The Algebraic Structure of Pencils and Block Toeplitz Matrices[1]

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Abstract

We prove several results majorizing the sequences of Kronecker and/or Jordan indices obtainable after small perturbations to a given matrix pencil. The proofs are simple consequences of a theory of majorization for semi-infinite integer sequences, developed in this paper. In particular, new simple bounds are proved on the indices obtainable after appending a single row or column to a matrix pencil. This corresponds to bounding the controllability and/or observability indices after adding a single input or a single output to a linear time-invariant dynamical system.

1 Introduction

The close links between the concepts of Controllability, Reachability, and Observability of linear time-invariant dynamical systems on the one hand, and Kronecker, Jordan indices of appropriate matrix pencils on the other hand have been well established in the literature (see e.g. [7, 12]). Though the complete structure of the Kronecker canonical form (KCF) is often not required, it has been found that the detailed structure of the KCF is needed in order to compute transmission zeroes or to know which zeroes may be placed by suitable inputs [5, 1]. Recently, several papers have appeared discussing the Kronecker/Jordan structure of matrix pencils under perturbations to the pencils and/or orbits of a given pencil. We define the *orbit* of the pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ as the set of all pencils of the form $\{\mathcal{P}(\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}})\mathcal{Q}\}$ such that $\mathcal{P}, \mathcal{Q}$ are any nonsingular matrices of appropriate dimensions. Then $\mathcal{E} - \lambda\mathcal{F}$ is in the closure of the orbit of $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ if and only if an arbitrarily small perturbation to $\mathcal{E} - \lambda\mathcal{F}$ yields a pencil $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$ with exactly the same Kronecker canonical form as $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$, which is equivalent to the condition that $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}} = \mathcal{P}(\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}})\mathcal{Q}$ for some nonsingular $\mathcal{P}, \mathcal{Q}$ [10].

Many of the results alluded to above are based on proving relations between the sequences of Jordan or Kronecker indices to sequences of nullities of special matrices with block Toeplitz structure defined as follows. Let $\mathcal{E} - \lambda\mathcal{F}$ be an $N_{\text{rows}} \times N_{\text{cols}}$ pencil. We form the sequence of

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constant block Toeplitz matrices ("Gantmacher matrices" [6])

$$\mathcal{A}_1 = \begin{pmatrix} \mathcal{E} \\ \mathcal{F} \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} \mathcal{E} & \\ \mathcal{F} & \mathcal{E} \\ & \mathcal{F} \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} \mathcal{E} & & \\ \mathcal{F} & \mathcal{E} & \\ & \mathcal{F} & \mathcal{E} \\ & & \mathcal{F} \end{pmatrix}, \mathcal{A}_4 = \begin{pmatrix} \mathcal{E} & & & \\ \mathcal{F} & \mathcal{E} & & \\ & \mathcal{F} & \mathcal{E} & \\ & & \mathcal{F} & \mathcal{E} \\ & & & \mathcal{F} \end{pmatrix}, \cdots \tag{1}$$

and their corresponding right nullities $\mathbf{A} = \{A_1, A_2, A_3, A_4, \ldots\}$. By constructing the same matrices for the $N_{\text{cols}} \times N_{\text{rows}}$ pencil $\mathcal{E}^T - \lambda \mathcal{F}^T$, we obtain the corresponding *left* nullities $\mathbf{A}^{\text{L}} = \{A_1^{\text{L}}, A_2^{\text{L}}, A_3^{\text{L}}, A_4^{\text{L}}, \ldots\}$.

We form also the following block Toeplitz matrices

$$\mathcal{G}_1 = (\, \mathcal{E} \,), \mathcal{G}_2 = \begin{pmatrix} \mathcal{E} & \\ \mathcal{F} & \mathcal{E} \end{pmatrix}, \mathcal{G}_3 = \begin{pmatrix} \mathcal{E} & & \\ \mathcal{F} & \mathcal{E} & \\ & \mathcal{F} & \mathcal{E} \end{pmatrix}, \mathcal{G}_4 = \begin{pmatrix} \mathcal{E} & & & \\ \mathcal{F} & \mathcal{E} & & \\ & \mathcal{F} & \mathcal{E} & \\ & & \mathcal{F} & \mathcal{E} \end{pmatrix}, \cdots \tag{2}$$

and their corresponding right nullities $\mathbf{G} = \{G_1, G_2, G_3, G_4, \ldots\}$.

Then we define the following quantities used throughout this paper.

K	refers to a right Kronecker block,
L	refers to a Left kronecker block,
J	refers to a Jordan block for eigenvalue zero, and
E	refers to the remaining rEgular part.
Specifically:	
K_i	refers to an $i \times (i{+}1)$ K-block (K for Kronecker block),
L_j	refers to a $(j{+}1) \times j$ L-block (L for Left Kronecker block),
J_j	refers to a $j \times j$ Jordan block for eigenvalue 0 (J for Jordan), and
and	
N_K	= number of rows occupied by all the K-blocks,
N_L	= number of columns occupied by all the L-blocks,
N_J	= number of columns occupied by all the J-blocks for eigenvalue 0, and
N_E	is the dimension of the entire remaining rEgular part (except for eigenvalue 0).
n_K, n_L, n_J	= total number of K, L, J, blocks, respectively.

For example, the algebraic and geometric multiplicities for eigenvalue zero are N_J, n_J, respectively, and $N_J + N_E$ is the dimension of the entire regular part. We also have the following identities for $N_{\text{rows}} \times N_{\text{cols}}$ pencils:

$$\begin{array}{lll} \text{(a)} & N_{\text{rows}} & = N_K + N_L + n_L + N_J + N_E \\ \text{(b)} & N_{\text{cols}} & = N_K + n_K + N_L + N_J + N_E \\ \text{(c)} & N_{\text{cols}} - N_{\text{rows}} & = n_K - n_L \end{array} \tag{3}$$

In this paper, we try to unify many of these results by developing a theory of majorization for infinite integer sequences, completely independent of any application to matrices or linear operators. Our theory is an extension of the theory of majorization for finite sequences in [9]. The semi-infinite sequences we will use are defined as follows. Let $\mathbf{a} = \{a_1, a_2, \ldots\}$ denote a semi-infinite sequence of integers. We implicitly define $a_i = 0$ for all $i \leq 0$. We define the set $\mathcal{S}$ as the set of all such sequences. We include sequences whose entries are infinite as well as ordinary integers. We define

$\mathcal{S}_0 \subset \mathcal{S}$ as the set of all sequences with *non-negative entries.* On $\mathcal{S}$ we define the *difference operator* $\mathbf{\Delta}$ as follows. Let $\mathbf{a} = \{a_1, a_2, \ldots\} \in \mathcal{S}$. Then $\dot{\mathbf{a}} \equiv \mathbf{\Delta a} = \{\dot{a}_1, \dot{a}_2, \ldots\} \in \mathcal{S}$ is the sequence defined by $\dot{a}_i = a_i - a_{i-1}$. We use this difference operator to define the sets $\mathcal{S}_k$ for $k > 0$ as follows: $\mathcal{S}_k = \{\mathbf{a} : \mathbf{\Delta a} \in \mathcal{S}_{k-1}\}$. For example, $\mathcal{S}_1$ is the set of *ascending* (i.e. non-decreasing) non-negative sequences. We use the special notation $\mathcal{S}_{\rm d}$ to denote the set consisting of non-negative *descending* (i.e. non-increasing) sequences $\mathbf{a}$: $\mathbf{a} \in \mathcal{S}_{\rm d} \subset \mathcal{S}_0$ if and only if $a_1 \geq a_2 \geq \cdots 0$.

A word on notation: we use bold letters (both upper and lower case) to denote sequences whose entries are given by the correspnding roman letter: viz. $\mathbf{a} = \{a_1, a_2, \ldots\}$. We use roman letters (both upper and lower case) to denote scalar quantities: viz. n, N. We use calligraphic upper case letters to denote matrices: viz. $\mathcal{A}, \mathcal{E}, \mathcal{M}$, except that the letter $\mathcal{S}$ is used to denote sets of sequences. The identity matrix is denoted $\mathcal{I}$. The greek letters ϵ, λ denote scalar quantities; all the other greek letters are used to denote operators on sequences.

In this paper we find that many of the results on the sequences of Kronecker and Jordan indices are simple consequences of this theory. Regarding the theory of Kronecker/Jordan indices under perturbations, this discovery helps separate those results which depend on the particulars of the linear operators from those results which are just properties inherent to the integer indices. We also use the theory of integer sequences to prove some new results bounding the Jordan or Kronecker indices obtainable when a single row or column is added to a pencil. In Control Theory, this corresponds to determining the reachability or observability indices obtainable by adding a single input or a single output.

The Jordan indices, when collected for each eigenvalue in descending order, are known as the Segré characteristics [11, p79-81] The relation of these to the so-called Weyr characterstics (the nullities of $(\mathcal{M} - s\mathcal{I})^k$, for $k = 1, 2, 3, \ldots$, where $\mathcal{M}$ is a square matrix, [11]) was extended in [8] to the case of semi-regular pencils, based on the fact that the Weyr characterstics are also exactly the nullities of the block Toeplitz matrices (2). A *semi-regular pencil* is a pencil whose normal rank equals $\min\{N_{\rm rows}, N_{\rm cols}\}$, or equivalently a pencil for which $\mathcal{E} - s\mathcal{F}$ achieves full row or column rank for some value of s. A semi-regular pencil is one which has right Kronecker blocks or left Kronecker blocks, but not both. The analogous construction (1) was used earlier by [6] to prove many basic properties for the Kronecker canonical form, including its existence. In Sec. 4 below, we present these results, extending the results of [8] to general pencils, not necessarily semi-regular.

Several papers discuss the effect of perturbations of pencils on the indices, or the structure of the Kronecker indices reachable in the closure of the orbit of a pencil. In [2], the effect on the Jordan indices for the eigenvalues of a matrix lying within a region of the complex plane under perturbations to a pencil was discussed. It was found that the sequence of Jordan indices for the perturbed pencil were majorized by the indices for the original pencil, in the sense that the leading sums of the former were bounded by the leading sums of the latter. In [10], the structure of the Kronecker indices within the closure of the orbit of a given pencil was analysed. It was found that one could apply a sequence of elementary perturbations to a pencil, each making a simple change to the Kronecker indices, to achieve any structure reachable within the closure of the orbit of the given pencil. In the last section of this paper, we illustrate those perturbations, showing that each one corresponds to a simple change to the sequence of nullities of the block Toeplitz matrices. In [3], the results of [10] and [4] were combined to define a stratification of the possible Kronecker structures, where each layer consisted of the structures reachable via arbitrarily small perturbations and/or within the closure of the orbit of a pencil in the neighboring layer.

The rest of this paper is organized as follows. In Sec. 2 we give explicit statements of the principal previous results on which this paper is based. In Sec. 3 we briefly sketch the basic results

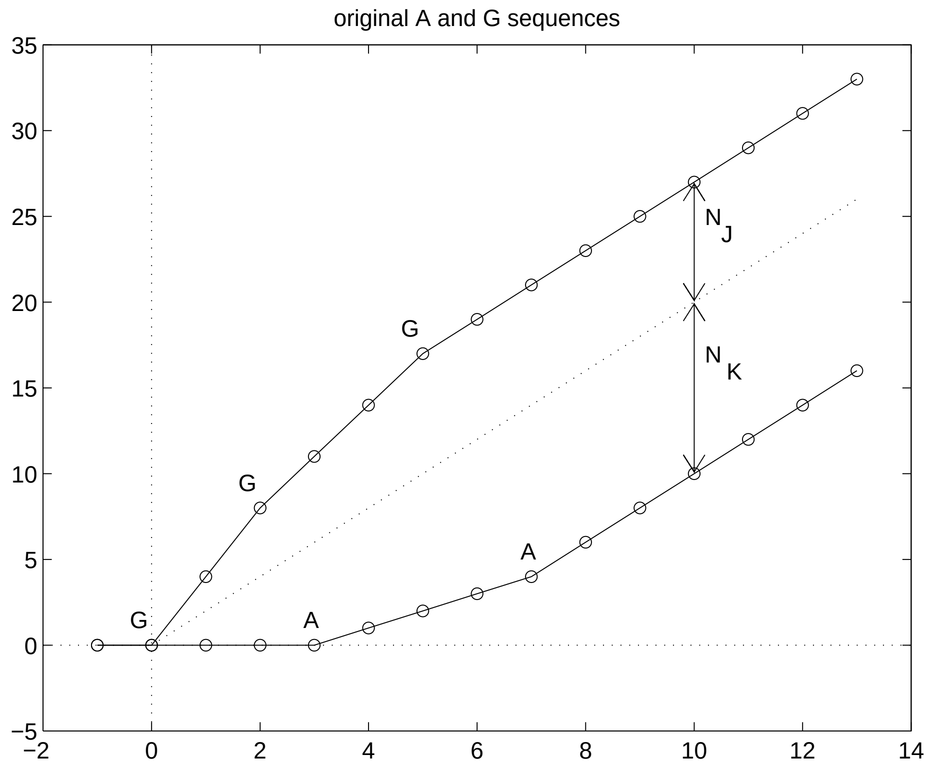

Figure 1: Sequences of nullities (14) with corners marked. The arrows indicate the distances according to equations (8) and (11).

needed from the algebra of integer sequences. In Sec. 4 we revisit and extend the theory relating the Jordan and Kronecker indicies to the nullities of the block Toeplitz matrices, expressed in terms of the notation of Sec. 3. In Sec. 5 we combine the majorization results of the previous section with the relations of Sec. 4 to prove our principal new majorization results for matrix pencils under various modifications. In Sec. 6 we discuss previous results regarding admissible perturbations in terms of the integer sequences. We collect into an Appendix the proofs of some of the theorems on integer sequences.

