we get the decomposition of the leading $r \times r$ part of the Hankel matrix as

$$H_r = PV_r = V_r^T F Q V_r.$$

Given that H is symmetric and V_r is nonsingular, we obtain

$$V_r^T F Q V_r = H = H^T = V_r^T (FQ)^T V_r.$$

Hence the matrix D, defined by

$$D \triangleq FQ,$$

must be symmetric; furthermore, it is block diagonal with blocks conforming to the Jordan blocks. Since each diagonal block of Q is upper triangular, we derive the following form for D:

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_s \end{pmatrix}_{r \times r}, \tag{15}$$

where
$$D_i = \begin{pmatrix} * & * & * & * & d_i \\ * & * & * & d_i & \\ * & * & \cdot^{\cdot^{\cdot}} & & \\ * & d_i & & 0 & \\ d_i & & & & \end{pmatrix}_{m_i \times m_i},$$

for $i = 1, 2, \ldots, s$; each block D_i is symmetric and upper anti-triangular, with a constant value along the main antidiagonal. Combining these formulas, we obtain both a relation between P and V_r:

$$P = V_r^T D,$$

and a symmetric decomposition for the leading $r \times r$ Hankel matrix:

$$H_r = PV_r = V_r^T D V_r = PD^{-1}P^T. \tag{16}$$

From (12) we get

$$V = \begin{pmatrix} V_r & J^r V_r & J^{2r} V_r & \cdots \end{pmatrix} = \begin{pmatrix} V_r & V_r(C^T)^r & V_r(C^T)^{2r} & \cdots \end{pmatrix}.$$

Since $CH_r = H_r C^T$, we obtain

$$H = \begin{pmatrix} I_r \\ C^r \\ C^{2r} \\ \vdots \end{pmatrix} \cdot H_r \cdot \begin{pmatrix} I_r & (C^T)^r & (C^T)^{2r} & \cdots \end{pmatrix} = V^T D V, \tag{17}$$

i.e., a factorization for the entire infinite Hankel matrix.

A further analysis of the structure of the matrices in (16) yields the fact that the diagonal blocks of D^{-1} and D have Hankel structure. For D^{-1}, we start with the identity

$$(PJP^{-1})(PD^{-1}P^T) = CH = HC^T = (PD^{-1}P^T)(P^{-T}J^T P^T),$$

which simplifies to $JD^{-1} = D^{-1}J^T$. The matrix D^{-1} is block diagonal with blocks conforming to those of J. So the i-th block of this last relation is

$$J_{m_i}(\lambda_i)D_i^{-1} = D_i^{-1}J_{m_i}^T(\lambda_i),$$

for $i = 1, \ldots, s$. Subtracting $\lambda_i I$ from both sides yields

$$Z_i D_i^{-1} = D_i^{-1} Z_i^T,$$

where Z has the form of an upshift matrix of appropriate size:

$$Z_i = J_{m_i}(\lambda_i) - \lambda_i I = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{m_i \times m_i}. \tag{18}$$

Hence D_i^{-1} must have a Hankel structure. For D, we carry out a similar argument using $V_r^T F$ instead of P and D instead of D^{-1}, and apply the factorization (14) to get a Hankel structure for the diagonal blocks D_i.

We now show how we may modify the Vandermonde matrix V_r so that its first column consists of all ones. Suppose that no component of $\mathbf{v}$ is zero. Define a new diagonal matrix D_v by

$$D_v \triangleq \operatorname{diag}(v_1, v_2, \dots, v_r),$$

and a new generalized Vandermonde matrix W_r by

$$W_r \triangleq D_v^{-1} V_r.$$

Then

$$D_v^{-1} \mathbf{v} = \mathbf{e} \triangleq \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^T,$$

and we can re-write (16) as

$$H_r = W_r^T (D_v D D_v) W_r,$$

where the product $D_v D D_v$ is a block diagonal matrix with blocks conforming to J. The matrix W_r has the following structure:

$$
\begin{aligned}
W_r &= \begin{pmatrix} \mathbf{e} & D_v^{-1} J D_v \mathbf{e} & D_v^{-1} J^2 D_v \mathbf{e} & \cdots & D_v^{-1} J^{r-1} D_v \mathbf{e} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e} & \widehat{J} \mathbf{e} & \widehat{J}^2 \mathbf{e} & \cdots & \widehat{J}^{r-1} \mathbf{e} \end{pmatrix},
\end{aligned}
\tag{19}
$$

with

$$\widehat{J} \triangleq D_v^{-1} J D_v.$$

We observe that the matrix $\widehat{J}$ possesses a sort of generalized Jordan structure.

