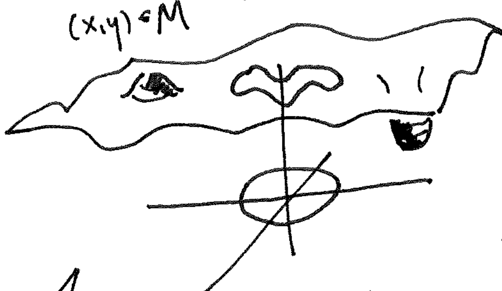
Last week, we explored local extrema for multivariable functions (or on a manifold):

If f is in $C^2(\mathcal{U})$, $\mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$, then find points $\underline{a} \in \mathcal{U}$ s.t. $[Df(\underline{a})] = 1 \times n$ 0-matrix. Examine quadratic form coming from $P^2_{f,\underline{a}}$. Check its signature.

This week: Optimize the function when constrained to submanifold $M \subseteq \mathbb{R}^n$.

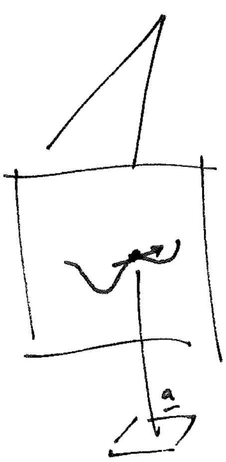
In pictures: $f: \mathbb{R}^2 \to \mathbb{R}$. M: unit circle, 1-dim'l manifold given by $x^2+y^2-1=0$ in $\mathbb{R}^2$

Find $\max\limits_{(x,y)\in M} f(x,y)$.



Problem: Max along M may not be a local max on graph of $f(x,y)$.

But, knowing it is a local extremum on M means that along vectors $\underline{v}$ tangent to M, it is true that $[D_{\underline{v}}(\underline{a})] = [0]$.



Rephrased: "derivative vanishes on tangent space to manifold" at constrained critical point $\underline{a}$

Theorem: $M \subseteq \mathbb{R}^n$ manifold. $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^1 function, $\underline{c} \in M \cap \mathcal{U}$ is a constrained local extremum of $f|_M$.

then $T_{\underline{c}} M \subset \ker[Df(\underline{c})]$

$T_{\underline{c}} M$ = tangent space to manifold $\overset{\text{def}}{=}$ "slope" in tangent line, in that its graph (viewed as linear trans.) is a subspace of $\mathbb{R}^n$

Main result in § 3.2 was elegant characterization of $T_{\underline{c}} M$ as

$\ker [DF(\underline{c})]$ where M is zero locus of F. (Don't be confused here:

In general, graph of $[D\phi(\underline{b})]$ where $\underline{c} = \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix}$ with $\underline{b}$ indep. vars.

ϕ implicitly defined with $\underline{a} = \phi(\underline{b})$.

$F(\underline{z}) = 0$ defines manifold, locally written as explicit function ϕ.)

So equivalently, the theorem says at constrained critical point.

$$\underset{\substack{\uparrow \\ (n-k)\times n \\ \text{if } M \text{ is a } k\text{-manifold}}}{\ker [DF(\underline{c})]} \subseteq \underset{\substack{\uparrow \\ 1\times n \text{ matrix}}}{\ker [Df(\underline{c})]}$$

Do first example in book (3.7.3)

$f(x,y) = xy$ on unit circle with $x,y > 0$. (i.e. in first quadrant)

open quarter of circle is manifold.
as is any open set of manifold

compute two kernels: $F = x^2 + y^2 - 1$

$DF(x_0, y_0) = [2x_0, 2y_0]$ $\qquad \ker [2x_0, 2y_0] = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$2x_0 a + 2y_0 b = 0.$

$Df(x_0, y_0) = [y_0, x_0]$ $\qquad (x_0 a + y_0 b) = 0.$

$\ker [y_0, x_0] = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$a y_0 + b x_0 = 0$

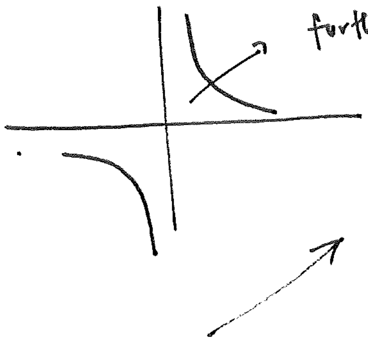
<u>Note</u>: In this case, only possibility is $\ker (DF(\underline{c})) = \ker (Df(\underline{c}))$

these are lines in $\mathbb{R}^2 = \{(a,b)\}$
when are they the same?