2 Background

In this section, we summarize the principal previous results on which this paper is based. The *normal rank* of a pencil $\mathcal{E} - \lambda\mathcal{F}$ is the maximum value attained by $rank(\mathcal{E} - s\mathcal{F})$ over all s on the extended complex plane. It is well known that this maximum rank is attained for all but finitely many values of s; these special values of s are the *eigenvalues* of the pencil. In terms of the Kronecker canonical form, the normal rank is equal to $N_K + N_L + N_J + N_E$.

In proving the existence of the Kronecker canonical form for an arbitrary pencil, Gantmacher [6, p30] proved that the order k of the smallest right Kronecker block is the smallest k such that the rank of $\mathcal{A}_{k+1}$ (1) is *strictly* less than $(k+1)N_{\text{cols}}$. Note that if there are any zero eigenvalues, we must replace the pencil $\mathcal{E} - \lambda\mathcal{F}$ with $(\mathcal{E} - s\mathcal{F}) - \lambda\mathcal{F}$ where the rank of $(\mathcal{E} - s\mathcal{F})$ is equal to the normal rank of the pencil. Since $(k+1)N_{\text{cols}}$ is exactly the number of columns in $\mathcal{A}_{k+1}$, This result is equivalent to

Theorem 1 [6]. If the $rank(\mathcal{E} - s\mathcal{F})$ equals the normal rank of the pencil $\mathcal{E} - \lambda\mathcal{F}$, then the order of the smallest Kronecker block is the smallest k such that the right nullity of $\mathcal{A}_{k+1}$ is bigger than zero. □

This is illustrated in Fig. 1, where the sequence of nullities of the matrices $\{\mathcal{A}_k\}_{k>0}$ are represented by the `A` curve. The first nonzero nullity appears for $k+1 = 4$. This "corner" in the `A` curve at $k = 3$ corresponds to the order of the smallest right Kronecker block, namely 3. Following the proof in [6], we can deflate out the smallest Kronecker block, removing the "corner" in Fig. 1 at $k = 3$; the effect on the curve is to subtract $k - 3$ from A_k for all $k > 3$. Then the next Kronecker block will correspond to the next "corner" in the `A` curve at $k = 7$. The remaining features shown in Fig. 1 will be developed in Sec. 4

In [8], a similar construction was used to relate the Jordan indices for eigenvalue zero to the block Toeplitz matrices (2). Any other eigenvalue can be handled by a suitable shift s as in $(\mathcal{E} - s\mathcal{F}) - \lambda\mathcal{F}$. Analogous to the above discussion for Kronecker blocks, the smallest Jordan block for eigenvalue zero is the smallest k such that the rank of $\mathcal{G}_{k+1}$ is *strictly greater* than $(k+1) \cdot rank(\mathcal{E})$. In [8], this result was used to prove relations between the Segré characteristics and the Weyr characteristics for a semi-regular pencil. There are many ways to describe the relationships between the between the Segré characteristics and the Weyr characteristics, but perhaps the simplest is the following.

Theorem 2 [8]. The Weyr characteristics, $\{G_k\}_{k>0}$, for a semi-regular pencil satisfy the property that $G_{k+1} - G_k$ is exactly the number of elementary divisors for eigenvalue zero of degree at least k (equivalently the number of Jordan blocks for zero of dimension at least k).

A separate set of papers was devoted to analyses of the possible structures achievable through arbitrarily small perturbations. An important example comes from [10], where the following result is proved.

Theorem 3 [10]. Consider the pencil $\mathcal{E} - \lambda\mathcal{F}$ with $\dot{a}_k$ right Kronecker blocks of dimension $k-1$, $\dot{a}_k^{\mathrm{L}}$ left Kronecker blocks of dimension $k-1$, and $h_k(s)$ Jordan blocks for eigenvalue s of dimension k, for all $k = 0, 1, \ldots$. Consider a second pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ with corresponding indices $\{\hat{\dot{a}}_k\}$, $\{\hat{\dot{a}}_k^{\mathrm{L}}\}$, $\{\widehat{h}_k(s)\}$. Then $\mathcal{E} - \lambda\mathcal{F}$ lies in the closure of the orbit of $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ if and only if the following conditions hold:

$$\begin{array}{rcl} \sum_{k=0}^{j} \mathrm{pos}(j-k)\hat{\dot{a}}_{k+1} & \leq & \sum_{k=0}^{j} \mathrm{pos}(j-k)\dot{a}_{k+1} \\ \sum_{k=0}^{j} \mathrm{pos}(j-k)\hat{\dot{a}}_{k+1}^{\mathrm{L}} & \leq & \sum_{k=0}^{j} \mathrm{pos}(j-k)\dot{a}_{k+1}^{\mathrm{L}} \\ j\hat{n}_K + \sum_{k=0}^{\infty} \min\{j,k\}\widehat{h}_{k+1}(s) & \leq & jn_K + \sum_{k=0}^{\infty} \min\{j,k\}h_{k+1}(s) \\ & & \text{for all } s. \end{array} \tag{4}$$

The proof of this theorem in [10] consists of the decomposition of any perturbation from the orbit of a given pencil into a sequence of "elementary perturbations" of one of small set of types. These elementary perturbations are discussed at greater length in Sec. 6. We remark that the first two conditions in (4) (involving $\dot{\mathbf{a}}_k$, $\dot{\mathbf{a}}_k^{\mathrm{L}}$) were also proved in [3]. In fact, Elmroth and Kågstrom [4] used this theory to produce a stratification of all the possible structures of 2×3 pencils. This stratification is used to determine which structures are reachable by arbitrarily small perturbations to given pencils and furthermore to produce the specific perturbation needed for each case.

In [2], they consider the sequence of Jordan indices in descending order for a matrix $\mathcal{M}$, or actually a general class of (linear) operators. Let $s_1(\lambda, \mathcal{M}) \geq \cdots \geq s_{n_J}(\lambda, \mathcal{M})$ denote the the dimensions of the Jordan blocks corresponding to the eigenvalue λ in descending order, with $s_j(\lambda, \mathcal{M}) = 0$ for $j > n_J$. Here n_J is the geometric multiplicity of the eigenvalue λ. If Γ is a contour in the complex plane such that there is no eigenvalue on Γ, then define $s_j(\Gamma) = \sum_\lambda s_j\lambda$ where the sum is taken over all eigenvalues λ inside the contour Γ. A typical result, expressed for matrix polynomials, is

Theorem 4[2]. If $\mathcal{M}(\lambda)$ is a $N_{\rm rows}\times N_{\rm cols}$ matrix polynomial with $N_{\rm rows} \le N_{\rm cols}$ with no eigenvalues on the contour Γ, then there exists an $\epsilon > 0$ and a matrix polynomial $\widetilde{\mathcal{M}}(\lambda)$ such that $\|\mathcal{M}-\widetilde{\mathcal{M}}\| < \epsilon$ on Γ and

$$\sum_{j\ge l} s_j(\Gamma,\widetilde{\mathcal{M}}) \le \sum_{j\ge l} s_j(\Gamma,\mathcal{M}) \text{ for all natural numbers } l. \tag{5}$$

Furthermore, if $N_{\rm rows} = N_{\rm cols}$, then we have equality for $l = 1$. Conversely, if $\mathcal{M}$ is as given above, and we have a sequence of prospective indices $s_j(\Gamma,\widetilde{\mathcal{M}})$ satisfying (5), then for every $\epsilon > 0$ there exists a corresponding matrix polynomial $\widetilde{\mathcal{M}}$ with the given indices $s_j(\Gamma,\widetilde{\mathcal{M}})$, with no eigenvalues on Γ, and such that $\|\mathcal{M}-\widetilde{\mathcal{M}}\| < \epsilon$.

All the results summarized in this section involve relations between various sequences of integers, differences between consecutive entries in such integer sequences, and leading partial sums of integers in such sequences. This motivated us to study the properties intrinsic to integer sequences independent of the relation between such sequences and any underlying matrix entity. In the next section we sketch the results arising from integer sequences and revisit the results of this section in light of the next in Sec. 4.

3 Integer Sequences

We sketch an algebra on semi-infinite integer sequences, defining several operations and transformations on such sequences. Though motivated by its application to linear algebra, this theory is completely independent of any particular application.

3.1 Basic Properties and Definitions

We define several operators on semi-infinite sequences of integers in Table 1 and summarize a few elementary properties in Table 2. We will also use the following *unit coordinate* sequences. Define the special sequence $\mathbf{e} = \{1,1,1,1,1,\ldots\}$ as a constant sequence, the sequence $\mathbf{e_1} = \mathbf{\Delta e} = \{1,0,0,0,\ldots\}$, and $\mathbf{E} = \mathbf{\Sigma e} = \{1,2,3,4,\ldots\}$. We also define the shifted coordinate sequence $\rho^{k-1}\mathbf{e_1} = \{0,\ldots,0,1,0,\ldots\}$ where the single "1" entry appears in the k-th position. We remark that if $0 \le i < j$, then $\rho^i\mathbf{e_1} + \rho^{i+1}\mathbf{e_1} + \cdots + \rho^{j-1}\mathbf{e_1} = \rho^i\mathbf{e} - \rho^j\mathbf{e}$.

Lemma 5. Tables 2 and 3 summarize some basic relations between the various operators defined in Table 1.

Proof: These properties are simple consequences of the definitions, as illustrated by the examples. □

Lemma 6. The operators $\rho, \sigma, \mathbf{\Sigma}, \mathbf{\Delta}$ are all linear in the sense that $\square(\alpha\mathbf{a}+\mathbf{b}) = \alpha\square\mathbf{a}+\mathbf{b}$ for all scalars α and sequences $\mathbf{a}, \mathbf{b}$, where $\square = \rho$, σ, $\mathbf{\Sigma}$, or $\mathbf{\Delta}$.

Proof: By direct calculation from the definitions. □

We now show the correspondance of our notation with the results of [10].

Lemma 7. Let $\mathbf{a} = \{a_1, a_2, a_3, \ldots, a_i, \ldots\}$ be any sequence, and let $\mathrm{pos}(x) = \max\{0, x\}$ be defined

operator	*description*	*example*
	operators for all integer sequences	
$\mathbf{x}$	any integer sequence	$1,1,3,5,5,5,\ldots$
$\mathbf{X}=\mathbf{\Sigma x}$	running sum	$1,2,5,10,15,20,\ldots$
$\dot{\mathbf{x}}=\mathbf{\Delta x}$	first difference	$1,0,2,2,0,0,\ldots$
$\sigma\mathbf{x}$	left shift	$1,3,5,5,5,5,\ldots$
$\rho\mathbf{x}$	right shift	$0,1,1,3,5,5,\ldots$
	operators for ascending integer sequences	
$\mathbf{a}$	an ascending sequence	$1,1,3,3,4,4,\ldots$
$\mathbf{a}^{\#}$	conjugate of an ascending sequence	$0,2,2,4,\infty,\infty,\ldots$
	[$a_k^{\#}$ is the no. of a_i's less than k]	
	operators for descending integer sequences	
$\mathbf{g}$	a descending sequence	$4,4,3,1,1,0,\ldots$
$\mathbf{g}^*$	conjugate of a descending sequence	$5,3,3,2,0,0,\ldots$
	[g_k^* is the no. of g_i's greater than or equal to k]	
	scalar values derived from sequences	
x_∞	final value (if it exists)	5
$(\mathbf{a}^{\#})_\infty$	number of finite entries in **a**	∞
G_∞	sum of all entries in **g**	12
g_1^*	number of positive entries in **g**	5

Table 1: Operators on Sequences.

