Review for exam -

Statements of Inverse function theorem, Implicit Function Theorem.

Quantitative versions not covered on exam. Applications: compute derivatives of inverse, implicitly defined functions

<u>Also</u>: check whether conditions are satisfied

(Is $Df(\underline{x_0})$ invertible? Is $Df(x_0)$ onto?)

2x2 example of Inverse Function Thm.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y \end{pmatrix} \qquad Df(x_0, y_0) = \begin{pmatrix} 2x_0 & 0 \\ 0 & 1 \end{pmatrix}$$

invertible if $x_0 \neq 0$.

Compute $Df^{-1}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$: $\quad f \circ f^{-1} = Id$

f^{-1} $\overset{\text{exists and}}{\text{diff.}}$ since $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

So $D(f \circ f^{-1})\begin{pmatrix} 4 \\ 1 \end{pmatrix} = Id.$

$\quad \parallel$

By chain rule: $Df(f^{-1}(y)) \cdot Df^{-1}(y)$

So
$$Df^{-1}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \left[Df\left(f^{-1}\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right)\right]^{-1}$$
$$= \left[Df\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]^{-1}$$
$$= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}$$

By hand $f^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^{1/2} \\ y_2 \end{pmatrix}$

with $Df^{-1} = \begin{pmatrix} \frac{1}{2} y_1^{-1/2} & 0 \\ 0 & 1 \end{pmatrix}$

$4^{-1/2} = 1/2$. ✓

For Implicit Function Theorem

$$F(\underline{c}) = \underline{0}$$

If we're given $F: U \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ with $DF(\underline{c})$ onto
(so has n pivots!)

Write vector in $\mathbb{R}^{n+m}$ so that first

n variables are pivot variables. $(\underline{x} \in \mathbb{R}^n)$

last m variables are independent (non-pivot variables) $(\underline{y} \in \mathbb{R}^m)$

(so rearranging columns in $DF(\underline{c})$ corresponding to this)

Example: $DF(\underline{c})$ reduces to

$F: \mathbb{R}^4 \to \mathbb{R}^2$

$$\begin{array}{cccc} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ \uparrow \; x_1 \qquad \uparrow \; x_3 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

pivot vars

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} \begin{array}{l} \} \text{ pivot vars. } \underline{x} \\ \} \text{ non-pivot vars } \underline{y} \end{array}$$

column is $D_1 F(\underline{c})$ $\quad D_3 F(\underline{c})$

in original matrix $DF(\underline{c})$

Consider new function $f: \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

$$f\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} = \begin{bmatrix} F\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} \\ \underline{y} \end{bmatrix}$$

if $\underline{c} = \begin{pmatrix} \underline{x_0} \\ \underline{y_0} \end{pmatrix}$ then $f\begin{pmatrix} \underline{x_0} \\ \underline{y_0} \end{pmatrix} = \begin{bmatrix} F\begin{pmatrix} \underline{x_0} \\ \underline{y_0} \end{pmatrix} = F(\underline{c}) \\ \underline{y_0} \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \end{bmatrix}$

Inverse function theorem says have inverse from nbhd of $\begin{bmatrix} 0 \\ y_0 \end{bmatrix}$. In particular,

for $\begin{bmatrix} 0 \\ y \end{bmatrix}$ in nbhd, $f^{-1}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} \phi(y) \\ y \end{bmatrix}$ $\quad \phi(y)$ desired implicit function.

Corollary: $\left[D\phi(y_0)\right] = -\left[D_{i_1}F(\underline{c}), \dots, D_{i_n}F(\underline{c})\right]^{-1} \cdot \left[D_{i_{n+1}}F(\underline{c}) \cdots D_{i_{n+m}}F(\underline{c})\right]$

pivot vars

non-pivot vars.

$\left(f \circ f^{-1} = \mathrm{Id}. \quad Df\left(f^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right) Df^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathrm{Id}. \right)$ work out block mult.

Example: For circle $\underline{c} = \begin{bmatrix} a \\ b \end{bmatrix}$

with $Df(\underline{c}) = [2a, 2b]$ since

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1 \quad : \mathbb{R}^2 \to \mathbb{R}^1$$

e.g. at $(1,0)$ then x is pivot var., y non-pivot. $Df\begin{pmatrix} 1 \\ 0 \end{pmatrix} = [2, 0]$

so x is implicit function of y there.

$$[Dx(0)] = -[2]^{-1} \cdot [0] = 0.$$

check: $x = \sqrt{1-y^2}$

$$\frac{dx}{dy} = \frac{1}{2}(1-y^2)^{-1/2} \cdot (-2y) = \frac{-y}{\sqrt{1-y^2}}$$

In general $\underline{c} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, then

$$Df(\underline{c}) = [2\cos\theta, 2\sin\theta]$$

if x pivot var, y non-pivot

$$[Dx(\sin\theta)] = -[2\cos\theta]^{-1} \cdot [2\sin\theta]$$

$$= -\frac{1}{2}\frac{2\sin\theta}{\cos\theta} = -\tan\theta$$

$$\frac{-\sin\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}}$$

✓

Example : $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 7 \end{bmatrix} = A$ with $\tilde{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

As linear transformation : $\mathbb{R}^4 \longrightarrow \mathbb{R}^3$

rank : # pivot vars = 2

ker : # of non-pivot vars = 2

Basis for image : columns with pivot vars : $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(if I asked you to prove from def'n these are linearly indep., how would you do it?)

column space = image !

Basis for kernel :

$$x_1 + 2x_2 + 3x_4 = 0$$
$$x_3 + x_4 = 0$$

Related question: Find basis for Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \right.$

Write pivot vars in terms of non-pivot vars.

$$x_1 = -2x_2 - 3x_4$$
$$x_3 = -x_4$$

To find basis, just pick solutions with non-pivot vars = $(1, 0, \dots, 0)$, $(0, 1, 0, \dots 0)$, $\vdots$, $(0, 0, \dots 1)$

guarantees linear independence.

In our example. sol'ns with $(x_2, x_4) = (1, 0)$
$= (0, 1)$

$(x_2, x_4) = (1, 0)$ then $x_1 = -2,\ x_3 = 0 \rightsquigarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $(x_2, x_4) = (0, 1)$ then $x_1 = -3,\ x_3 = 1 \rightsquigarrow \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$