We conclude this section with three notes.

First, consider the special case that all the eigenvalues of C are simple. In this case, the Jordan matrix $J = \widehat{J}$ is diagonal, F is the identity, $D = Q$ is "scalar" diagonal, and

$$P = V_r^T D.$$

The decomposition (16) simplifies to

$$H_r = V_r^T (D) V_r = W_r^T (D D_v^2) W_r,\tag{20}$$

where the parts enclosed in parentheses are diagonal matrices. The last expression on the right applies if $\mathbf{v}$ has no zero component, which is guaranteed in this case by the nonsingularity of V_r. In fact, the matrix W_r of (19) for this case has the usual Vandermonde structure. The above holds for any choice of P, but if we fix P as given in (10), we see that $P = W_r^T$.

Second, the factorization (16) can be used to factor any nonsingular $r \times r$ Hankel matrix H_r. This matrix is filled by the entries $h_1, h_2, \dots, h_{2r-1}$; hence to fix the polynomial (4) and carry out the rest of the development above, we must choose some value for h_{2r}. With such a choice, the rest of the development above goes through unchanged.

Third, we address the issue of factoring a singular $n \times n$ Hankel matrix H_n. One way to do this is to embed this $n \times n$ matrix inside an infinite Hankel matrix H_∞ of a finite rank r by extending it with infinitely many zeros. We could use a different choice for the extension to avoid a nilpotent C, and one open issue is how to choose the extension to minimize the resulting rank. Factor the infinite Hankel matrix as

$$H_\infty = V^T D V,$$

where D is $r \times r$ and V is $r \times \infty$, and extract the first n rows and columns of this decomposition to get

$$H_n = V_n^T D V_n,$$

where V_n is the $r \times n$ matrix consisting of the first n columns of V. Note that it could be that $r < n$ or $r > n$. The former case occurs, for example, if the rank of H_n is $r < n$ and the leading $r \times r$ part of H_n happens to be nonsingular.

3   Algorithms (Generic case)

We briefly discuss choices of algorithms that can be used to compute the Vandermonde decomposition. Many of the individual pieces to the algorithms are off-the-shelf methods; some are quite experimental and some have received very little attention in the literature. The methods we present are based on the use of the Lanczos algorithm or its derivatives. Most details can be found in [11].

We begin with an outline of the basic steps:

1. Compute the "modes" generating the Hankel matrix, viz., the roots of the polynomial $p(\lambda)$ of (4).
2. Compute the "diagonal" matrix D:

$$D = V^{-T} H V^{-1},$$

   where V is the Vandermonde matrix generated by the eigenvalues in step 1. The diagonal structure of D follows from the theory developed in the previous section.
3. Optionally scale the columns of V to unit norm, scaling the entries in D appropriately.

For each of these steps there are choices for the algorithm to use. For step 1, we could solve for the coefficients in the recurrence (3) by simply plugging the values for h_i $(i = 1, \ldots, 2n - 1)$, yielding a special set of n equations in n unknowns originally proposed by Prony [10] and popularized by Yule [14] and Walker [12]. Then we must find all the roots of the polynomial $p(\lambda)$.

Vandevoorde [11] proposed an alternative for step 1. We use a variant of a nonsymmetric Lanczos process to generate a tridiagonal matrix T whose eigenvalues match those of C. Then we compute the eigenvalues of T. It turns out that both these steps (generating T and computing its eigenvalues) can be performed very efficiently. Space does not permit a full description of the process, but we can give a hint on the basic ideas used. We have already seen that the $n \times n$

Hankel matrix H_n can be thought of as a Krylov sequence generated by C and $\mathbf{h}_1$ (cf. (7)). If we let H_{2n} be the $2n \times 2n$ Hankel matrix obtained by extending the "signal" $\{h_k\}$ with all zeros, then H_{2n} can similarly be thought of as the Krylov sequence generated by the $2n \times 2n$ upshift matrix Z of the form (18).