	Expression	*= this expression*	*Example*
If **a** is any integer sequence			$\mathbf{a}=\{1,1,3,3,4,4,\ldots\}$
(a)	$\mathbf{\Delta a}$	$=\mathbf{a}-\rho\mathbf{a}$	$=\{1,0,2,0,1,0,0,\ldots\}$
(b)	$\mathbf{\Sigma\Delta a}$	$=\mathbf{\Delta\Sigma a}=\mathbf{a}$	$=\{1,1,3,3,4,4,\ldots\}$
(c)	$\sigma\rho\mathbf{a}$	$=\mathbf{a}$	$=\{1,1,3,3,4,4,\ldots\}$
(d)	$\rho\sigma\mathbf{a}$	$=\mathbf{a}-a_1\mathbf{e_1}$	$=\{0,1,3,3,4,4,\ldots\}$
(e)	$\rho\mathbf{\Sigma a}$	$=\mathbf{\Sigma}\rho\mathbf{a}$	$=\{0,1,2,5,8,12,16,\ldots\}$
(f)	$\rho\mathbf{\Delta a}$	$=\mathbf{\Delta}\rho\mathbf{a}$	$=\{0,1,0,2,0,1,0,\ldots\}$
(g)	$\sigma\mathbf{\Sigma a}$	$=\mathbf{\Sigma}\sigma\mathbf{a}+a_1\mathbf{e}$	$=\{2,5,8,12,16,20,\ldots\}$
(h)	$\sigma\mathbf{\Delta a}$	$=\mathbf{\Delta}\sigma\mathbf{a}-a_1\mathbf{e_1}$	$=\{0,2,0,1,0,0,\ldots\}$

Table 2: Basic Properties of shift, sum, and difference operators.

	Expression	*= this expression*	*Example*
If $\mathbf{a}$ is ascending but bounded			$\mathbf{a} = \{1, 1, 3, 3, 4, 4, \ldots\}$
(i)	$\operatorname{Neg}\mathbf{a}$	$= a_{\max}\mathbf{e} - \mathbf{a}$	$= \{3, 3, 1, 1, 0, 0, \ldots\}$
(j)	$\operatorname{neg}\operatorname{Neg}\mathbf{a}$	$= \mathbf{a} - a_1\mathbf{e}$	$= \{0, 0, 2, 2, 3, 3, \ldots\}$
(k)	$(\mathbf{a} + \mathbf{e})^{\#}$	$= \rho(\mathbf{a}^{\#})$	$= \{0, 0, 2, 2, 4, \infty \ldots\}$
(l)	$(\operatorname{Neg}\mathbf{a})^*$		$= \{4, 2, 2, 0, 0, 0, \ldots\}$
	[reverse of the finite entries of $\mathbf{a}^{\#}$*, followed by 0's]*		
If $\mathbf{g}$ is descending with $g_\infty = 0$			$\mathbf{g} = \{4, 4, 3, 1, 1, 0, 0, \ldots\}$
(m)	$\operatorname{neg}\mathbf{g}$	$= g_1\mathbf{e} - \mathbf{g}$	$= \{0, 0, 1, 3, 3, 4, 4, \ldots\}$
(n)	$\operatorname{Neg}\operatorname{neg}\mathbf{g}$	$= \mathbf{g} - g_\infty\mathbf{e}$	$= \{4, 4, 3, 1, 1, 0, 0, \ldots\}$
(o)	$(\mathbf{g} + \mathbf{e})^*$	$= \rho\mathbf{g}^* + \infty\mathbf{e_1}$	$= \{\infty, 5, 3, 3, 2, 0, 0, \ldots\}$
(p)	$\mathbf{g}^* + \mathbf{e_1}$	$= (\mathbf{g} + \rho^{g_1^*}\mathbf{e_1})^*$	$= \{4, 4, 3, 1, 1, 1, 0, \ldots\}^*$
			$= \{6, 3, 3, 2, 0, 0, \ldots\}$
(q)	$(\operatorname{neg}\mathbf{g})^{\#}$		$= \{2, 3, 3, 5, \infty, \infty, \ldots\}$
	[reverse of the nonzero entries of $\mathbf{g}^*$*, followed by* ∞*'s]*		

Table 3: Basic Properties of conjugate and negation operators.

as the *positive part* of x. Then the i-th entry of the double sum of $\mathbf{a}$ is

$$[\mathbf{\Sigma}^2\mathbf{a}]_i = \sum_{k=0}^{\infty} \operatorname{pos}(i-k)a_{k+1}.$$

Also, let x and $y_1, y_2, \ldots, y_m$ be a collection of $m+1$ non-negative integers (setting $y_k = 0$ for $k > m$), let $Y = y_1 + y_2 + \cdots + y_m$, and define the sequence $\mathbf{z} = \{x+Y, -y_1, -y_2, \ldots, -y_m, 0, 0, \ldots\}$. Then the i-th entry of $\mathbf{\Sigma}^2\mathbf{z}$ is $[\mathbf{\Sigma}^2\mathbf{z}]_i = ix + \sum_{k=1}^{\infty} \min\{i,k\}y_k$.

Proof: By direct calculation by noting that

$$\begin{aligned}\mathbf{\Sigma}^2\mathbf{a} = \{&a_1,\\ &2a_1 + a_2,\\ &3a_1 + 2a_2 + a_3,\\ &\ldots,\\ &ia_1 + (i-1)a_2 + \cdots + a_i,\\ &\ldots\},\end{aligned}$$

and

$$[\mathbf{\Sigma}^2\mathbf{z}]_i = ix + y_1 + 2y_2 + \cdots + iy_i + iy_{i+1} + \cdots + iy_m.$$

□

3.2 Ferrer's Diagrams

The sequences can be illustrated by so called Ferrers diagrams [11, 8]. Consider the sequence $\mathbf{a}$ defined in Table 1. It can be modeled by the histogram in (6a) in which the k-th column has a_k

X's. Likewise, the descending sequence $\mathbf{g}$ from Table 1 can be represented by (6b).

a_k							
$\uparrow$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	o	o	o	o	o	o	$\cdots$
5	o	o	o	o	o	o	$\cdots$
4	o	o	o	o	X	X	$\cdots$
3	o	o	X	X	X	X	$\cdots$
2	o	o	X	X	X	X	$\cdots$
1	X	X	X	X	X	X	$\cdots$
	1	2	3	4	5	6	$\to k$

(a)

g_k							
$\uparrow$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
6	o	o	o	o	o	o	$\cdots$
5	o	o	o	o	o	o	$\cdots$
4	X	X	o	o	o	o	$\cdots$
3	X	X	X	o	o	o	$\cdots$
2	X	X	X	o	o	o	$\cdots$
1	X	X	X	X	X	o	$\cdots$
	1	2	3	4	5	6	$\to k$

(b)

Ferrer's Diagrams (6)

From (6) we can also read the conjugate sequence by reading across. From (6a) we can read $\mathbf{b} = \mathbf{a}^{\#}$ by reading the number of o's across, and from (6b) we can read off the sequence $\mathbf{g}^*$ by reading the number of X's across in each row. From (6a) we can also see that the entries of $\mathbf{a}$ and $\mathbf{b}$ fill up leading rectangles anchored at the origin. This particular property can be formalized in the following Lemma.

Lemma 8. Suppose we have sequences $\mathbf{a} \in \mathcal{S}_1$, $\mathbf{b} = \mathbf{a}^{\#}$, $\mathbf{A} = \mathbf{\Sigma a}$, $\mathbf{B} = \mathbf{\Sigma b}$. Then $B_k + A_{b_k} = kb_k$, where $\mathbf{A}$, $\mathbf{B}$ are any sequences in the space $\mathcal{S}_2$ related by $\mathbf{b} = \mathbf{\Delta B} = (\mathbf{\Delta A})^{\#}$. Analogously, $A_k + B_{a_k} = ka_k$.

Proof: in the appendix. □

Remark. Let $f(x)$ be a strictly increasing non-negative function of x, defined for all non-negative x, and $g(y)$ be its conjugate, i.e. $g(f(x)) = x$ and $f(g(y)) = y$ for all non-negative x, y. Then Lemma 8 is analogous to the continuous theorem $\int_0^x f(x)dx + \int_0^y g(y)dy = xy$. This remark is easily proved via integration by parts, and indeed the proof of Lemma 8 can be thought of as a discrete analog to integration by parts.

3.3 Comparison and Majorization of Sequences

We define what it means to for a sequence to be *less than* another or to be *majorized* by another sequence.

Comparison of sequences: Given two sequences $\mathbf{A}$, $\mathbf{B}$, we say that $\mathbf{A} \le \mathbf{B}$ if $A_i \le B_i$ for all i. We say that $\mathbf{A} = \mathbf{B}$ if $A_i = B_i$ for all i. We say that $\mathbf{A} < \mathbf{B}$ if $\mathbf{A} \le \mathbf{B}$ and $\mathbf{A} \ne \mathbf{B}$.

Majorization of sequences: Let $\mathbf{a}$ and $\mathbf{b}$ be two sequences in $\mathcal{S}_0$. We say that $\mathbf{b}$ weakly majorizes $\mathbf{a}$, denoted by $\mathbf{b} \succ_w \mathbf{a}$ or $\mathbf{a} \prec_w \mathbf{b}$, if $\mathbf{\Sigma a} \le \mathbf{\Sigma b}$. If in addition, the sum of all the entries of the two sequences $\mathbf{a}$ and $\mathbf{b}$ agree and are finite, then we say that $\mathbf{b}$ strictly majorizes $\mathbf{a}$, written $\mathbf{b} \succ \mathbf{a}$. In other words, $\mathbf{b} \succ \mathbf{a}$ iff $\mathbf{\Sigma b} \ge \mathbf{\Sigma a}$ and $\max\{\mathbf{\Sigma b}\} = \max\{\mathbf{\Sigma a}\} < \infty$.

Since majorization of sequences plays a critical role in the results of this paper, we state here the two fundamental results we will use.

Theorem 9. Suppose we have the sequences

$$\mathbf{a} \in \mathcal{S}_1, \quad \mathbf{A} = \mathbf{\Sigma a}, \quad \mathbf{b} = \mathbf{a}^{\#}, \quad \mathbf{B} = \mathbf{\Sigma b},$$

as well as

$$\mathbf{g} \in \mathcal{S}_1, \quad \mathbf{G} = \mathbf{\Sigma g}, \quad \mathbf{h} = \mathbf{g}^{\#}, \quad \mathbf{H} = \mathbf{\Sigma h}.$$

If $\mathbf{g} \prec_w \mathbf{a}$, then (a) the counts $(\mathbf{a}^{\#})_\infty \geq (\mathbf{g}^{\#})_\infty$, and (b) $\mathbf{h} \succ_w \mathbf{b}$. If the first majorization is strict, then so is the second, among the finite entries.

Proof: in the appendix. □

An analog of this theorem for descending sequences appeared in [9] for the case of "strong" majorization. This case can be proved as a special case of Theorem 9, or proved directly using the analog of Lemma 8. We now state and prove a theorem on weak majorization for descending sequences. We see that the conjugate sequences must be adjusted slightly in order to satisfy the inequalities.

Theorem 10. ([9, p174] for strong majorization). If $\mathbf{x}, \mathbf{y}$ are sequences in $\mathcal{S}_\mathrm{d}$, then their conjugates $\mathbf{x}^*, \mathbf{y}^*$ are also in $\mathcal{S}_\mathrm{d}$. For any such sequences, $\mathbf{x} \prec \mathbf{y}$ iff $\mathbf{y}^* \prec \mathbf{x}^*$. If $\mathbf{x} \prec_w \mathbf{y}$, then $(\mathbf{x} + \rho^k \mathbf{e_1} + \cdots \rho^{k+j-1}\mathbf{e_1}) \prec \mathbf{y}$ and $\mathbf{y}^* \prec \mathbf{x}^* + j\mathbf{e_1}$, where $k = x_1^*$ and $j = Y_\infty - X_\infty \geq 0$.

