Tangent spaces : Recall tangent hyperplane to a function $f(\underline{x})$ at $\underline{x_0}$,

$f: \mathbb{R}^k \to \mathbb{R}^n$, is $T(\underline{x}) - f(\underline{x_0}) = [Df(\underline{x_0})](\underline{x} - \underline{x_0})$

with $Df(x_0) : \mathbb{R}^k \to \mathbb{R}^n$
linear transformation.

We can make the same definition for a smooth k-manifold. By defn, for any $\underline{z_0} \in M$, locally the graph of C^1 function. For $\underline{z} \in B_\varepsilon(\underline{z_0})$,
$M \subset \mathbb{R}^n$

Write $\underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ y_1 \\ \vdots \\ y_{n-k} \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ f_1(x_1,\dots,x_k) \\ \vdots \\ f_{n-k}(x_1,\dots,x_k) \end{bmatrix} \begin{matrix} \} \underline{x} \\ \} \underline{f}(\underline{x}) \end{matrix}$. Then

Note: reordered variables here to make it pretty.

tangent hyperplane is again $T(\underline{x}) - \underline{f}(x_0) = [Df(\underline{x_0})](\underline{x} - \underline{x_0})$.

Definition : The tangent space to manifold M at point $\underline{z_0} \in M$, denoted $T_{z_0}(M)$, is the graph of the linear transformation $Df(\underline{x_0})$ with f as above.

e.g. $y = f(x)$, then $Df(\underline{x_0}) = f'(\underline{x_0})$
and graph is linear function $y = f'(\underline{x_0}) \cdot x$ (not tangent line. This we get by shifting $(0,0)$ to $(x_0, f(x_0))$)

Nicer from linear algebra point of view. Tangent space is vector space, but it is good to think about it as "anchored" to point $(\underline{x_0}, \underline{f}(\underline{x_0})) \in \mathbb{R}^n$.

You might worry that Df depends on choice of pivot variables, so tangent line is not independent of coordinates.

Even a problem for 1-manifold in $\mathbb{R}^2$: $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y)=0$.

If $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \neq 0$ (at (x_0,y_0)), we have choice of pivot vars. in nbhd of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ f(a) \end{pmatrix} \text{ or } = \begin{pmatrix} g(b) \\ b \end{pmatrix}$$

for all $\begin{pmatrix} a \\ b \end{pmatrix} \in B_\epsilon\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

parametrizing same points so $g(b)=a$, $f(a)=b$

i.e. $g \circ f(x) = x$, so by chain rule $g'(f(a)) \; (= g'(b)) = \frac{1}{f'(a)}$.

so tangent line w.r.t. x as pivot:

$$x - g(b) = g'(b)(y-b).$$

w.r.t. y as pivot:

$$y - f(a) = f'(a)(x-a)$$

same equation since $\frac{1}{f'(a)} = g'(b)$.

Better (and more general) argument:

If M : k-manifold in $\mathbb{R}^n$ $\longleftrightarrow$ F : zero locus with $DF(\underline{c})$ onto for every $\underline{c} \in M$

then $T_c M = \operatorname{Ker}[DF(\underline{c})]$

Kernel is intrisic subspace assoc. to point $\underline{c} \in M$. Doesn't depend on choice of pivot vars.

pf: if F: zero locus, implicit function theorem says we may write implicit function $\phi(\underline{y}) = \underline{x}$

$\underline{x}$: pivot vars.

$\underline{y}$: non-pivot vars.

Then implicit function thm also tells us how to find derivative $\phi(\underline{b})$ if point $\underline{c} = \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix}$ is on locus (i.e. Manifold) with $\underline{a} = \phi(\underline{b})$. It is:

$$[D\phi(\underline{b})] = -\begin{bmatrix} | & & | \\ D_{j_1}F(\underline{c}) & \cdots & D_{j_{n-k}}F(\underline{c}) \\ | & & | \end{bmatrix}^{-1} \cdot \begin{bmatrix} D_{i_1}F(\underline{c}) \cdots D_{i_p}F(\underline{c} \end{bmatrix}$$

pivot vars $\underline{x}$

non-pivot vars $\underline{y}$

Now equation of tangent space is just

$$\underline{x} = [D\phi(b)]\,\underline{y}, \text{ so substituting:}$$

$$\underline{x} = -[\cdots]^{-1}[\cdots]\,\underline{y} \iff$$

$$\begin{bmatrix} | & & | \\ D_{j_1}F(\underline{c}) & \cdots & D_{j_{n-k}}F(\underline{c}) \\ | & & | \end{bmatrix}\underline{x} + \begin{bmatrix} | & & | \\ D_{i_1}F(\underline{c}) & \cdots & D_{i_k}F(\underline{c}) \\ | & & | \end{bmatrix}\underline{y} = \underline{0}$$

$$\text{i.e. } [DF(\underline{c})]\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \underline{0}$$

so sol'ns are $\underline{z} = \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix}$ in ~~[illegible]~~ $\mathrm{Ker}(DF(\underline{c}))$

What if, instead, your manifold is given as a parametrization. Have some $U \subseteq \mathbb{R}^k$ s.t. $\gamma: U \to \mathbb{R}^n$ parametrizes points of k-manifold M in $\mathbb{R}^n$.

Rather not translate to 0-locus $F(\underline{z})=0$. Can be hard to do.

Proposition: $T_{\gamma(u)} M = \text{Im}\,[D\gamma(u)]$.

pf: In nbhd. V of $\gamma(u)$, know $\exists$ F with DF onto at all pts in V, F is C^1

so that points in $V \cap M$ given by $F(\underline{z})=0$.

$\text{Ker}(DF(\underline{z})) = n - \text{rank} = n-(n-k)$ since DF onto for all $\underline{z} \in V \cap M$.

$= k$.

On inverse image $\gamma^{-1}(V) \subset U$ we have $F \circ \gamma = 0$. Chain rule

gives $[D(F\circ\gamma)(u)] = [DF(\gamma(\underline{u}))] \circ [D\gamma(\underline{u})]$ (= 0 by above)

so $\text{Im}(D\gamma(\underline{u}))$ is in $\text{Ker}(DF(\gamma(\underline{u})))$

γ parametrization of M $\Rightarrow$ $\text{Im}(D\gamma(\underline{u}))$ is of dim k

So must have $\text{Im}(D\gamma(\underline{u})) = \text{Ker}(DF(\gamma(\underline{u})))$

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$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_{\gamma(u)} M$

by previous theorem.

built into definition of param.

3.1.18 in book

Book shows either approach valid in example (rare) where we have both a parametrization and implicit function:

$$\gamma : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix} \quad \text{for} \quad \begin{matrix} 0 < u < \infty \\ 0 < v < \infty \end{matrix}$$

with implicit function $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xz - y^2 = 0$

$$[D\gamma(u,v)] = \begin{bmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{bmatrix} \quad \text{so} \quad D\gamma(1,1) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

whose image is $T_{\underbrace{\gamma(1,1)}_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}} M$

Via equation,

must compute

$$\ker \left[DF\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] = \ker [1 \;\; -2 \;\; 1]$$

$$\operatorname{Im} D\gamma(1,1) = \operatorname{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

check they agree, either explicitly or by showing $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ are in kernel.