Remember, given $\{f_k\} : \mathbb{R}^n \to \mathbb{R}$ integrable with

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\underline{x})| \, |d^n \underline{x}| < \infty \quad , \text{ then } f_k \to f \text{ a.e.}$$

and define

$$\int_{\mathbb{R}^n} f \, |d^n \underline{x}| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\underline{x}) \, |d^n x| .$$

Easier properties

1. Lebesgue integral is linear. $\left(af + bg \overline{\overline{L}} \sum_k af_k + bg_k \right)$

2. If f is L-integrable, g is R-integrable, then fg is L-integrable

$$\left(fg = \sum_k f_k \cdot g \quad , \text{ now } \sum \int |f_k g| \, |d^n x| \le \underbrace{\sup(|g|)}_{\text{finite since } g \text{ bounded}} \cdot \underbrace{\sum \int |f_k|}_{\text{finite since } f \text{ is L-integrable}} \right.$$

3. If f, g are L-integrable with $f \le g$ a.e.

then $$\int_{\mathbb{R}^n} f \le \int_{\mathbb{R}^n} g .$$

(use linearity : need to show $\int_{\mathbb{R}^n} g - f \ge 0$ if $g - f \ge 0$ a.e.

but rest of pf. is a little trickier.)

Harder properties:

1. Various convergence thms. p. 509
2. Fubini's theorem
3. Change of vars. theorem.
4. Differentiation under integral sign.

Fubini's theorem: if know $f: \underbrace{\mathbb{R}^n \times \mathbb{R}^m}_{\mathbb{R}^{n+m}} \to \mathbb{R}$ is L-integrable

then Fubini's thm applies. (don't have to check $\underline{y} \mapsto \int f(\underline{x}, \underline{y})\, |d^n \underline{x}|$ is integ.)

if ~~...~~ don't know, then check:

① f is integrable on some ball at every point $(\underline{x}, \underline{y}) \in \mathbb{R}^{n+m}$

② $\underline{x} \mapsto |f(\underline{x}, \underline{y})|$ is integrable on $\mathbb{R}^n$ for a.e. $\underline{y}$.

③ $\underline{y} \mapsto \int |f(\underline{x}, \underline{y})|\, |d^n \underline{x}|$ integrable on $\mathbb{R}^m$.

then f integrable, and can do Fubini with inner integration in $\underline{x}$.

(symmetric version for inner integral in $\underline{y}$)

Change of Vars formula:

$$\int_V f(\underline{v})\, |d^n \underline{v}| = \int_U (f \circ \Phi)(\underline{u})\, |\det D\Phi(\underline{u})|\, |d^n \underline{u}|$$

for sufficiently nice $\Phi : U \to V$.

Φ: bijective, C^1, with Φ^{-1} C^1 and Φ, Φ^{-1} Lipschitz derivatives.

(much uglier for Riemann integration.)
invertibility of Jacobian at interior points, etc.

(need to know one side is L-integrable initially)

Differentiating under integral sign:

Define $F(t) = \int_{\mathbb{R}^n} f(t, \underline{x}) \, |d^n \underline{x}|$ $f(t, \underline{x}) : \mathbb{R}^{n+1} \to \mathbb{R}$ integrable for any fixed t.

If $\frac{\partial}{\partial t} f(t, \underline{x})$ exists a.e. $\underline{x}$ and

$\exists$ $\delta > 0$ ~~then~~, integrable g s.t. (for all t_1, t_2) if $|t_1 - t_2| < \delta$ then

$$\left| \frac{f(t_1, \underline{x}) - f(t_2, \underline{x})}{t_1 - t_2} \right| \le g(x)$$ then F is diff. in t with

derivative $$\frac{d}{dt} F(t) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} f(t, \underline{x}) \, |d^n \underline{x}|.$$

this gives us chance to use generalized Dominated conv. thm in proof.

(*)

Do simple example.

If compact region, then (*) easily satisfied

if $\frac{\partial}{\partial t} f(t, x)$ is continuous.

Fourier transform: $$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \, e^{i\xi x} \, dx$$

$i = \sqrt{-1}$.

$$e^{i\xi x} = \sum_{k=0}^{\infty} (i\xi x)^k / k! = \cos \xi x + i \sin \xi x.$$

If you prefer just think of writing function $g = g_1 + i g_2$ with g_1, g_2 real

Analyze behavior of Fourier transform under differentiation.

proof of differentiating under integral sign:

$$F(t) := \int_{\mathbb{R}^n} f(t, \underline{x}) \, |d^n x| \quad \text{so}$$

$$\frac{d}{dt} F(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h}$$

$$\left(= \lim_{h \to 0} \frac{F(t + 1/k) - F(t)}{1/k} \right)$$

(or sequence: $h = \frac{1}{k}$, $k \to \infty$)

$$= \lim_{k \to \infty} \int_{\mathbb{R}^n} \underbrace{\frac{f(t + 1/k, \underline{x}) - f(t, \underline{x})}{1/k}} \, |d^n \underline{x}|$$

$$= \int_{\mathbb{R}^n} \frac{\partial}{\partial t} f(t, \underline{x}) \, |d^n \underline{x}|$$

assumption that for $k \gg 0$, this is bounded allows us to apply dominated convergence theorem: $\{f_k\} \to f$ a.e.

and $|f_k(x)| \leq \underbrace{R(\underline{x})}$

then move limit inside integral

bounding function that is L-integrable, not constant.