Proof: in the appendix. □

Unlike ascending sequences in Theorem 9, the precedence relation for descending sequences requires adjustment to the sequences if the sequence sums differ: $X_\infty \neq Y_\infty$. We give a example to illustrate what happens when we weaken $\prec$ to $\prec_w$. Consider the following sequences $\mathbf{x}, \mathbf{y}$, each with 7 positive entries, with their sums and conjugates:

$\mathbf{x}$	$\widehat{\mathbf{x}}$	$\mathbf{y}$	$\Sigma\mathbf{x}$ $\Sigma\widehat{\mathbf{x}}$ $\Sigma\mathbf{y}$	$\mathbf{x}^*$	$\widehat{\mathbf{x}}^*$	$\mathbf{y}^*$	$\Sigma\widehat{\mathbf{x}}^*$ $\Sigma\mathbf{y}^*$ $\Sigma\mathbf{x}^*$
6	6	6	$06 = 06 = 06$	7	8	7	$08 > 07 = 07$
4	4	6	$10 = 10 < 12$	4	4	3	$12 > 10 < 11$
4	4	5	$14 = 14 < 17$	4	4	3	$16 > 13 < 15$
3	3	1	$17 = 17 < 18$	3	3	3	$19 > 16 < 18$
1	1	1	$18 = 18 < 19$	1	1	3	$20 > 19 = 19$
1	1	1	$19 = 19 < 20$	1	1	2	$21 = 21 > 20$
1	1	1	$20 = 20 < 21$	0	0	0	$21 = 21 > 20$
0	1	0	$20 < 21 = 21$	0	0	0	$21 = 21 > 20$
0	0	0	$20 < 21 = 21$	0	0	0	$21 = 21 > 20$
0	0	0	$20 < 21 = 21$	0	0	0	$21 = 21 > 20$
⋮	⋮	⋮	⋮ ⋮ ⋮	⋮	⋮	⋮	⋮ ⋮ ⋮

(7)

We see that $\mathbf{x} \prec_w \mathbf{y}$, but $\mathbf{x}^*$ and $\mathbf{y}^*$ do not majorize each other in either direction. But we do obtain majorization with the modified sequence: $(\widehat{\mathbf{x}})^* = \mathbf{x}^* + \mathbf{e_1} \succ \mathbf{y}$, which limits the amount by which $\mathbf{x}^*$ misses majorizing $\mathbf{y}^*$. A further example is given later in (16), where $\widetilde{\mathbf{g}}, \widehat{\widetilde{\mathbf{g}}}, \mathbf{g}$ there play the role of $\mathbf{x}, \widehat{\mathbf{x}}, \mathbf{y}$ here, respectively.

4 Jordan, Kronecker Indices and Sequences of Nullities

We review some results relating the Jordan and Kronecker indices and the Weyr characteristics to the nullities of the block Toeplitz matrices. We illustrate some of these results with some examples involving the sequences of nullities.

4.1 Notation and Basic Results

We state the following theorem regarding the Kronecker indices.

Theorem 11. Define the sequences

$$\mathbf{a} = \mathbf{\Delta A}, \dot{\mathbf{a}} = \mathbf{\Delta}^2\mathbf{A} \text{ and } \mathbf{b} = (\mathbf{\Delta A})^{\#},$$

where A_i is the nullity of $\mathcal{A}_i$ in (1). Then $\dot{a}_k = [\mathbf{\Delta}^2\mathbf{A}]_k$ is exactly the number of Kronecker indices *equal* to $k-1$ and a_k is the number of indices *less than* k. Hence $a_\infty = n_K$ is the total number of Kronecker indices. The first n_K entries of $\mathbf{b}$: $b_1, \ldots, b_{n_K}$, are the non-negative integers consisting of the right Kronecker indices in ascending order, and $b_k = \infty$ for all $k > n_K$. □

Corollary 12. For any integer k, the K-blocks up to size $(k{-}1) \times k$ occupy $k \cdot a_k - A_k = (k{-}1)A_k - kA_{k-1}$ rows and $(k{+}1) \cdot a_k - A_k$ columns. In particular, if q is the order of the largest K-block and n_K is the total number of K-blocks, then the entire right Kronecker part of the pencil occupies, for any $k > q$,

$$N_K = k \cdot n_K - A_k = (k{-}1)A_k - kA_{k-1} \tag{8}$$

rows and $(k{+}1) \cdot n_K - A_k = kA_k - (k{+}1)A_{k-1}$ columns (illustrated in Fig. 1).

Proof:

$$\begin{aligned} A_k &= a_1 + a_2 + \cdots + a_k \\ &= \dot{a}_1 + (\dot{a}_1{+}\dot{a}_2) + \cdots + (\dot{a}_1{+}\cdots{+}\dot{a}_k) \\ &= k \cdot \dot{a}_1 + (k{-}1) \cdot \dot{a}_2 + \cdots + 1 \cdot \dot{a}_k \\ &= k \cdot a_k - [0 \cdot \dot{a}_1 + 1 \cdot \dot{a}_2 + \cdots + (k{-}1) \cdot \dot{a}_k] \\ &= k \cdot (A_k - A_{k-1}) - [0 \cdot \dot{a}_1 + 1 \cdot \dot{a}_2 + \cdots + (k{-}1) \cdot \dot{a}_k]. \end{aligned} \tag{9}$$

But in the last expression, the part within square brackets is exactly the rows occupied by the K-blocks up to size $(k{-}1) \times k$. The total number of such K-blocks is $\dot{a}_1 + \cdots + \dot{a}_k = a_k$. Hence the number of columns occupied is exactly a_k more than the number of rows. Also, if $k > q$, we have $a_k = n_K$ yielding (8), and the corollary is proved. □

We remark that (8) is equivalent to saying

$$N_K = [n_K\mathbf{E} - \mathbf{A}]_\infty = \text{sum of all the entries in } [n_K\mathbf{e} - \mathbf{a}] \tag{10}$$

We also remark that we could also define the sequence of left nullities $\mathbf{G}^{\mathrm{L}}$ of the matrices (2), but this is equivalent to $\mathbf{G}$ since they differ only by a sequence fixed by the dimensions of the overall pencil, by (5c):

$$\mathbf{G} - \mathbf{G}^{\mathrm{L}} = (n_K - n_L)\mathbf{E} = (N_{\text{cols}} - N_{\text{rows}})\mathbf{E}.$$

We now turn our attention to the Jordan indices. We prove that the nullities $\mathbf{G} = \{G_1, G_2, \ldots\}$ of the matrices (2) yield the dimensions of the Jordan chains associated with the zero eigenvalue of the pencil $\mathcal{E} - \lambda\mathcal{F}$, independent of the presence of any Kronecker blocks. Without loss of generality, we can examine the pencil $\mathcal{E} + \lambda\mathcal{F}$.

Theorem 13. The Jordan indices for eigenvalue zero for the pencil $\mathcal{E} + \lambda\mathcal{F}$ are related to the nullities $\mathbf{G}$ of the matrices (2) as follows. Define $\mathbf{g} = \mathbf{\Delta G}$. Let n_K be the total number of right Kronecker blocks for the pencil, and let h_i be the number of Jordan blocks (indices) *equal* to i, for $i = 1, 2, \ldots$. Let $n_J = h_1 + h_2 + \cdots$ be the total number of Jordan blocks for eigenvalue 0. Then $n_K = g_\infty = a_\infty$ and we have the following sequences (different ways of expressing the same result):

(a)	$\mathbf{G} - n_K\mathbf{E}$	Extended Weyr Characteristics
(a)	$\mathbf{g} - n_K\mathbf{e}$	
	(Number of Jordan indices greater than or equal to i, for $i = 1, 2, 3, \ldots$)	
(b)	$\mathbf{s} = (\mathbf{g} - n_K\mathbf{e})^*$	Jordan indices in descending order, followed by 0's (Segré characteristics for eigenvalue 0 [11, 8])
(c)	$\operatorname{neg}\mathbf{g}$	Number of Jordan indices less than i, for $i = 1, 2, 3, \ldots$
(d)	$\mathbf{d} = (\operatorname{neg}\mathbf{g})^{\#}$	Jordan indices in ascending order, followed by ∞'s
(e)	$\dot{\mathbf{g}} = \{n_K + n_J, -h_1, -h_2, \ldots, -h_r, 0, 0, \ldots\} = (n_K + n_J)\mathbf{e_1} - \rho\mathbf{h},$	

where r is the largest Jordan index.

We also state the following result regarding the tail of the sequence of nullities:

Corollary 14. Let r be the index of the largest J-block and $n_K = g_\infty$ be the total number of K-blocks. Then the entire part corresponding to eigenvalue zero has order

$$N_J = G_k - k \cdot n_K = kG_{k-1} - (k-1)G_k, \tag{11}$$

for any $k > r$ (illustrated in Fig. 1).

Proof: By (9), we have for any k

$$\begin{aligned} G_k &= k \cdot (G_k - G_{k-1}) - [0 \cdot \dot{g}_1 + 1 \cdot \dot{g}_2 + 2 \cdot \dot{g}_3 + \cdots + (k-1)\dot{g}_k] \\ &= k \cdot (G_k - G_{k-1}) + [1 \cdot h_1 + 2 \cdot h_2 + \cdots + (k-1)h_{k-1}] \\ &= k \cdot (g_k) + [1 \cdot h_1 + 2 \cdot h_2 + \cdots + (k-1)h_{k-1}] \\ &= k \cdot (\dot{g}_1 + \ldots + \dot{g}_k) + [1 \cdot h_1 + 2 \cdot h_2 + \cdots + (k-1)h_{k-1}] \\ &= k \cdot (n_K + n_J - h_1 - \ldots - h_{k-1}) + [1 \cdot h_1 + 2 \cdot h_2 + \cdots + (k-1)h_{k-1}] \\ &\quad \text{(by part (e) of Theorem 13),} \end{aligned}$$

where the h_i are defined as in Theorem 13. When $k > r$, the second line above becomes $G_k = k \cdot (G_k - G_{k-1}) + N_J$, and the last line becomes $G_k = k \cdot (n_K + n_J - n_J) + N_J$, proving the corollary. □

We remark that (11) is equivalent to

$$N_J = [\mathbf{G} - n_K \cdot \mathbf{E}]_\infty = \text{sum of all the entries in } [\mathbf{g} - n_K \cdot \mathbf{e}]. \tag{12}$$

4.2 Example

We illustrate the relation between the Jordan/Kronecker indices and the sequences of nullities with an example. Consider the pencil (where .'s denote zeroes), corresponding to Fig. 1:

$$
\mathcal{E} + s\mathcal{F} = \left(\begin{array}{ccccccccccccccccccc}
s & 1 & 0 & 0 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & s & 1 & 0 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & s & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & s & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & s & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & 0 & s & 1 & 0 & 0 & 0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & 0 & 0 & s & 1 & 0 & 0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & 0 & 0 & 0 & s & 1 & 0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & 0 & 0 & 0 & 0 & s & 1 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & s & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & s & 1 & 0 & 0 & 0 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & s & 1 & 0 & 0 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & 0 & s & 1 & 0 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 & s & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 & 0 & 0 & 0 & s & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & s & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & s \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\multicolumn{19}{r}{(\mathcal{M}+s\mathcal{N})}
\end{array}\right), \tag{13}
$$

where $\mathcal{M} + s\mathcal{N}$ is regular with no zero eigenvalues, or is empty (equivalently, $\mathcal{M}$ is a square nonsingular matrix, or empty). This pencil has right Kronecker blocks of sizes 3×4 and 7×8, a left Kronecker block of size 1×0, and two Jordan blocks for eigenvalue 0 of orders 5 and 2. In this pencil, there are

$n_K = 2$	right Kronecker blocks occupying	$N_K = 10$	rows,
$n_L = 1$	left Kronecker blocks occupying	$N_L = 0$	columns, and
$n_J = 2$	Jordan blocks for eigenvalue 0 occupying	$N_J = 7$	rows.
		N_E is the dimension of $\mathcal{M}$.	

