Compute k-volume of manifold from parametrization $\gamma : \mathcal{U} \subseteq \mathbb{R}^k \longrightarrow M \subseteq \mathbb{R}^n$

by

$$\int_{\mathcal{U}} \sqrt{\det \left(D\gamma(\underline{u})^T D\gamma(\underline{u}) \right)} \; |d^k \underline{u}|$$

(or better from "relaxed parametrization" $\gamma : \mathcal{U} \longrightarrow \mathbb{R}^n$

with $\gamma(\mathcal{U}) \supseteq M$, $\gamma(\mathcal{U} - X) \subseteq M$, X : set of volume 0.

and $\gamma : \mathcal{U} - X \longrightarrow M$ is nice (one-one, $\underbrace{C^1, \text{ locally Lipschitz}}_{\text{satisfied if } C^2.}$ deriv.)

with $[D\gamma(\underline{u})]$ one-one for all $\underline{u} \in \mathcal{U} - X$

$\ker (D\gamma(\underline{u})) = 0$

$D\gamma$: $n \times k$ matrix.

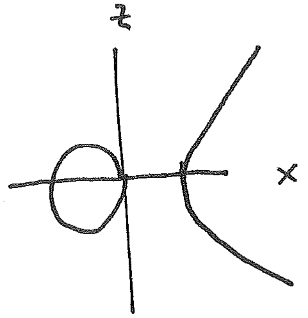
(all columns have pivots))

Note: Don't have easy criterion yet for what k-volume 0 in $\mathbb{R}^n$ means.
Only n-volume 0 in $\mathbb{R}^n$. Back to this after example.

Example: surface of revolution. Curve in xz plane rotated around z-axis.

$z^2 = x^3 - x$ in x-z plane, rotated about z-axis

(portion with $x \leq 0$)



surface has equation $z^2 = \left(\sqrt{x^2+y^2} \right)^3 - \left(\sqrt{x^2+y^2} \right)$

$= \sqrt{x^2+y^2} \cdot (x^2 + y^2 - 1)$

Express x, z coordinates via parametrization

Try: $t \longmapsto \begin{pmatrix} x = t \\ z = \sqrt{t^3 - t} \end{pmatrix}$

Issues? Only captures positive z's

Proposition if $0 \le m < k \le n$, M : an m-manifold in $\mathbb{R}^n$.

then k-volume of ~~[illegible]~~ any compact subset $X \subset M$ is 0.

~~Key lemma: prove that f: $\mathbb{R}^n \to \mathbb{R}$ integrable ...~~

pf: For each $x \in X$, $\exists\ \delta(x) > 0$ s.t. $\underbrace{B_{2\delta(\underline{x})}}(x) \cap M$ is graph of C^1 function $f : \mathbb{R}^m \to \mathbb{R}^{n-m}$.

ball of radius $2\delta(\underline{x})$ centered at $\underline{x}$. or equally good: cube of side $2\delta(\underline{x})$ cent. at $\underline{x}$.

These ~~B~~ $B_{2\delta(\underline{x})}(\underline{x})$ cover X if we take union over $\underline{x} \in X$. But X compact,

so (Heine-Borel prop) $\exists$ finite subcover.

So suffices to prove proposition on $X \cap B_{2\delta(\underline{x})}(\underline{x})$ for ~~one~~ any such box in finite cover

Rescale volume so that $B_{2\delta(\underline{x})}(x) = \underbrace{Q_1}_{\text{unit cube in } \mathbb{R}^m} \times \underbrace{Q_2}_{\text{unit cube in } \mathbb{R}^{n-m}} \subset \mathbb{R}^n$ (won't matter what scaling is if we can prove volume is 0.)

$|Df|$ bounded on Q_1

since f is C^1 so Df continuous on compact set.

Say $|Df| \le L$, some L.

Then # cubes of form $\underbrace{C_1 \times C_2}_{C}$ with $C_1 \subset \overline{Q}_1$ fixed s.t. $C \cap M \neq \emptyset$

is $\left(L\sqrt{m}/2\right)^{n-m}$ (

So to cover $M \cap Q$ takes at most $2^{mN}\left(\frac{L\sqrt{m}}{2}\right)^{n-m}$ cubes with volume $\left(\frac{1}{2^N}\right)^k$

$\to 0$ as $N \to \infty$.

Can use symmetry to get around this. $t \in [-1,0]$

$$\gamma : \begin{pmatrix} t \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} t\cos\theta \\ t\sin\theta \\ \sqrt{t^3-t} \end{pmatrix}$$

C^1 : continuous first partials.

$$\frac{d}{dt}\,(t^3-t)^{1/2} = \frac{1}{2}\,\underbrace{(t^3-t)}_{\text{problem if } t^3-t=0 \text{ occurs at } t=-1,0,1.}{}^{-1/2}\cdot(3t^2-1)$$

$D\gamma$: 3×2 matrix

$$\begin{pmatrix} \cos\theta & -t\sin\theta \\ \sin\theta & t\cos\theta \\ \text{mess in } t & 0 \end{pmatrix}$$

not one-one if $t=0$.

$=0$ if $3t^2-1=0$

$t = \pm\sqrt{1/3}$.

<u>10.11's</u>: ~~measure~~ k-volume 0 in $\mathbb{R}^n$, volume integral is well-defined (independent of parametrization)

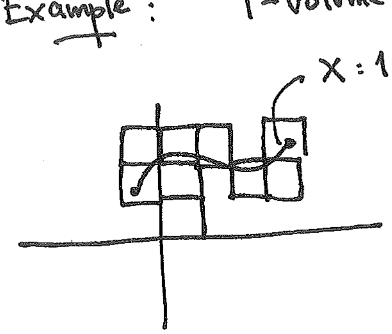
Definition of <u>k-volume</u> 0: If X : bounded subset of $\mathbb{R}^n$, then

X has k-volume 0 if

$$\lim_{N\to\infty} \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n) \\ C\cap X\neq\emptyset}} \left(\frac{1}{2^N}\right)^k = 0$$

<u>Example</u>: 1-volume 0 in $\mathbb{R}^2$

X : 1-manifold



Approximated by widths of all cubes intersecting X

~~Proposition~~ If X arbitrary subset of $\mathbb{R}^n$, say it has k-volume 0 if, $\forall R$, the set (bounded) $X\cap B_R(0)$ has k-volume 0.