Exam Wednesday.

Covers Section 4.11 - Lebesgue integration + Ch. 5 (Sects. 1-4) on integration on manifolds.

all exams are cumulative, but really only in Lebesgue int.
that we require some facts from Riemann integration.

Major topics in Lebesgue integration:

- definition of Lebesgue integral
- proving that functions are L-integrable
- easy properties ("Some elementary properties of Lebesgue integral")
- career-making hard theorems
- differentiation under integral sign (bringing limits inside)

Definition: Write $f \underset{L}{=} \sum_{k=1}^{\infty} f_k$ $\quad f_k$: R-integrable (In particular bounded with bounded supp.)
ALSO: cont. except on set of meas. 0.

(Note need $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(x)| \, |d^n x| < \infty$)

ensures sum is convergent a.e.

Set $$\int_{\mathbb{R}^n} f(x) \, |d^n x| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(x) \, |d^n x|.$$

Hard part: Find these functions f_k, R-integrable.

Prototypical examples : (0) R - integrable

(1) unbounded support (but decays fast enough)

e.g. $\mathbb{R}^2 - B_1(0)$ with $|x|^p$ for various p.

find f_k supported on larger and larger domains as function of k

(2) unbounded values

e.g. $|x|^p$ with unit ball and $p < 0$.

find thinner and thinner regions to sample f_k, depending on k, as we approach undefined value(s)

non-examples : oscillating functions for which $|f|$ not integrable.

e.g. $\sin x / x$

How do we show that $\int_{\mathbb{R}} 1/x \, dx$ is not Lebesgue integrable ? Can't take improper integral of it, so we're suspicious, but don't know how to connect the improper & Lebesgue ints.

Use fancier property.

e.g. $\int_0^1 \frac{1}{x} dx$ send $x \mapsto u^2$

(nice map on $(0,1) \mapsto (0,1)$)

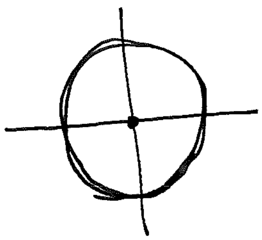
$$= \int_0^1 \frac{1}{u^2} |2u| |du| = \int_0^1 \frac{2}{u} du = 2 \cdot \int_0^1 \frac{1}{x} dx$$

Example : function has a singularity (place where blows up) at origin.

Integral is over compact region. e.g. unit ball.

In $\mathbb{R}^2$

then might try chopping up unit disk into annuli.



Q. Does width of annuli matter?

If width $\left[\frac{1}{k+1}, \frac{1}{k} \right]$ $k \geq 1$,

$$\text{area} = \pi \left(\frac{1}{k}\right)^2 - \pi \left(\frac{1}{k+1}\right)^2$$

which functions can we handle?

suppose f increasing toward 0, so

takes max at $1/k+1$.

If we set $f_k = f$ on interval $\left[\frac{1}{k+1}, \frac{1}{k}\right]$ then it is R-integrable if f is C^1 elsewhere.

Now have to examine whether

$$\sum_{k=1}^{\infty} \int |f_k| \, |d^n x| < \infty \, ?$$

Estimate integrals using max or actually try to compute using Fubini.

If we use the max, we get

$$\int |f_k| \, |d^n \underline{x}| \leq f\left(\frac{1}{k+1}\right) \cdot \left[\pi \left(\underbrace{\frac{1}{k^2} - \frac{1}{(k+1)^2}} \right) \right]$$

$$\frac{(k+1)^2 - k^2}{k^2 (k+1)^2} = \frac{2k+1}{k^2 (k+1)^2} \sim \frac{1}{k^3}$$

If we use $\left[2^{-(k+1)}, 2^{-k} \right]$

Get $\pi \left((2^{-k})^2 - (2^{-(k+1)})^2 \right) \leq \pi \, 2^{-2k}$

More accurate if we can integrate f_k over annulus, e.g. using polar coords

Differentiation under integral sign.

<u>Example</u>: $\displaystyle\int_0^\infty \cos(tx)\, e^{-x^2/2}\, dx = ?$

First justify that differentiating under integral sign is possible.

Integral must exist for each fixed t, + some version of dominated convergence theorem

$|s-t| < \epsilon \Rightarrow$

$$\left| \frac{f(s,x) - f(t,x)}{s-t} \right| \le g(x).$$

g: Lebesgue integrable.

Then differentiating with respect to t,

call integral above $F(t)$:

$$\frac{d}{dt} F(t) = \int_0^\infty -x \overset{\sin}{\cancel{\cos}}(tx)\, e^{-x^2/2}\, dx$$

now do integration by parts:

$u = \sin tx \qquad dv = -x \cdot e^{-x^2/2}$

$du = t \cos tx \qquad v = e^{-x^2/2}$

$$\frac{d}{dt} F(t) = \underbrace{\sin tx\, e^{-x^2/2} \Big|_0^\infty}_{0} - t\, F(t)$$

So $F'(t) = -t F(t) \Rightarrow F(t) = C \cdot e^{-t^2/2}$

Solve for C by setting $t = 0$.

Then

$$C = \int_0^\infty e^{-x^2/2}\, dx = \sqrt{\frac{\pi}{2}}.$$