The nullities of the block Toeplitz matrices (1) and (2) are the following, together with the nullities (1) corresponding to $(\mathcal{E} + s\mathcal{F})^T$ which yield the left nullities:

$$
\begin{array}{ll}
\mathbf{A} & = \{0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad \cdots\}, \\
\mathbf{G} & = \{4 \quad 8 \quad 11 \quad 14 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31 \quad 33 \quad \cdots\}, \\
\mathbf{A}^{\mathrm{L}} & = \{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad \cdots\}.
\end{array} \tag{14}
$$

The first two sequences above are pictured in Fig. 1. Notice that the **A** sequence is concave up, the **G** sequence is concave down (except for the virtual corner at the origin). The final slope of both sequences are the same (both equal to $n_K = 2$, which is also exactly the number of corners in the **A** sequence, counting multiplicities), and when the sequences reach this final slope, they are $N_K + N_J = 17$ apart. The reader will notice the corners in the `A`'s in entries 3 and 7, and in the `g`'s in entries 2 and 5 (plus an implicit one at the origin), marked by `A` and `G` in the Figure, respectively. The corners correspond exactly to the nonzero entries in the second differences: $\dot{a}_4, \dot{a}_8$, $\dot{g}_3, \dot{g}_6$, and

$\dot{a}_1^{\mathrm{L}}$:

$$\begin{aligned}
\dot{\mathbf{a}} &= \{0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots\},\\
\dot{\mathbf{g}} &= \{4 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots\},\\
\dot{\mathbf{a}}^{\mathrm{L}} &= \{1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots\}.
\end{aligned}$$

So one sees that the right Kronecker and Jordan indices in the pencil represented by these sequences are, respectively $\{3, 7\}$ and $\{2, 5\}$ (one less than the corresponding subscripts). The Kronecker and Jordan indices appear in the conjugate sequences:

$$\begin{aligned}
\mathbf{a}^{\#} &= \{3 \;\; 7 \;\; \infty \;\; \infty \;\; \infty \;\; \cdots\} && \text{Kronecker indices (ascending)},\\
(\mathbf{g} - g_\infty \mathbf{e})^{*} &= \{5 \;\; 2 \;\; 0 \;\; 0 \;\; 0 \;\; \cdots\} && \text{Jordan indices (descending)},\\
(\operatorname{neg} \mathbf{g})^{\#} &= \{2 \;\; 5 \;\; \infty \;\; \infty \;\; \infty \;\; \cdots\} && \text{Jordan indices (ascending)},\\
(\mathbf{a}^{\mathrm{L}})^{\#} &= \{0 \;\; \infty \;\; \infty \;\; \infty \;\; \infty \;\; \cdots\} && \text{left Kronecker indices (ascending)}.
\end{aligned}$$

5 Effect of Modifying Pencils

In this section, we use some of the theory above to extend some results regarding the effects of perturbations on the Kronecker and on the Jordan indices.

5.1 Perturbations

Let $\mathbf{A}$ be the sequence of nullities of (1), $\mathbf{b}$ be the sequence of the Kronecker indices in ascending order (followed by ∞'s), and let $\mathbf{B} = \mathbf{\Sigma b}$. Hence also $B_1, \ldots, B_{n_K}$ are also non-negative integers and $B_{n_K+1} = B_{n_K+2} = \cdots = \infty$. With this identification, we immediately obtain a result on the initial sums of the Kronecker indices as the matrices are perturbed, using Theorem 9. We use $A_1, A_2, \ldots$ to denote the nullities of the matrices (1), corresponding to the pencil $\mathcal{E} - \lambda\mathcal{F}$. Denote by $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$ a slightly perturbed pencil and let $\widetilde{\mathbf{A}} = \{\widetilde{A}_1, \widetilde{A}_2, \ldots\}$ be the sequence of nullities of the resulting perturbed matrices of the form (1). Denote the sequence of Kronecker indices of the perturbed pencil by $\widetilde{\mathbf{b}} = \{\widetilde{b}_1, \widetilde{b}_2, \ldots\}$. If the perturbation is sufficiently small, the nullities will satisfy $\widetilde{\mathbf{A}} \leq \mathbf{A}$ so that Theorem 3 yields the result $\widetilde{\mathbf{B}} \geq \mathbf{B}$. This yields one of the basic theorems linking the perturbations of pencils to majorization of sequences of nullities.

Theorem 15. Let $\mathbf{b}$ be the sequence of right Kronecker indices in ascending order for the pencil $\mathcal{E} - \lambda\mathcal{F}$ (followed by ∞'s), and $\widetilde{\mathbf{b}}$ be likewise for the new pencil $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$. If the new pencil $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$ is formed from $\mathcal{E} - \lambda\mathcal{F}$ by taking a sufficiently small perturbation, by appending an additional row, or by deleting a column, or if $\mathcal{E} - \lambda\mathcal{F}$ lies in the closure of the orbit of $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$, then

$$\widetilde{\mathbf{a}} \prec_w \mathbf{a}.$$

Proof: For the case of orbits or perturbations, this has been proved in [10, 3]. The proof depends on the fact that for a sufficiently small perturbation, the nullities of the matrices (1) can only decrease. Since the nullities are affected in the same way by the addition of a row or deletion of a column, we can arrive at the same conclusion for these cases too. □

By a similar argument, we have the similar theorem for the Jordan indices, where $\mathbf{d}$ is the sequence of Jordan indices in ascending order (followed by ∞'s), and $\mathbf{G}$ is the sequence of nullitites of the matrices (2).

Theorem 16. Under the same conditions as Theorem 15,

$$\widetilde{\mathbf{g}} \prec_w \mathbf{g}.$$

□

Theorem 17. Under the same conditions as Theorem 15,

$$\mathbf{b} \prec_w \widetilde{\mathbf{b}},$$

where $\mathbf{b}, \widetilde{\mathbf{b}}$ are, respectively, the sequences of right Kronecker indices in ascending order for the original and perturbed pencils. Furthermore, $(G_1 - \widetilde{G}_1) \geq 0$ and

$$\rho^{(G_1 - \widetilde{G}_1)} \widetilde{\mathbf{d}} \prec_w \mathbf{d},$$

where $\mathbf{d}$ and $G_1 = n_K + n_J$ are the sequence of Jordan indices in *ascending* order and the combined count of right Kronecker and Jordan blocks, respectively, for the original pencil; and $\widetilde{\mathbf{d}}$ and $\widetilde{G}_1 = \widetilde{n}_K + \widetilde{n}_J$ are the corresponding items for the perturbed pencil.

Proof: For the case of $\mathbf{b} \prec_w \widetilde{\mathbf{b}}$, this is a simple consequence of the theorems 15 and 9. For the rest, we go through the following derivation, using the identities $\operatorname{neg} \mathbf{g} = g_1 \mathbf{e} - \mathbf{g}$ and $\operatorname{neg} \widetilde{\mathbf{g}} + (g_1 - \widetilde{g}_1)\mathbf{e} = g_1 \mathbf{e} - \widetilde{\mathbf{g}}$:

$$\begin{array}{lcll}
\widetilde{\mathbf{g}} & \prec_w & \mathbf{g} & \text{by assumption} \\
\operatorname{neg} \widetilde{\mathbf{g}} + (g_1 - \widetilde{g}_1)\mathbf{e} & \succ_w & \operatorname{neg} \mathbf{g} & \\
(\operatorname{neg} \widetilde{\mathbf{g}} + (g_1 - \widetilde{g}_1)\mathbf{e})^{\#} & \prec_w & (\operatorname{neg} \mathbf{g})^{\#} & \text{by Theorem 9} \\
(\operatorname{neg} \widetilde{\mathbf{g}} + (g_1 - \widetilde{g}_1)\mathbf{e})^{\#} & = & \rho^{(g_1 - \widetilde{g}_1)}((\operatorname{neg} \widetilde{\mathbf{g}})^{\#}) & \text{by item (p) of Fig. 3.}
\end{array}$$

□

We remark that this theorem was proved in [2] for the case where $\widetilde{n}_K = n_K$, but generalized to eigenvalues lying within a contour of the complex plane.

The following example illustrates this theorem, in which $g_1 - \widetilde{g}_1 = 1$ and $\widetilde{n}_K = n_K = 0$.

$\widetilde{\mathbf{G}}$		$\mathbf{G}$	$\widetilde{\mathbf{g}}$	$\mathbf{g}$	$(\operatorname{neg} \widetilde{\mathbf{g}})$	$(\operatorname{neg} \mathbf{g})$	$\widetilde{\mathbf{d}}$	$\rho\widetilde{\mathbf{d}}$	$\mathbf{d}$	$\mathbf{\Sigma}\rho\widetilde{\mathbf{d}}$		$\mathbf{\Sigma d}$		$\mathbf{\Sigma}\widetilde{\mathbf{d}}$
06	$<$	07	6	7	0	0	1	0	1	00	$<$	01	$=$	01
10	$<$	13	4	6	2	1	1	1	2	01	$<$	03	$>$	02
11	$<$	13	1	0	5	7	2	1	2	02	$<$	05	$>$	04
12	$<$	13	1	0	5	7	2	2	2	04	$<$	07	$>$	06
12	$<$	13	0	0	6	7	2	2	2	06	$<$	09	$>$	08
12	$<$	13	0	0	6	7	4	2	2	08	$<$	11	$<$	12
12	$<$	13	0	0	6	7	∞	4	2	12	$<$	13	$<$	∞
12	$<$	13	0	0	6	7	∞	∞	∞	∞		∞		∞
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮		⋮

(15)

(where $\mathbf{G} = \mathbf{\Sigma g}$ = nullities of (2); $\mathbf{d} = (\operatorname{neg} \mathbf{g})^{\#}$ = Jordan indices (ascending))

It is easy to construct a 7×7 matrix $\mathcal{M}$ in Jordan Canonical Form whose Jordan indices are given by the sequence $\mathbf{d}$ in (15), and for which an arbitrarily small perturbation yields a matrix $\widetilde{\mathcal{M}}$ whose Jordan indices are $\widetilde{\mathbf{d}}$. In the small perturbation, one 2×2 Jordan block of that matrix is changed to:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & \epsilon \end{pmatrix},$$

and two 2×2 Jordan blocks coalesce into one 4×4 block. We see that for this case, the number of Jordan blocks for eigenvalue zero has been reduced by 1, so the new sequence of Jordan indices has been shifted 1 position. The last column in (15) shows that the shift is necessary to achieve majorization.

We remark in the above theorem that $g_1 - g_\infty$, $\tilde{g}_1 - \tilde{g}_\infty$ are, respectively, the number of finite entries in $\mathbf{d}$, $\tilde{\mathbf{d}}$. Hence $\rho^{(g_1-\tilde{g}_1)}\tilde{\mathbf{d}}$ has $g_1 - \tilde{g}_\infty$ finite entries, exactly $g_\infty - \tilde{g}_\infty$ more than $\mathbf{d}$ has. It always the case that $g_\infty \geq \tilde{g}_\infty$ (otherwise it would be that $G_k < \tilde{G}_k$ for some sufficiently large index k).

But the case where the number of right Kronecker blocks remains unchanged is of particular interest. This was the case proved in [2], but generalized to eigenalues within a contour on the complex plane. A particular example of this is the ordinary eigenvalue problem in which there are no Kronecker indices, as illustrated above with $\mathcal{M}$. Then $\mathbf{d}$ and $\rho^{(g_1-\tilde{g}_1)}\tilde{\mathbf{d}}$ have the same number of finite entries. In other words, the sequence of Jordan indices in ascending order for the perturbed pencil must be shifted to line up its last Jordan index with that of the original pencil. Then this theorem states that when so lined up, the original Jordan indices in ascending order majorizes the shifted perturbed Jordan indices.

In the general case where the number of right Kronecker blocks does change, the last Jordan index for the perturbed pencil is lined up $g_\infty - \tilde{g}_\infty > 0$ positions *past* the last Jordan index of the original pencil. Then this theorem states that when so lined up, the original Jordan indices in ascending order majorizes the shifted perturbed Jordan indices.

Now we consider the Jordan indices in descending order. If $n_K = \tilde{n}_K$ then $\tilde{\mathbf{G}} - \tilde{n}_K\mathbf{E} \leq \mathbf{G} - n_K\mathbf{E}$. We obtain the following more limited theorem regarding the Jordan indices in descending order, as a simple consequence of Theorem 10.

