Goal : Give a definition of dimension of a subspace $V \subset \mathbb{R}^n$.

(big part of this is proving that all subspaces look like $\mathbb{R}^k$,

$k \in [0, n]$.)

So far, exploring Spans - $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, $\vec{v}_i \in \mathbb{R}^n$.

when is $\vec{v}$ in Span? when is it represented by unique linear comb.?

Proposition: If $\vec{v} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ has unique rep'n as linear comb, then every vector in the span has a unique rep'n as linear comb.

(pf: analyze known facts about echelon form of $\left[\begin{array}{ccc|c} | & & | & | \\ \vec{v}_1 & \cdots & \vec{v}_k & \vec{v} \\ | & & | & | \end{array}\right]$)

Our assumptions imply if elt in span, then last column doesn't contain pivot, and if $\vec{v}$ has unique sol'n, all columns of reduced form of $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{bmatrix}$ contain a pivot.

~~[illegible]~~ for elements of Span$\{\vec{v}_1, \dots, \vec{v}_k\}$, if one ~~[illegible]~~ has a unique linear combination, all have unique linear combination.

In particular, it suffices to check if $\vec{0}$ can be represented as a unique linear combination. $\underline{0}$ is a nice choice since it is <u>ALWAYS</u> in Span$\{\vec{v}_1, \dots, \vec{v}_k\}$ for any choice of $\vec{v}_i$: Just pick all $c_i = 0$.

So <u>Key question</u>: Is there a linear combination of $\vec{v}_i$'s such that

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \underline{0} \quad \text{and } \underline{\text{not all}} \ c_i = 0 ?$$

If not, we say the set of vectors $\vec{v}_1, \dots, \vec{v}_k$ is linearly independent.

If so, we say the set is linearly dependent.

So in our example from Monday, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ was linearly dependent $\quad (\vec{v}_1 = 2\vec{v}_3 - \vec{v}_2)$

but $\{\vec{v}_2, \vec{v}_3\}$ was linearly independent.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

this is true in general:

<u>Theorem</u>: Given $\{\vec{v}_1, \dots, \vec{v}_k\}$ linearly independent in $\mathbb{R}^n$

then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}\}$ is linearly indep. iff $\vec{v} \notin \text{Span}\{\vec{v}_i\}_{i=1}^{k}$

<u>pf</u>: easy exercise. if $v \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then $\vec{v} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$. Subtract $\vec{v}$ from both sides.

other direction also simple enough.

Example of linear indep / dependence:

Are $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ linearly indep.?

Set up linear equation to give linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \underline{0}$

Row reduce: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \underline{c} = \underline{0} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & \\ 0 & 1 & 1 & ? \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right]$

∞-ly many sol'ns.

More abstract example: If $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent, what about $\{\vec{u}_1 + \vec{u}_2, \vec{u}_2 + \vec{u}_3, \vec{u}_1 + \vec{u}_3\}$?

Is it linearly independent / dependent?

In $\mathbb{R}^n$, how many vectors can be linearly indep? How many are needed to span $\mathbb{R}^n$? Consider collection of vectors

$$S = \{\vec{v}_1, \dots, \vec{v}_k\} \quad \vec{v}_i \in \mathbb{R}^n.$$

<u>Proposition</u>: (a) if $k > n$, then S is linearly dependent.

(b) if $k < n$, then $\mathrm{Span}(S) \subsetneq \mathbb{R}^n$. (proper subset)

<u>pf</u>: just think about echelon form ...

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just write out linear combination we obtain.

Another <u>example</u>: standard basis vectors $\{\vec{e}_1, \dots, \vec{e}_n\}$ are linearly independent in $\mathbb{R}^n$. (and span $\{\vec{e}_1, \dots, \vec{e}_n\} = \mathbb{R}^n$)

Make a general definition: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is called a basis of a subspace $V \subseteq \mathbb{R}^n$ if

(1) $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = V$

(2) $v_1, \dots, v_k$ are linearly independent.

Examples: $\{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$.

$\{\vec{e}_1, \vec{e}_2\}$ is a basis of the plane: $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$ in $\mathbb{R}^n$.

Or sometimes subspace is implicitly defined.

e.g. solutions in $\mathbb{R}^3$ (x_1, x_2, x_3): $x_1 + x_2 + x_3 = 0$.

this is $A \cdot \underline{x} = \underline{0}$ with $A = [1 \; 1 \; 1]$.

(in echelon form. one pivot var, two non-pivot vars., so sol'ns are in bijection with $\mathbb{R}^2$. Find basis for this subspace.)

This will be simpler once we have more results at our disposal. For now, give rough idea – find two vectors not multiples of each other.

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ always subspace of $\mathbb{R}^n$, so it is in 1-1 correspondence with $\mathbb{R}^l$ some $l \in [0, k]$.

Two important theorems:

Theorem 1: Any subspace $V \neq (0)$ has a basis. (Suppose $V \subseteq \mathbb{R}^n$)

pf: Pick $\vec{v}_1 \neq \underline{0}$ in V. Is $\mathrm{Span}(\vec{v}_1) = V$? If yes, done

if not, pick $v_2 \notin \mathrm{Span}(\vec{v}_1)$. Then $\vec{v}_1, \vec{v}_2$ linearly indep.

by earlier result. Repeat. Must terminate eventually since $n+1$

vectors in $\mathbb{R}^n$ are always linearly dependent, so every $\underline{x} \in \mathbb{R}^n$ will

be in $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_n)$ if $\{\vec{v}_1, \dots, \vec{v}_n\}$ indep. (by earlier theorem on $\vec{v} \notin \mathrm{Span}$ iff $\{v_1, \dots, v_k, v\}$ indep.)

Theorem 2: If $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ and $\{\vec{w}_1, \dots, \vec{w}_k\}$ are bases for $V \subseteq \mathbb{R}^n$,

then $\ell = k$.

pf: if $\ell > k$, write $\vec{w}_1, \dots, \vec{w}_\ell$ as linear combs of $\vec{v}_i$'s.

encode this in $k \times \ell$ matrix A.

with
$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \cdot A = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_\ell \\ | & & | \end{bmatrix}$$

$\ell > k$ implies A has column w/o pivot, so $A\underline{x} = \underline{0}$ has inf. many solutions.

multiplying both sides by $\underline{x}$ we see $\vec{w}_i$'s linearly dependent.

(if $k > \ell$, same pf with roles of $\vec{v}_i$'s, $\vec{w}_j$'s reversed)

Definition: Size of basis of V is the dimension of V.