On Wednesday, anticipating second derivative test for extrema of

$f: U \subseteq \mathbb{R}^n \to \mathbb{R}$, studying quadratic forms (e.g. in $\mathbb{R}^2$: $Q(x,y) = c_1 x^2 + c_2 xy + c_3 y^2$, $c_i \in \mathbb{R}$.)

Theorem: (1) $Q(\underline{x})$ can be put in form:

$$Q(\underline{x}) = (\alpha_1(\underline{x}))^2 + \cdots + (\alpha_k(\underline{x}))^2 - (\alpha_{k+1}(\underline{x}))^2 - \cdots - (\alpha_{k+l}(\underline{x}))^2$$

for linearly indep. linear functions $\alpha_1, \dots, \alpha_{k+l}$ with $k+l \le n$.

(2) Call (k,l) signature of Q. It is independent of choice of α_i's.

pf: part (1) is completing the square.

part (2): Given Q show that we can characterize (k,l) without using coord. change α_i.

New language: A quadratic form Q is positive definite if $Q(\underline{x}) > 0$ $\forall\ \underline{x} \neq \underline{0}$.

(examples: x^2+y^2, $(\alpha_1(\underline{x}))^2 + \cdots + (\alpha_k(\underline{x}))^2$ for any linear α_i)

similarly, Q is negative definite if $Q(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$.

(examples: $-x^2-y^2$, $-(\alpha_{k+1}(\underline{x}))^2 - \cdots - (\alpha_{k+l}(\underline{x}))^2$)

Proposition: Given Q with signature (k,l) for some α_i, k: largest dimension of subspace of $\mathbb{R}^n$ on which Q is positive definite

l: largest on which Q is negative definite.

Corollary: Signature of Q is independent of α_i. (If not, get (k_1,l_1) with one set of α_i, and (k_2,l_2) with another. If $k_1 > k_2$ contradicts proposition. Can't both be largest...)

pf. of proposition is nice use of linear algebra. Just do one piece of it —

show Q can't be positive def. on space of dimension $> k$ if

$$Q(\underline{x}) = \alpha_1(\underline{x})^2 + \cdots + \alpha_k(\underline{x})^2 - \alpha_{k+1}(\underline{x})^2 - \cdots - \alpha_{k+l}(\underline{x})^2.$$

—

If W is subspace of $\mathbb{R}^n$ of dim $> k$, consider linear transformation

$$T : W \longrightarrow \mathbb{R}^k$$

$$\underline{w} \longmapsto \begin{bmatrix} \alpha_1(\underline{w}) \\ \vdots \\ \alpha_k(\underline{w}) \end{bmatrix}$$

It has non-trivial kernel since $\dim(W) > k$.

Pick $\underline{w} \in \mathrm{Ker}(T)$.

$$Q(\underline{w}) = \underbrace{\alpha_1(\underline{w})^2 + \cdots \alpha_k(\underline{w})^2}_{=\underline{0}} - \alpha_{k+1}(\underline{w})^2 - \cdots - \alpha_{k+l}(\underline{w})^2 \leq 0.$$

So shows that Q isn't positive definite on any such W.

Now back to our classification of local extrema.

Prove two theorems 1. Extrema occur at $\underline{a}$ for which $Df(\underline{a}) = \underline{0}$.

2. "Second derivative test" using signatures of quadratic forms.

For 1., book reduces pf. to one-variable case. Just as easy to use definition of derivative:

e.g. Suppose $\underline{a}$ is a local minimum. Then on some $B_\delta(\underline{a})$, δ small enough

for any $\underline{v} \in \mathbb{R}^n$: $f(\underline{a} + t\underline{v}) - f(\underline{a}) \geq 0$ if $|t| < \delta$.

So $\lim_{t \to 0^+} \dfrac{f(\underline{a} + t\underline{v}) - f(\underline{a})}{t} = D_{\underline{v}} f(\underline{a}) \geq 0$ while

since f diff.

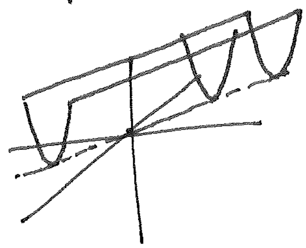
$\lim_{t \to 0^-} \dfrac{f(\underline{a} + t\underline{v}) - f(\underline{a})}{t} = D_{\underline{v}} f(\underline{a}) \leq 0.$

so $= 0$ $\forall\, \underline{v}$.

Return to subtle point in second derivative test:

Given $Q(x,y) = x^2 + 2xy + y^2 = (x+y)^2$. one linear function: $x+y$.

graph this:



Problem: along line $x+y=0$, (corresponding to Q in $P^2_{f,\underline{a}}$) $\sim$ Taylor expansion of f is 0, in quadratic terms.

Can't predict behavior.

This is captured by notion of "positive definite" – $Q(x,y)=0$ only when $x=y=0$.

Not so in above example. In fact Q can only be positive definite if signature is $(n,0)$, not for $(m,0)$ with $m<n$, then matrix of α_i's has non-triv. kernel.

Q positive definite, let

T_Q : linear transformation $\mathbb{R}^n \to \mathbb{R}^n$

$\underline{x} \mapsto (\alpha_1(\underline{x}), \dots, \alpha_n(\underline{x}))$

it is invertible.

Then $Q(\underline{x}) = |T_Q(\underline{x})|^2$.

Now $|\underline{x}| = |T^{-1}T\underline{x}| \leq |T^{-1}||T\underline{x}|$

so $|T(\underline{x})|^2 \geq \dfrac{|\underline{x}|^2}{|T^{-1}|^2}$

Hence $Q(\underline{x}) \geq \underbrace{\dfrac{1}{|T^{-1}|^2}}_{\substack{\text{non-zero constant.} \\ \text{dep. on } Q,\ C_Q.}} |\underline{x}|^2$

Correct formulation: Given f diff. with $P^2_{f,\underline{a}}$ assoc. to quad. form $Q_{f,\underline{a}}$

If signature of $Q_{f,\underline{a}}$ is:
1. $(n,0) \rightsquigarrow \underline{a}$ is local min
2. $(0,n) \rightsquigarrow \underline{a}$ is local max
3. (k,l), $k,l<n \rightsquigarrow \underline{a}$ is neither

($\underline{a}$ must also be critical point of f, of course...)

pf: if $Q_{f,\underline{a}}$ has signature $(n,0)$, show $\underline{a}$ is local min.

i.e. show $f(\underline{a}+\underline{h}) - f(\underline{a})$ is positive for $\underline{h}$ suffic. small (but $\underline{h} \neq \underline{0}$)

But $f(\underline{a}+\underline{h}) = f(\underline{a}) + \underbrace{Df(\underline{a})}_{\text{since } \underline{a} \text{ critical point}}^{\to 0} \cdot h + Q_{f,\underline{a}}(\underline{h}) + \underbrace{R(\underline{h})}_{\text{remainder}}$

$$\Rightarrow \quad \frac{f(\underline{a}+\underline{h}) - f(\underline{a})}{|h|^2} = \underbrace{\underbrace{\frac{Q_{f,a}(h)}{|h|^2}}_{\substack{Q \text{ pos. definite} \\ \Rightarrow \text{ this term} \\ \geq C_Q}} + \underbrace{\frac{R(h)}{|h|^2}}_{\substack{\to 0 \\ \text{as } |\underline{h}| \to 0}}}_{\substack{\text{so for } |\underline{h}| \text{ sufficiently small} \\ \text{this side is positive!}}}$$