Theorem 18.. Under the assumptions of Theorem 17, if $n_K = \tilde{n}_K$ then $N_J \geq \tilde{N}_J$, and

$$(\hat{\tilde{\mathbf{g}}})^* \equiv (\tilde{\mathbf{g}} - n_K\mathbf{e})^* + (N_J - \tilde{N}_J)\mathbf{e_1} \succ (\mathbf{g} - n_K\mathbf{e})^*,$$

where $(\mathbf{g}-n_K\mathbf{e})^*$, $(\tilde{\mathbf{g}}-n_K\mathbf{e})^*$ are the sequences of Jordan indices in *descending* order for the original and new pencil, respectively, and $\hat{\tilde{\mathbf{g}}}$ is defined to be the result of appending $(N_J - \tilde{N}_J)$ "1"'s to the end of $\tilde{\mathbf{g}} - n_K\mathbf{e}$. □

We remark that this adjustment (appending $(N_J - \tilde{N}_J)$ "1"'s) is needed following the result of Theorem 10. Essentially, we implicitly increase the order of the largest Jordan block to make the sums coincide. An example of this effect is given by (7), in which $\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}$ there play the role of $(\tilde{\mathbf{g}} - n_K\mathbf{e}), \hat{\tilde{\mathbf{g}}}, (\mathbf{g} - n_K\mathbf{e})$ here. We give another example in (16) for which a matrix example having the appropriate Jordan indices, $(\tilde{\mathbf{g}} - \tilde{n}_K\mathbf{e})^*, (\mathbf{g} - n_K\mathbf{e})^*$, can be easily constructed, where we assume for simplicity that $n_k = \tilde{n}_k = 0$ (or equivalently we compute these sequences for just the regular part of the pencil). We see in (16) that though $\tilde{\mathbf{g}}^*$ does not majorize $\mathbf{g}^*$, the discrepancy is limited by $N_J - \tilde{N}_J$.

$\tilde{\mathbf{g}}$	$\hat{\tilde{\mathbf{g}}}$	$\mathbf{g}$	$\boldsymbol{\Sigma}\tilde{\mathbf{g}}$		$\boldsymbol{\Sigma}\hat{\tilde{\mathbf{g}}}$		$\boldsymbol{\Sigma}\mathbf{g}$	$\tilde{\mathbf{g}}^*$	$\hat{\tilde{\mathbf{g}}}^*$	$\mathbf{g}^*$	$\boldsymbol{\Sigma}(\hat{\tilde{\mathbf{g}}}^*)$		$\boldsymbol{\Sigma}(\mathbf{g}^*)$		$\boldsymbol{\Sigma}(\tilde{\mathbf{g}}^*)$
6	6	7	06	=	06	<	07	3	5	2	05	>	02	<	03
4	4	6	10	=	10	<	13	2	2	2	07	>	04	<	05
1	1	0	11	=	11	<	13	2	2	2	09	>	06	<	07
0	1	0	11	<	12	<	13	2	2	2	11	>	08	<	09
0	1	0	11	<	13	=	13	1	1	2	12	>	10	=	10
0	0	0	11	<	13	=	13	1	1	2	13	>	12	>	11
0	0	0	11	<	13	=	13	0	0	1	13	=	13	>	11
0	0	0	11	<	13	=	13	0	0	0	13	=	13	>	11
⋮	⋮	⋮	⋮		⋮		⋮	⋮	⋮	⋮	⋮		⋮		⋮

(16)

(where $\mathbf{G} = \boldsymbol{\Sigma}\mathbf{g}$ = nullities of (2) with $n_K = 0$; $\mathbf{g}^*$ = Jordan indices (descending))

5.2 Appending a Single Row or Column

Finally we state and prove two new corollaries that provide bounds on the Kronecker indices when a single row or column are appended to a pencil, or a rank-one change is made to one of the matrices.

Corollary 19. Consider a pencil $\mathcal{E} - \lambda\mathcal{F}$ with corresponding block Toeplitz matrices (1) whose nullities form the sequence $\mathbf{A}$, and let $\mathbf{b}$ be the ascending sequence of Kronecker indices for this pencil. If a single row is appended to the pencil $\mathcal{E} - \lambda\mathcal{F}$ to obtain the pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$, then the corresponding new sequence of Kronecker indices $\widehat{\mathbf{b}}$ satisfies

$$\mathbf{b} + \mathbf{e_1} \succ_w \rho\widehat{\mathbf{b}} \succ_w \rho\mathbf{b} \quad \text{or equivalently} \quad \mathbf{B} + \mathbf{e} \geq \rho\widehat{\mathbf{B}} \geq \rho\mathbf{B},$$

where $\mathbf{B} = \mathbf{\Sigma b}$. If the row appended is constant (i.e. the row appended to $\mathcal{F}$ is zero) then $\widehat{\mathbf{b}}$ satisfies

$$\mathbf{b} \succ_w \rho\widehat{\mathbf{b}} \succ_w \rho\mathbf{b}. \quad \text{or equivalently} \quad \mathbf{B} \geq \rho\widehat{\mathbf{B}} \geq \rho\mathbf{B}.$$

In either case, the number of right Kronecker blocks ($\widehat{n}_K$) in the new pencil is either the same or one less than that in the original pencil:

$$n_K - 1 \leq \widehat{n}_K \leq n_K.$$

Proof: In going from the original pencil to the modified pencil, the nullities of the block Toeplitz matrices (1) cannot go up, and cannot go down by more than the number of rows appended to each individual matrix $\mathcal{A}_i$. Hence we have the inequality

$$\mathbf{A} - \{2, 3, 4, 5, \ldots\} \leq \widehat{\mathbf{A}} \leq \mathbf{A}.$$

We can then apply a sequence of identities just using the algebra of integer sequences, beginning with the above inequality, using the definition $\mathbf{b} = \mathbf{a}^{\#}$:

$$\begin{aligned}
&\mathbf{A} - \sigma\mathbf{E} \leq \widehat{\mathbf{A}} \leq \mathbf{A} \\
&\rho\mathbf{A} - \mathbf{E} \leq \rho\widehat{\mathbf{A}} \leq \rho\mathbf{A} \\
&\rho\mathbf{A} \leq \rho\widehat{\mathbf{A}} + \mathbf{E} \leq \rho\mathbf{A} + \mathbf{E} \\
&\rho\mathbf{a} \prec_w \rho\widehat{\mathbf{a}} + \mathbf{e} \prec_w \rho\mathbf{a} + \mathbf{e} \\
&(\mathbf{b} + \mathbf{e}) \succ_w \rho\widehat{\mathbf{b}} + \rho\mathbf{e} \succ_w \rho\mathbf{b} + \rho\mathbf{e} \\
&(\mathbf{B} + \mathbf{E}) \geq \rho\widehat{\mathbf{B}} + \rho\mathbf{E} \geq \rho\mathbf{B} + \rho\mathbf{E} \\
&(\mathbf{B} + \mathbf{e}) \geq \rho\widehat{\mathbf{B}} \geq \rho\mathbf{B},
\end{aligned}$$

where we have used the identity from item (k) of Fig. 3:

$$\begin{aligned}
(\rho\mathbf{a} + \mathbf{e})^{\#} &= \rho[(\rho\mathbf{a})^{\#}] \\
&= \rho[(\mathbf{a}^{\#} + \mathbf{e})] \\
&= \rho\mathbf{b} + \rho\mathbf{e}.
\end{aligned}$$

If the row appended to the $\mathcal{F}$ matrix is zero, then the nullities of the matrices (1) can go down by at most one less than the general case, so we have the identities

$$\begin{aligned}
&\mathbf{A} - \mathbf{E} \leq \widehat{\mathbf{A}} \leq \mathbf{A} \\
&\mathbf{A} - \mathbf{E} \leq \widehat{\mathbf{A}} \leq \mathbf{A} \\
&\mathbf{A} \leq \widehat{\mathbf{A}} + \mathbf{E} \leq \mathbf{A} + \mathbf{E} \\
&\mathbf{a} \prec_w \widehat{\mathbf{a}} + \mathbf{e} \prec_w \mathbf{a} + \mathbf{e} \\
&\mathbf{b} \succ_w \rho\widehat{\mathbf{b}} \succ_w \rho\mathbf{b} \\
&\mathbf{B} \geq \rho\widehat{\mathbf{B}} \geq \rho\mathbf{B}.
\end{aligned}$$

As for the number of Kronecker blocks, from the above and (8) we have

$$\mathbf{A} - \sigma\mathbf{E} \le \widehat{\mathbf{A}} \le \mathbf{A}k \cdot n_K - N_K - (k+1) = A_k - (k+1) \le \widehat{A}_k = k \cdot \widehat{n}_K - \widehat{N}_K \le A_k = k \cdot n_K - N_K$$

for all k sufficiently large. Extracting the terms in k yields

$$k \cdot (n_K - 1) \le k \cdot \widehat{n}_K \le k \cdot n_k,$$

for all k sufficiently large, yielding the result. □

Corollary 20. Under the same conditions and notation as the previous theorem, if the new pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ is formed by appending an extra column, then the new sequence of Kronecker indices satisfies $\mathbf{b} \succ_w \widehat{\mathbf{b}} \succ_w \rho\mathbf{b}$. The number of the Kronecker blocks satisfies $n_K \le \widehat{n}_K \le n_k + 1$.

Proof: The nullities of the matrices (1) cannot go down, and can go up only by the the number of columns appended, so we obtain the identities

$$\begin{aligned} &\mathbf{A} \le \widehat{\mathbf{A}} \le \mathbf{A} + \mathbf{E} \\ &\mathbf{a} \prec_w \widehat{\mathbf{a}} \prec_w \mathbf{a} + \mathbf{e} \\ &\mathbf{b} \succ_w \widehat{\mathbf{b}} \succ_w (\mathbf{a} + \mathbf{e})^{\#} = \rho\mathbf{b} \\ &\mathbf{B} \ge \widehat{\mathbf{B}} \ge \rho\mathbf{B}. \end{aligned}$$

The bound on n_K is proved by the same technique as in the previous Corollary, using (8). □

6 Admissible Perturbations

6.1 General Result

We now examine the converse of theorems 15, 16, 17, proved in [10]. That is, we restate a result which guarantees when given Kronecker structure is reachable from a given pencil via arbitrarily small perturbations in terms of conditions on the sequences of nullities. Equivalently, the result guarantees when a pencil lies in the closure of the orbit of a second pencil in terms of conditions on the sequences of nullities. This is presented for completeness and to illustrate the simple effect on the sequences of nullities of the block Toeplitz matrices (1) and (2).

To do this, we extend our notation as follows. Let s be a scalar complex-valued parameter, which can take the value ∞. Then we can define the sequence of nullities $\mathbf{G}(s)$ corresponding to the nullities of the matrices of the form (2), but formed from the modified pencil $(\mathcal{E} - s\mathcal{F}) - \lambda\mathcal{F}$. That is, $\mathbf{G}(s)$ is the sequence of nullities corresponding to the (possibly empty) sequence of Jordan indices for eigenvalue s, where s can be any complex number or infinity.

Theorem 21. Consider two pencils $\mathcal{E} - \lambda\mathcal{F}$, and $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$, with corresponding configurations of Kronecker blocks leading to the corresponding sequences of nullities $\mathbf{A}, \mathbf{B}, \mathbf{G}(s)$, and $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{G}}(s)$. If the sequences of nullities satisfy $\widehat{\mathbf{A}} \le \mathbf{A}$, $\widehat{\mathbf{B}} \le \mathbf{B}$, and $\widehat{\mathbf{G}}(s) \le \mathbf{G}(s)$ for all complex s, then $\mathcal{E} - \lambda\mathcal{F}$ lies in the closure of the orbit of $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$, or equivalently, an arbitrarily (infinitesimally) small perturbation to $\mathcal{E} - \lambda\mathcal{F}$ suffices to obtain a pencil with the same Kronecker configuration as that of $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$.

Proof: Theorem 3 states that $\mathcal{E} - \lambda\mathcal{F}$ lies in the closure of the orbit of $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ if and only if the conditions (4) hold. But by Lemma 7, the right hand sides in (4) are exactly the sequences of nullities $\mathbf{A}$,, $\mathbf{A}^{\mathrm{L}}$,, $\mathbf{G}(s)$, and the left hand sides are the sequences $\widehat{\mathbf{A}}$,, $\widehat{\mathbf{A}}^{\mathrm{L}}$,, $\widehat{\mathbf{G}}(s)$, respectively. So we can rewrite the above conditions as $\widehat{\mathbf{A}} \le \mathbf{A}$, $\widehat{\mathbf{A}}^{\mathrm{L}} \le \mathbf{A}^{\mathrm{L}}$, $\widehat{\mathbf{G}}(s) \le \mathbf{G}(s)$ for all s, respectively □

We remark that the first two conditions (involving $\mathbf{A}$, $\mathbf{A}^{\mathrm{L}}$) were proved in [3].