The nonsymmetric Lanczos process in [11] is equivalent to a procedure that bi-orthogonalizes the Krylov sequence we just mentioned against another sequence (called the "left" Krylov sequence). If the coefficients computed by the enforcement of the bi-orthogonalization conditions involve only the first n entries of each vector, the Lanczos coefficients generated by C and by H_{2n} will be identical. This can happen, for example, by making the left Krylov sequence upper triangular, so that the right Krylov sequence will be bi-orthogonalized to lower triangular. This can be arranged by starting the left Krylov sequence with the coordinate unit vector $\mathbf{e}_1$. As a result, each new vector is generated by shifting up the previous vector and then orthogonalizing it against the two previous vectors. This takes linear time and linear space, and there are at most $O(n)$ steps so that the total time is $O(n^2)$. The resulting algorithm is described in detail in [2]. There is also a symmetric variant originally proposed in [9] that can generate a symmetrized tridiagonal matrix directly from this non-symmetric recursion. This symmetric variant has similar costs.

Once the tridiagonal matrix has been generated, the task is to find its eigenvalues. There are two variants of the QR-type algorithm that can be applied here. One is the complex symmetric QR algorithm proposed in [4], for which the matrix T must be symmetrized. Even when T is real, if the signs of the corresponding superdiagonal and subdiagonal entries of T are opposite, then the symmetrized matrix will be complex. The resulting QR algorithm is a direct analog of the ordinary Hermitian QR method, but it uses complex orthogonal rotations and complex symmetric matrices instead of unitary rotations and Hermitian matrices, respectively. Another option is to use the LR algorithm [13], which is based on the LU factorization without pivoting to preserve the tridiagonal structure. The LR algorithm can break down, but if a random shift is applied when zero pivot occurs during the LU factorization, the process can still exhibit very rapid convergence. If T is real, an implicit double-shift LR algorithm can in principle be carried out in real arithmetic [13]. Both algorithms require linear time for each iteration in a manner very similar to the Hermitian analog, and the number of iterations is generally $O(n)$ in a manner very similar to the QR algorithm usually employed. The relative merits between these alternative algorithms have not been studied in detail.

The other major task is finding the diagonal matrix D in step 2. Because of the structure of V, the diagonal entries of D appear in the first column of the product DV. But $DV = V^{-T}H$. Hence this first column is the solution $\mathbf{d}$ to the Vandermonde system:

$$V^T \mathbf{d} = \mathbf{h}_1,$$

where $\mathbf{h}_1$ is the first column of H. This can be solved with a fast $O(n)$ Vandermonde solver [1], where to maintain stability Higham [7, p. 438] recommends arranging the eigenvalues with a so-called Leja ordering.

4   Analysis of a Signal

Consider a signal $\{h_k\}$ which suffers from the presence of noise. How can we recover the principal modes that generate the signal? A popular method by Kung [8] based on the singular value decomposition (SVD) is known to be an effective method for this purpose, but it suffers from the need to carry out both an SVD and a matrix eigensolution, each costing $O(n^3)$ operations. A second popular approach is to form the Hankel matrix generated by the signal, and then proceed to find a nearby Hankel matrix of a lower rank [3]. The Vandermonde decomposition of this nearby low-rank Hankel matrix yields the parameters in (2). The method of [3] iterates until it converges to a nearby Hankel matrix. Unfortunately, this method requires the repeated use of the SVD and hence costs up to $O(n^3)$ operations per iteration.

We indicated in Section 3 how the Vandermonde decomposition can be computed quickly. An obvious way to obtain a nearby Hankel matrix of a lower rank is to set to zero all the diagonal entries in D that are smaller than a certain tolerance. Although this crude method does not always yield the best approximation, a judicious combination of this approach with other criteria can yield a good result. We conclude this paper with an illustration of one such approach in the next paragraph.

