Defined convergence of sequences in $\mathbb{R}^n$ and showed $\iff$ convergence of components in $\mathbb{R}$.

Didn't need any "topology" — definition of open/closed sets

Just used open balls.

Discuss properties, then on to limits of functions.

for this return to topological properties — closure.

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functions : $\mathbb{R}^n \to \mathbb{R}^m$ Do examples

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topology — closure Given subset $\mathcal{U} \subseteq \mathbb{R}^n$, define

closure $\bar{\mathcal{U}} \stackrel{\text{def}}{=} \{ \underline{x} \in \mathbb{R}^n \mid B_r(\underline{x}) \cap \mathcal{U} \neq \phi \quad \forall r > 0 \}$

example : $\{ \underline{x} \mid |\underline{x}| < 1 \}$ has closure $\{ \underline{x} \mid |\underline{x}| \leq 1 \}$

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back to definition of limit

Do non-example + example.

first mention criterion about sequences of points approaching.

Given $\epsilon > 0$

Just Pick M so $|\underline{x}_i - \underline{x}_0| < \delta(\epsilon)$

if $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \underline{a}$ then given $\underline{x}_i \to \underline{x}_0$ $\lim_{i \to \infty} f(\underline{x}_i) = \underline{a}$

then $|f(\underline{x}_i) - \underline{a}| < \epsilon$.

We can also combine this result (Prop. 1.5.13 in H–H) with our knowledge of how limits of sequences behave under addition/mult. in one variable (i.e. in $\mathbb{R}$) to prove:

(1) if $\{\underline{a}_m\}$, $\{\underline{b}_m\}$ converges, then so does $\{\underline{a}_m + \underline{b}_m\}$

$$\text{and} \quad \lim_{m\to\infty} \underline{a}_m + \underline{b}_m = \left(\lim_{m\to\infty} \underline{a}_m\right)\left(\lim_{m\to\infty} \underline{b}_m\right)$$

(2) same for $\{\underline{a}_m\}$ and $\{c \cdot \underline{a}_m\}$ c: scalar mult.

(3) same for dot product

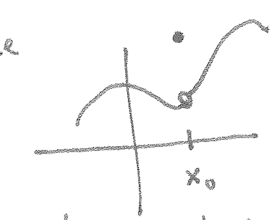
$\boxed{\text{Alternate definition of closed (Prop. 1.5.17): every convergent sequence in } C \subseteq \mathbb{R}^n \text{ converges to a limit in } C.}$

Limits of functions: $X \subset \mathbb{R}^n$, $f: X \longrightarrow \mathbb{R}^m$. $\underline{x}_0 \in \overline{X}$ remember: closure of X

say $\lim_{\underline{x}\to\underline{x}_0} f(\underline{x}) = \underline{a}$ if, for every $\epsilon > 0$, there exists $\delta > 0$

s.t., for all $\underline{x} \in X$, $|\underline{x} - \underline{x}_0| < \delta$ implies $|f(\underline{x}) - \underline{a}| < \epsilon$.

Remarks: The condition is for all $\underline{x} \in X$. So if $\underline{x}_0 \in X$, requires $f(\underline{x}_0) = \underline{a}$ (Other sources put $0 < |\underline{x} - \underline{x}_0| < \delta$ to avoid this, so functions like



have limit at x_0 but ~~[illegible]~~.)

our definition forbids this.

On other hand, if we remove extra point, then $\underline{x}_0 \notin X$, so can have a limit there.

functions on $X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Examples: linear functions (rep'd by $m \times n$ matrix) $\longrightarrow$ hard to draw if m, n not very small

parametric curves $\mathbb{R} \longrightarrow \mathbb{R}^m$

$$t \longmapsto \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

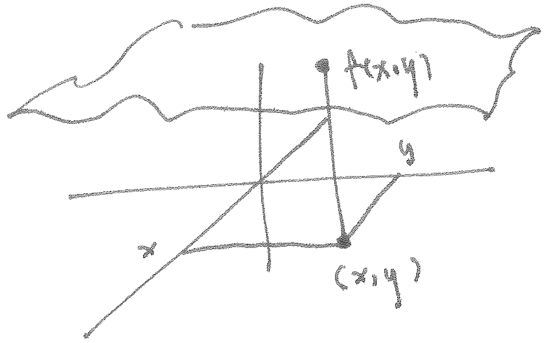
(only drawing image in $\mathbb{R}^m$)

e.g. $t \longmapsto \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

gives unit circle. Compose with stretch $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ to make ellipse, etc.

graph of (smooth) function from $\mathbb{R}^2 \to \mathbb{R}$

might look like wavy curtain above xy-plane.



$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto f(x,y)$$

Examples: (1) $f(x) = |\underline{x}|^2 : \mathbb{R}^n \longrightarrow \mathbb{R}$. Want to show:

$\lim_{x \to \underline{a}} f(x) = |\underline{a}|^2$. We need to show $\big| |\underline{x}|^2 - |\underline{a}|^2 \big| < \varepsilon$

when $|\underline{x} - \underline{a}| < \underbrace{\delta(\varepsilon)}_{\text{a } \delta \text{ depending on } \varepsilon}$.

Work backwards: Have

$$\big| \underline{x} \cdot \underline{x} - \underline{a} \cdot \underline{a} \big| = \big| (\underline{x} - \underline{a}) \cdot (\underline{x} + \underline{a}) \big|$$

$$\underset{\text{Cauchy-Schwarz}}{\leq} |\underline{x} - \underline{a}| \, |\underline{x} + \underline{a}| < \delta(\varepsilon) \cdot |\underline{x} + \underline{a}|$$

$$|\underline{x} + \underline{a}| \leq |\underline{x}| + |\underline{a}| = |\underline{x} - \underline{a}| + |\underline{a}| < \delta(\varepsilon)$$

Given ε, what should we choose for δ?

e.g. $\delta = \min\left(|\underline{a}|, \frac{\varepsilon}{3|\underline{a}|} \right)$.

(unless $\underline{a} = 0$. (then $\delta = 0$) Go back and do proof for this remaining case!)

Pick $\delta(\varepsilon) = \sqrt{\varepsilon}$. according to $\delta(\varepsilon) \cdot (\delta(\varepsilon) + 2|\underline{a}|)$

(2) $f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2}{x^2+y^2}$ Approach along x-axis, y-axis to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to see if limit exists.

(No! get 1, 0 as limits along $\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}$ respectively.)

or see Example 1.5.25 in book ...