We also remark that in computing the nullities in the example (14), we used a zero tolerance (set by MATLAB) equal to the $\epsilon \cdot N \cdot \|\mathcal{M}\|$, where $\epsilon = 2^{-52}$ is the unit round-off of the machine, N is the dimension of the block Toeplitz matrix, and $\|\mathcal{M}\| < 2$ is the norm of the matrix involved. Any singular value less than this tolerance was considered zero. We further remark that in this example, the smallest singular value considered nonzero for any block Toeplitz matrix encountered was .2091, well separated from the zero singular values. This smallest nonzero singular value is a lower bound on the perturbation necessary to *increase* the nullity of any block Toeplitz matrix. In view the Theorem 26, this value is also a lower bound on the perturbation to (13) needed to obtain a pencil whose orbit-closure does *not* contain the original given pencil (13). This deserves further investigation.

6.2 Types of Fundamental Transitions

The proof in [10] of the result cited above was based on proving that one can always apply at least one of the *transitions* described below to the original pencil such that the sequences of nullities of the resulting intermediate pencil still majorize the corresponding sequences of the target perturbed pencil. Hence, after a sequence of such transitions, one must eventually reach a pencil whose sequences of nullities exactly match those of the target pencil. In other words, the perturbations need to reach an attainable Kronecker structure can be decomposed into a sequence of *fundamental transitions* listed in this section. We will see that each transition corresponds to a simple modification to the sequence of nullities, illustrated by a simple movement of one or two corners in the curves shown in Fig. 1.

In what follows, we denote the starting "source" pencil $\mathcal{E} - \lambda\mathcal{F}$, the target pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$, and the intermediate pencil after a transition $\widetilde{\mathcal{E}} - \lambda\widetilde{\mathcal{F}}$. We denote by $\mathbf{A}$, etc. the sequence of nullities of (1) before the transition, together with $\dot{\mathbf{a}} = \mathbf{\Delta}^2\mathbf{A}$, and $\widehat{\mathbf{A}}, \hat{\dot{\mathbf{a}}}$, etc. the corresponding desired target sequences after the transition. and $\widetilde{\mathbf{A}}, \tilde{\dot{\mathbf{a}}}$, etc. the sequences after one transition.

We now list the different types of transitions one can apply to a pencil. Each transition can be applied to a pencil using an arbitrarily small perturbation, yielding a new pencil with the indicated new structure of nullities. We claim that given a source pencil $\mathcal{E} - \lambda\mathcal{F}$ with sequence of nullities $\mathbf{A}, \mathbf{G}$, and target pencil $\widehat{\mathcal{E}} - \lambda\widehat{\mathcal{F}}$ with corresponding sequence of nullities $\widehat{\mathbf{A}}, \widehat{\mathbf{G}}$, both majorized by the source sequences, one can always find one of these transitions to apply to $\mathbf{A}$ and/or $\mathbf{G}$ such that the resulting sequences $\widetilde{\mathbf{A}}, \widetilde{\mathbf{G}}$ still majorize the target sequences, unless the source and target sequences are identical. Each transition corresponds to a simple change to the sequence of nullities as illustrated in Figures 2, 3.

1. `(KL->J)` $K_i + L_j$ are replaced by J_{i+j+1} (i.e. a left & a right Kronecker block coalesce to form a new Jordan block for eigenvalue zero.) The sequences are then affected as follows:

$$\begin{aligned}
\tilde{\dot{\mathbf{a}}} &= \dot{\mathbf{a}} - \rho^i \mathbf{e_1} &&= \dot{\mathbf{a}} + \{0, \ldots, 0, -1, 0, 0, 0, \ldots\} \\
\widetilde{\mathbf{A}} &= \dot{\mathbf{a}} - \rho^i \mathbf{E} &&= \mathbf{A} + \{0, \ldots, 0, -1, -2, -3, -4, \ldots\} \\
\tilde{\dot{\mathbf{g}}} &= \dot{\mathbf{g}} - \rho^{i+j+1} \mathbf{e_1} &&= \dot{\mathbf{g}} + \{0, \ldots, 0, -1, 0, 0, 0, \ldots\} \\
\widetilde{\mathbf{G}} &= \mathbf{G} - \rho^{i+j+1} \mathbf{E} &&= \mathbf{G} + \{0, \ldots, 0, -1, -2, -3, -4, \ldots\}
\end{aligned}$$

where the leading string of zeroes might be empty. A similar effect occurs on the sequence $\mathbf{B}$ corresponding to the L-blocks.

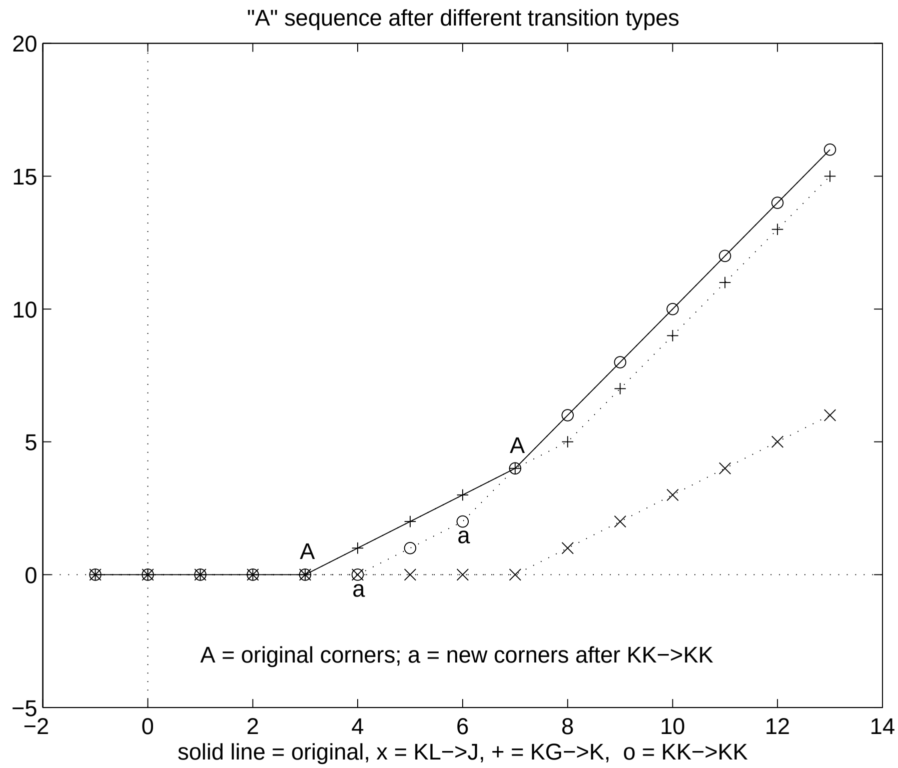

Figure 2: The `A` sequence together with the effect of certain transitions.

2. `(KE->K)` $K_i + (N_E)$ are replaced by $K_{i+1} + (N_E{-}1)$ (i.e. the regular part is reduced by 1 dimension, which is appended to one Kronecker block) The L part is left unchanged. The sequences are then affected as follows:

$$\begin{aligned}
\tilde{\dot{\mathbf{a}}} &= \dot{\mathbf{a}} + \rho^{i+1}\mathbf{e_1} - \rho^{i}\mathbf{e_1} &&= \dot{\mathbf{a}} + \{0,\ldots,0,-1,+1,0,0,\ldots\} \\
\widetilde{\mathbf{A}} &= \mathbf{A} - \rho^{i}\mathbf{e} &&= \mathbf{A} + \{0,\ldots,0,-1,-1,-1,-1,\ldots\} \\
\tilde{\dot{\mathbf{g}}} & &&= \dot{\mathbf{g}} \\
\widetilde{\mathbf{G}} & &&= \mathbf{G}
\end{aligned}$$

where the leading string of zeroes might be empty.

3. `(KK->KK)` $K_i + K_j$ are replaced by $K_{i+1} + K_{j-1}$ where $i{+}1 \le j{-}1$. (i.e. two K-blocks are replaced with two other more generic K-blocks) The L and E parts remain unchanged. The sequences are then affected as follows:

if $j > i+2$

$$\begin{aligned}
\tilde{\dot{\mathbf{a}}} &= \dot{\mathbf{a}} - \rho^{i}\mathbf{e_1} + \rho^{i+1}\mathbf{e_1} + \rho^{j-1}\mathbf{e_1} - \rho^{j}\mathbf{e_1} &&= \dot{\mathbf{a}} + \{0,\ldots,0,-1,+1,0,\ldots,0,+1,-1,0,\ldots\} \\
\widetilde{\mathbf{A}} &= \mathbf{A} - \rho^{i}\mathbf{e} + \rho^{j}\mathbf{e} &&= \mathbf{A} + \{0,\ldots,0,-1,-1,-1,\ldots,-1,0,0,0,\ldots\}
\end{aligned}$$

or if $j = i+2$

$$\begin{aligned}
\tilde{\dot{\mathbf{a}}} &= \dot{\mathbf{a}} - \rho^{i}\mathbf{e_1} + 2\rho^{i+1}\mathbf{e_1} - \rho^{i+2}\mathbf{e_1} &&= \dot{\mathbf{a}} + \{0,\ldots,0,-1,+2,-1,0,\ldots\} \\
\widetilde{\mathbf{A}} &= \mathbf{A} - \rho^{i}\mathbf{e_1} &&= \mathbf{A} + \{0,\ldots,0,-1,0,0,0,\ldots\}
\end{aligned}$$

and in either case

$$\begin{aligned}
\tilde{\dot{\mathbf{g}}} &= \dot{\mathbf{g}} \\
\widetilde{\mathbf{G}} &= \mathbf{G}
\end{aligned}$$

where the leading string of zeroes might be empty.

4. `(LE->L)` $L_i + N_E$ are replaced by $L_{i+1} + (N_E{-}1)$ (i.e. the rEgular part is reduced by 1 dimension, which is appended to one Kronecker block) The K part is left unchanged. Only the sequences corresponding to the L-blocks are affected in a way analogous to the type 2 `(KE->K)`.

5. `(LL->LL)` $L_i + L_j$ are replaced by $L_{i+1} + L_{j-1}$ where $i{+}1 \leq j{-}1$ (i.e. two L-blocks are replaced with two other more generic L-blocks) The K and E parts remain unchanged. Only the sequences corresponding to the L-blocks are affected in a way analogous to the type 3 `(KK->KK)`.

The effect of these transitions on the example of Sec. 4.2 is illustrated by Fig. 2. Notice that transition `KL->J` removes a corner in the **A** sequence, the transition `KE->K` moves a corner one position to the left, and the transition `KK->KK` moves two corners each one step toward the other.

The remaining transition types apply specifically to the Jordan chains.

6. `(JJ->JJ)` $J_i + J_j$ are replaced by $J_{i-1} + J_{j+1}$ (i.e. two J-blocks are replaced with two other more generic J-blocks), leaving the K- and L-blocks unchanged. The sequences are then affected as follows:

$$
\begin{aligned}
&\text{if } j > i \\
&\quad \tilde{\dot{\mathbf{g}}} = \dot{\mathbf{g}} - \rho^{i-1}\mathbf{e_1} + \rho^{i}\mathbf{e_1} + \rho^{j}\mathbf{e_1} - \rho^{j+1}\mathbf{e_1} && = \dot{\mathbf{g}} + \{0,\ldots,0,-1,+1,0,\ldots,0,+1,-1,0,\ldots\} \\
&\quad \widetilde{\mathbf{G}} = \mathbf{G} - \rho^{i-1}\mathbf{e} + \rho^{j}\mathbf{e} && = \mathbf{G} + \{0,\ldots,0,-1,-1,-1,\ldots,-1,0,0,0,\ldots\} \\
&\text{else if } j = i \\
&\quad \tilde{\dot{\mathbf{g}}} = \dot{\mathbf{g}} - \rho^{i-1}\mathbf{e_1} + 2\rho^{i}\mathbf{e_1} - \rho^{i+1}\mathbf{e_1} && = \dot{\mathbf{g}} + \{0,\ldots,0,-1,+2,-1,0,\ldots\} \\
&\quad \widetilde{\mathbf{G}} = \mathbf{G} - \rho^{i-1}\mathbf{e_1} && = \mathbf{G} + \{0,\ldots,0,-1,0,0,0,\ldots\}
\end{aligned}
$$

where the leading strings of zeroes has at least one entry (i.e. $j \geq i \geq 2$).