Start with a signal generated by five modes, shown by circles on the complex plane in Figure 1, to which has been added white noise with a signal-to-noise ratio of 3.55dB. Form the 128×128 Hankel matrix H and compute its Vandermonde decomposition $H = V^T D V$. Figure 2 shows the absolute values of the diagonal entries of D in descending order. It turns out that selecting the modes corresponding to the five largest values of D does not yield satisfactory results, but we can almost recover the correct modes by the following simple procedure. Choose the modes corresponding to the largest entries in D (also called weights), specifically those that are within 10% of the largest entry (in absolute value); in this case fourteen modes remained. Then choose a subset of these fourteen using a second criterion based on the Discrete Fourier Transform (DFT) of the signal. The DFT of the original signal is shown by the dotted line in Figure 3. As most of the modes lie relatively close to the unit circle, their argument (angle on the complex plane) maps to the horizontal axis of Figure 3. In fact, we have marked the angles corresponding to the five original "unknown" modes by means of circles along the x-axis. This leads to our second criterion, viz., select those modes for which the DFT is larger than a certain threshold (in this case 30%) of the largest value in the DFT (in absolute value). This selection criterion is applied only to those modes that survived the first selection process. In this example, out of the fourteen modes only seven survived the second selection process. These final seven modes are marked by $*$'s in Figure 1, and the resulting DFT using these seven modes is shown by the solid line in Figure 3. We remark that one can still distinguish the two close peaks in this DFT corresponding to the two very close original modes. We should emphasize that the choice of criteria requires further study. Indeed, a more sophisticated selection criterion is presented in [11].

Computed Vandermonde Modes (eigenvalues)

o=true modes; *=selected modes; x=unselected modes (7 selected)
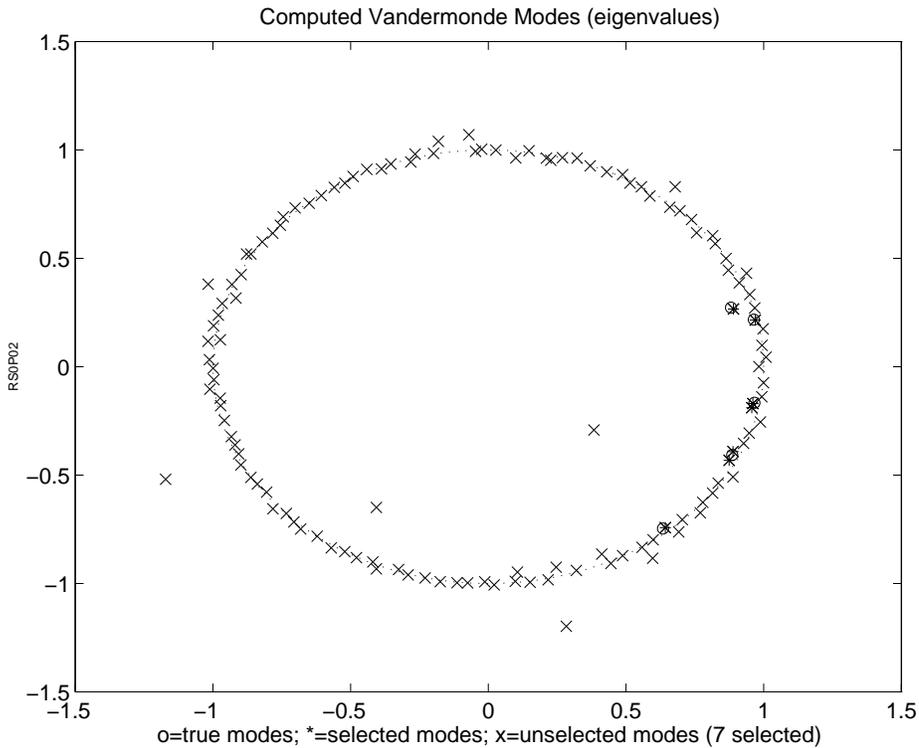
Fig. 1. Original modes (o) and those computed from the tridiagonal matrix discussed in in Section 3 (* & x).