We could have predicted this by thinking about when graph $f(x,y) = xy = m$.

$xy = m$ is level curve in xy-plane on which $f(x,y) = m$.

Hyperbola.



further we go from origin, larger function becomes.

When $xy = m$ is tangent to unit circle, this should be max.

(no min on open quarter of circle, since mins occur at points in closure $x=0$ or $y=0$)

When does $xy=m$ intersect $x^2+y^2=1$ in single point?

substitute: $x^2 + \left(\frac{m}{x}\right)^2 = 1$

solve: $x = \frac{\sqrt{2}}{2} = \cos \pi/4$.

then y, on unit circle, is also $\frac{\sqrt{2}}{2} = \sin \pi/4$.

i.e. $x_0 = y_0$ as predicted above.

Criterion seems good start, but seems it may be hard to figure out what $\underline{c}$ to choose to get $\text{Ker}[DF(\underline{c})] \subset \text{Ker}[Df(\underline{c})]$

(this is where Lagrange multipliers come in. Give us a checkable condition for when this occurs)

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pf. of theorem: M is locally representable as $\underbrace{\begin{bmatrix} g(\underline{y}) \\ \underline{y} \end{bmatrix}}_{\tilde{g}:\ \mathbb{R}^k \to \mathbb{R}^n}$ $\underline{y}$ indep. vars in $\mathbb{R}^k$

$g(\underline{y}): \mathbb{R}^k \to \mathbb{R}^{n-k}$

Write $\underline{c} = \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix}$ as usual with $\underline{a} = g(\underline{b})$.

points of M are image$(\tilde{g})$

then if $\underline{c}$ is constrained critical point we have $[D(f\circ\tilde{g})(\underline{b})] = [0]$.

In other words, $\underline{b}$ is honest local extremum for composition $f \circ \tilde{g}$.

On other hand, we compute $D(f \circ \tilde{g})$ by chain rule:

$$= \Big[Df\big(\underbrace{\tilde{g}(\underline{b})}_{\underline{c}}\big) \Big] \Big[D\tilde{g}(\underline{b}) \Big]$$ so if this $= 0$,

$$\text{Ker}\,[\, Df(\underline{c})\,] \supseteq \text{Im}\,[\, D\tilde{g}(b)\,] = \text{graph of } [Dg(b)]$$

$$= T_c M. \quad //$$

Problem yet to be solved: Find $\underline{c} \in \mathcal{U} \cap M$

$\mathcal{U}$: open set w/ f defined

M: smooth k-Manifold in $\mathbb{R}^n$

such that $\text{Ker}\,[\, Df(\underline{c})\,] \supseteq \text{Ker}\,[\, DF(\underline{c})\,]$.

$f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ function we're optimizing so $Df(\underline{c})$ is $1 \times n$

$F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ so $DF(\underline{c})$ is $n-k \times n$.

As linear transformations, when is $\text{Ker}\,[\beta] \supseteq \text{Ker}\,[A]$ $\quad \beta : 1 \times n$, $A : m \times n$. ?

Think of A is m $1 \times n$ matrices $\alpha_1, \dots, \alpha_m$

claim: $\text{ker}\,[\beta] \supseteq \text{Ker}\,[A] \iff \exists\ \lambda_1, \dots, \lambda_m \in \mathbb{R}$ s.t.

$$\beta = \lambda_1 \alpha_1 + \dots + \lambda_m \alpha_m$$

($\Leftarrow$) is immediate

($\Rightarrow$) next time...