7. `(JJ1->J)` $J_i + J_1$ are replaced by J_{i+1} (i.e. a 1×1 Jordan block is absorbed into another Jordan block, increasing the latter's dimension by one). The sequences are then affected as follows:

$$
\begin{aligned}
&\text{if } i > 1 \\
&\quad \tilde{\dot{\mathbf{g}}} = \dot{\mathbf{g}} - \mathbf{e_1} + \rho\mathbf{e_1} + \rho^{i}\mathbf{e_1} - \rho^{i+1}\mathbf{e_1} && = \dot{\mathbf{g}} + \{-1,+1,0,\ldots,0,+1,-1,0,\ldots\} \\
&\quad \widetilde{\mathbf{G}} = \mathbf{G} - \mathbf{e} + \rho^{i}\mathbf{e} && = \mathbf{G} + \{-1,-1,-1,\ldots,-1,0,0,0,\ldots\} \\
&\text{else if } i = 1 \\
&\quad \tilde{\dot{\mathbf{g}}} = \dot{\mathbf{g}} - \mathbf{e_1} + 2\rho\mathbf{e_1} - \rho^{2}\mathbf{e_1} && = \dot{\mathbf{g}} + \{-1,+2,-1,0,\ldots\} \\
&\quad \widetilde{\mathbf{G}} = \mathbf{G} - \mathbf{e_1} && = \mathbf{G} + \{-1,0,0,0,\ldots\}
\end{aligned}
$$

where latter case occurs when two J_1 blocks coalesce to form a J_2 block.

8. `(J->JE)` $J_i + (N_E)$ are replaced by $J_{i-1} + (N_E{+}1)$ (i.e. one zero eigenvalue becomes nonzero, reducing the order of one Jordan block for zero). The sequences are then affected as follows:

$$
\begin{aligned}
\tilde{\dot{\mathbf{g}}} &= \dot{\mathbf{g}} - \rho^{i-1}\mathbf{e_1} + \rho^{i}\mathbf{e_1} && = \dot{\mathbf{g}} + \{0,\ldots,0,-1,+1,0,0,\ldots\} \\
\widetilde{\mathbf{G}} &= \mathbf{G} - \rho^{i-1}\mathbf{e} && = \mathbf{G} + \{0,\ldots,0,-1,-1,-1,-1,\ldots\}
\end{aligned}
$$

where the leading strings of zeroes has at least one entry.

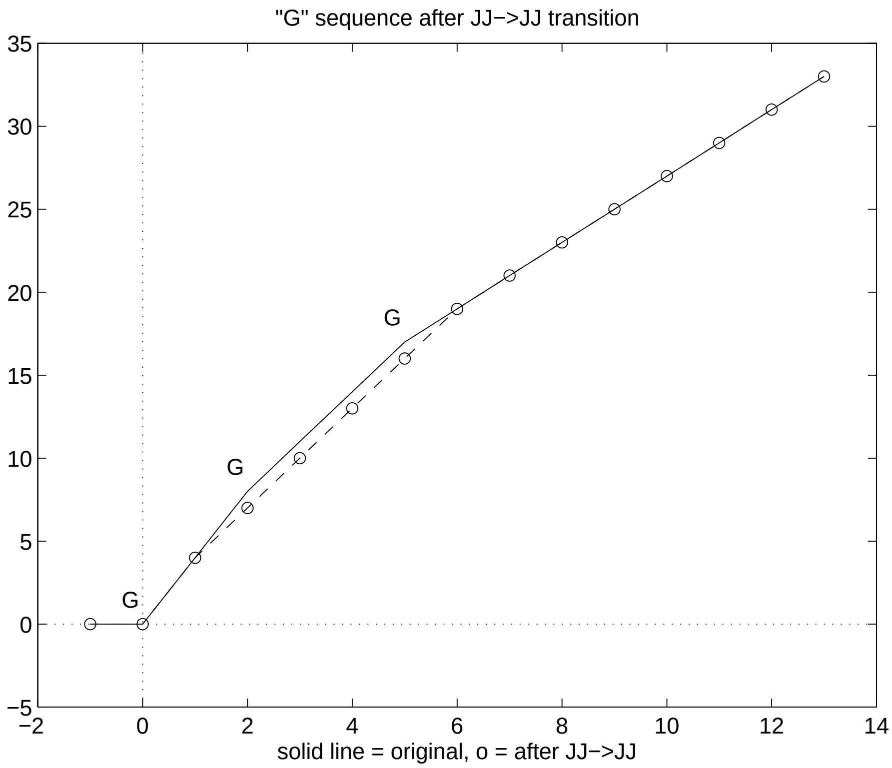

Figure 3: The `G` sequence together with the effect of the `JJ->JJ` transition.

9. `(J1->E)` $J_1 + (N_E)$ is replaced by $(N_E{+}1)$ (i.e. the zero eigenvalue in a 1×1 Jordan block becomes nonzero). The sequences are then affected as follows:

$$\begin{aligned} \tilde{\dot{\mathbf{g}}} &= \dot{\mathbf{g}} - \mathbf{e_1} + \rho\mathbf{e_1} &&= \dot{\mathbf{g}} + \{-1, +1, 0, 0, 0, \ldots\} \\ \tilde{\mathbf{G}} &= \mathbf{G} - \mathbf{e} &&= \mathbf{G} + \{-1, -1, -1, -1, -1, \ldots\} \end{aligned}$$

The effect of transition `JJ->JJ` on the example of Sec. 4.2 is illustrated by Fig. 3. Notice that this transition moves two corners each one step away from each other. This transition is marked by the solid line. The transition `KL->J` (not shown) would have the effect of adding a new corner to the **G** sequence. The other transitions would have effect analogous to those for the **A** sequence, moving corners around in appropriate ways.

To prove Theorem 3, Pokrzywa [10] proved that if one is given "source" and "target" sequences of nullities corresponding to two valid pencil configurations, where the source sequence majorizes the target sequence, then a sequence of transitions of exactly the types outlined above can be applied to the source pencil to reach the target pencil. Another way to state this is: as long as the target sequences are majorized by the source sequences, one can always apply one of the given transition types in reverse to find a new set of sequences, also majorized by the source sequence and also corresponding to a valid pencil configuration. The existence of a transition is a consequence of the fact that every valid transition must modify the sequence at the corners, and that the sequences delimit convex regions in the plane.

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Appendix – Proofs of Majorization Theorems

We prove some of the theorems from Sec. 3.

Proof of Theorem 8. We prove the second form $A_k + B_{a_k} = ka_k$. Define $\dot{\mathbf{a}} = \mathbf{\Delta a} = \mathbf{\Delta}^2\mathbf{A}$.

$$
\begin{aligned}
A_k \;= a_1 + \cdots + a_k \;=\; & \quad \dot{a}_1 \\
& + \dot{a}_1 + \dot{a}_2 \\
& \quad \vdots \\
& + \dot{a}_1 + \dot{a}_2 + \cdots + \dot{a}_k .
\end{aligned}
$$

Rearrange terms to get

$$A_k = k\dot{a}_1 + (k-1)\dot{a}_2 + \cdots + 2\dot{a}_{k-1} + \dot{a}_k = k(\dot{a}_1 + \cdots + \dot{a}_k) - \sum_{1\le j\le k} (j-1)\dot{a}_j .$$

Each term in the last summation is just the sum of all the b_i's that are exactly equal to $j-1$. Hence the last summation is just the sum of all the b_j's that are less than k. But a_k is exactly the number of such b_j's, so the summation equals

$$\sum_{1\le j\le k} (j-1)\dot{a}_j = b_1 + \cdots + b_{a_k} = B_{a_k}.$$

Hence we have the result $A_k = ka_k - B_{a_k}$. □

Proof of Theorem 9. Let m be the number of finite entries in the sequence $\mathbf{b}$, i.e. $b_m < \infty = b_{m+1}$, and let n be the corresponding number for the sequence $\mathbf{h}$. We must show that $H_k - B_k$ is not negative, for any k, and strictly positive if $\mathbf{G} < \mathbf{A}$. In order to do this, we must divide the proof into several cases. We prove the theorem for $k \le n$, proving in the process that $n \le m$. This case also applies if $n = \infty$. For $k > n$, $H_k = \infty$ so the theorem is vacuously true.

Let $k \le n$. We have that $h_k = l < \infty$. Then it must be that $b_k < \infty$. For if b_k were infinite, we would have from Lemma 7 that

$$k \le g_j \text{ for all } j > l, \quad \text{but} \quad a_j \le \cdots < k \text{ for all } j.$$

Hence for large enough j, $G_j > A_j$, contradicting the assumption. Hence $k \le m$ and so $n \le m$.

Next we use Theorem 8 to show that $H_k - B_k$ cannot be negative. We have that

$$\begin{aligned} H_k - B_k = & \; k(h_k - b_k) + A_{b_k} - G_{h_k} \\ = & \; [k(h_k - b_k)] + [A_{b_k} - G_{b_k}] + [G_{b_k} - G_{h_k}] \\ = & \; x + y + z, \end{aligned}$$

where $x = [k(h_k - b_k)]$, $y = [A_{b_k} - G_{b_k}]$, and $z = [G_{b_k} - G_{h_k}]$. By assumption $y \ge 0$. If we have the strict inequality $\mathbf{G} > \mathbf{A}$, then $y > 0$. We then show that $x + z \ge 0$ and hence $B_k - B_k \ge 0$, with strict inequality if $\mathbf{G} > \mathbf{A}$.

Case I: $b_k = h_k$. Then $x = z = 0$.

Case II: $b_k < h_k$. Then we use the fact that $\mathbf{G}$ is the summation of $\mathbf{g}$ to obtain the expression $-z = g_{b_k+1} + \cdots + g_{h_k}$. Since $\mathbf{g} = \{g_i\}$ is a nondecreasing (*ascending*) nonnegative sequence, this expression must be less than the last term times the number of terms: $-z \le g_{h_k} \cdot (h_k - b_k)$. From Lemma 7 we have that $g_{h_k} \le k$, so we may conclude that $-z \le x$.

Case III: $b_k > h_k$. Then $z = g_{h_k+1} + \cdots + g_{b_k}$. As in case 2, we obtain the inequality $z \ge g_{h_k} \cdot (b_k - h_k) \ge k \cdot (b_k - h_k) \ge -x$.

□

Proof of Theorem 10. Define $l = y_1^*$, the number of positive entries in $\mathbf{y}$. Assume that l is finite, otherwise this theorem is vacuous. This implies that k (the number of positive entries in $\mathbf{x}$) is also finite.

If $Y_\infty = X_\infty$, then $\prec_w \equiv \prec$. In this case this theorem is identical to Fact B.5 in [9, p174], or can be proved using techniques similar to above. So let us suppose that $\mathbf{x} \prec_w \mathbf{y}$, but $Y_\infty \ne X_\infty$.

The precedence relation implies that $Y_\infty > X_\infty$ and $j > 0$. Form the sequence $\widehat{\mathbf{x}} = \mathbf{x} + \rho^k \mathbf{e_1} + \cdots + \rho^{k+j-1}\mathbf{e_1}$. This consists of appending j ones after the last positive entry (the k-th entry) in $\mathbf{x}$. It is clear that $\widehat{X}_\infty = Y_\infty$ by construction.

We claim that $\widehat{\mathbf{x}} \prec \mathbf{y}$. To see this, note that the first k entries of $\widehat{\mathbf{X}} = \mathbf{\Sigma}\widehat{\mathbf{x}}$ and $\mathbf{X} = \mathbf{\Sigma}\mathbf{x}$ coincide. Denote $\mathbf{Y} = \mathbf{\Sigma}\mathbf{y}$. For all the terms beyond the l-th, we have $\widehat{X}_i \le Y_\infty = Y_i$, for $i \le l$. So if $l \le k$, the claim is true. So suppose $l > k$. Then we must examine the terms from the k-th to the l-th. For the k-th term, we have $\widehat{X}_k = X_\infty \le Y_k < Y_\infty$. For all the terms between the k-th and the l-th, we have $y_{k+i} \ge 1$, and hence $X_\infty + i \le (Y_k) + i \le Y_{k+i} \le Y_\infty$, for $1 \le i \le l - k$. This implies that $j \equiv Y_\infty - X_\infty \ge l - k$, by setting $i = l - k$. This in turn implies that for every $i = 1, 2, \cdots, l - k$, $\widehat{x}_{k+i} = 1$, and $\widehat{X}_{k+i} = X_\infty + i$ and hence $\widehat{X}_{k+i} \le Y_{k+i}$. We conclude that $\widehat{X}_i \le Y_i$ for all i.

For the effect on the conjugate sequences, we note that $\rho^k\mathbf{e} - \rho^{k+j}\mathbf{e} = \rho^k\mathbf{e_1} + \cdots + \rho^{k+j-1}\mathbf{e_1}$, and apply Fig. 3 (p) to see that $\widehat{\mathbf{x}}^* = \mathbf{x}^* + j\mathbf{e_1}$. □