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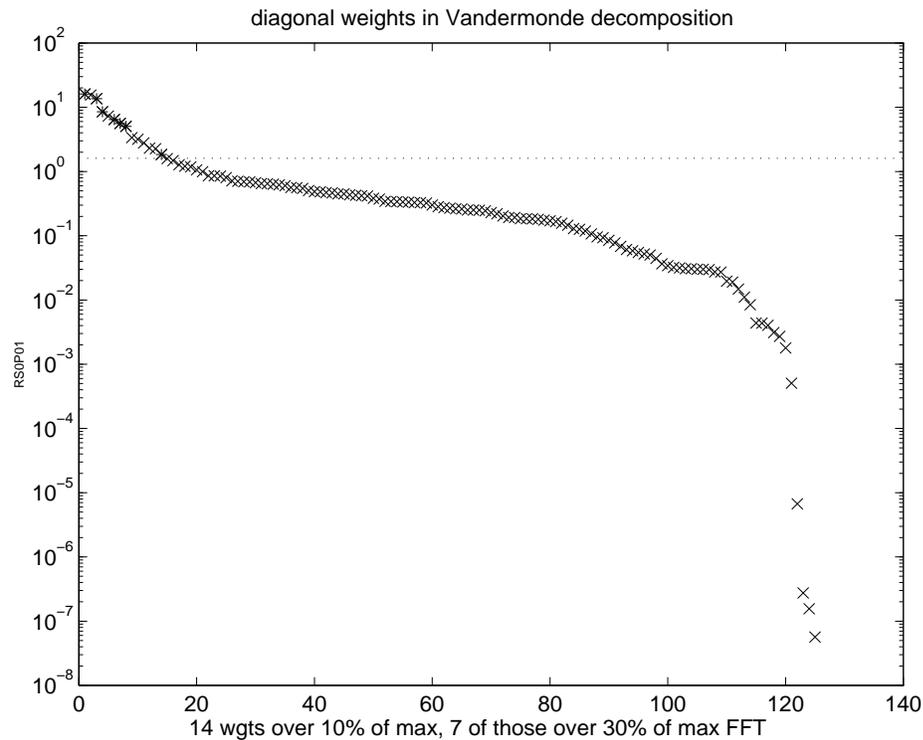
Fig. 2. Diagonal entries from Vandermonde decomposition

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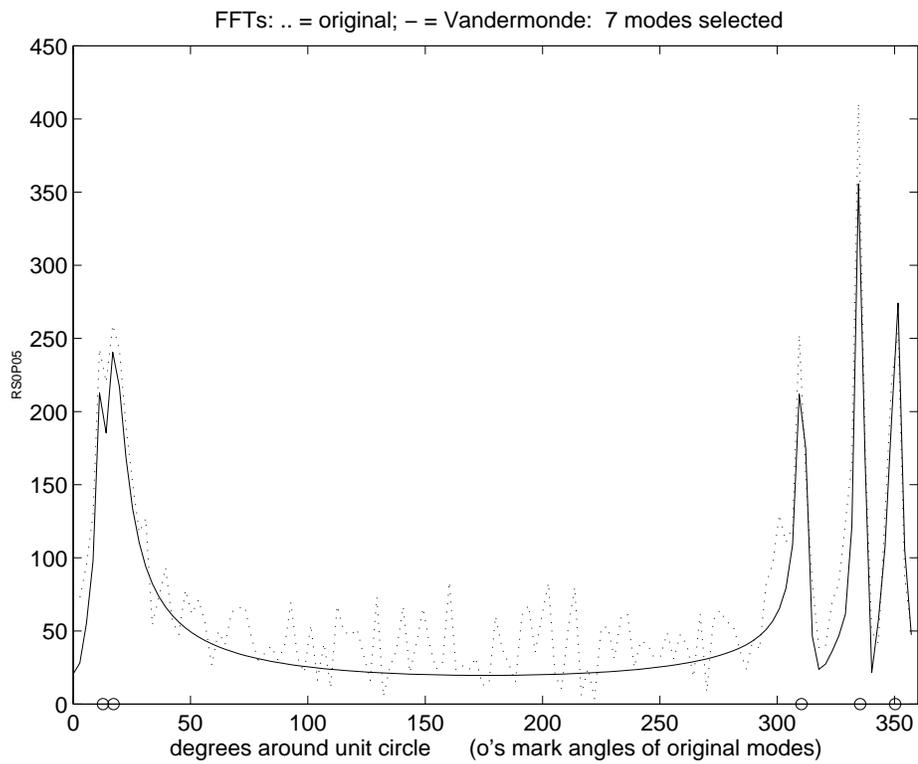
Fig. 3. Discrete Fourier Transform (DFT) of the original signal (dotted) and the reconstructed reduced-order signal (solid). Small circles mark the angles corresponding the original modes.