On Friday, introduced orientation - choice of sign on manifold determined by basis vectors in its tangent space at a point.

Wanted integrands that behaved according to this orientation (changed signs when we change order of basis vectors)

introduce $k$-forms on $\mathbb{R}^n$ (motivated by determinant)

- multilinear
- antisymmetric

as function $\phi : \underbrace{(\mathbb{R}^n)^k}_{\substack{k \text{ vectors} \\ \text{in } \mathbb{R}^n}} \longrightarrow \mathbb{R}$.

Space of all $k$-forms on $\mathbb{R}^n$ is a vector space. Ended Friday claiming that this vector space has dimension $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ with

basis $\{ dx_{i_1} \wedge \cdots \wedge dx_{i_k} \}_{i_1 < i_2 < \cdots < i_k \leq n}$.

Remember $dx_{i_1} \wedge \cdots dx_{i_k} \, (v_1, \ldots, v_k) =$ det of matrix made by selecting $i_1, \ldots, i_k$ rows from

$$\underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}}_{} \Big\} n$$

Ex: if $v_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

then $dx_1 \wedge dx_3 \, (v_1, v_2)$

$$= \det \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix} = 2.$$

(these are all $k$-forms on $\mathbb{R}^n$ since det is $k$-form on $\mathbb{R}^k$ )

What does $dx_1 \wedge dx_3 \,(v_1, v_2)$ mean geometrically?

Start with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 4 & 2 \end{pmatrix}$, then project into x-z plane (or $x_1$-$x_3$ plane)

this is what selecting first/third rows does.

then we take det of resulting $k \times k$ matrix.

$\downarrow$

volume of $k$-parallelogram spanned by projections (or more precisely, signed volume depending on ordering of basis vectors. $dx_1 \wedge dx_3 (v_1, v_2) = 2$ and $dx_3 \wedge dx_1 (v_1, v_2) = -2$. )

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<u>Theorem</u>: (elementary $k$-forms are a basis)

Every $k$-form can be written uniquely as

$$\phi = \sum_{1 \le i_1 < \cdots < i_k \le n} c_{i_1, \dots, i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{with } c_{i_1, \dots, i_k} = \phi(e_{i_1}, \dots, e_{i_k})$$

standard basis elts.

book does nice simple example: dot product with fixed vector $w$.

$\phi_{\underline{w}}(\underline{v}) := \underline{v} \cdot \underline{w}$. so if $\underline{w} = (w_1, \dots, w_n)$, $\underline{v} = (v_1, \dots, v_n)$

$\phi_{\underline{w}}(\underline{v}) = w_1 v_1 + \cdots + w_n v_n$. (linear in $v_i$, nothing to swap with so trivially antisymm.)

clear that $= w_1 \, dx_1 + \cdots + w_n \, dx_n$. Indeed $c_1 = w_1 = \phi(e_1)$. ✓

so far $dx_1 \wedge dx_3$ was just notation for a 2-form. Now want to explain that there is a product operation "$\wedge$" for multiplying a $k$-form and an $l$-form. (so $dx_1 \wedge dx_3$ is the "wedge product" of 1-forms $dx_1$ and $dx_3$.)

Definition is messy, properties are more useful.

<u>Definition</u>: Given $k$-form $\phi_1$ and $l$-form $\phi_2$ form $\phi_1 \wedge \phi_2$, a $k+l$-form defined by

$$\phi_1 \wedge \phi_2 (v_1, \dots, v_{k+l}) = \sum_{\substack{\text{permutations} \\ \sigma}} (-1)^* \phi_1(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \phi_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

sum ranges over "shuffles" $\sigma$ s.t.

$\sigma(1) < \dots < \sigma(k)$ <u>AND</u> $\sigma(k+1) < \dots < \sigma(k+l)$.

Sign $(-1)^* = \pm 1$ is called "sign of the permutation $\sigma$" $\leftarrow$ whether $\sigma$ is composed of odd/even number of swaps of pairs of elts.

<u>Example</u> show $dx_1 \wedge dx_3$ is really given by $2\times2$ det.

$$dx_1 \wedge dx_3 (v_1, v_2) \underset{\substack{\text{prod} \\ \text{def}}}{\overset{\text{wedge}}{=}} \sum_{\substack{\sigma \\ \text{permuting} \\ 1 \text{ and } 2}} (\cdots)$$

$\leftarrow$ only two permutations of 2 elts.

$(v_1, v_2) \mapsto (v_1, v_2)$ or $(v_1, v_2) \mapsto (v_2, v_1)$ both ok since $k = l = 1$.

$$= dx_1(v_1)\, dx_3(v_2) - dx_1(v_2)\, dx_3(v_1)$$

this agrees with $2\times2$ determinant.

<u>Properties</u>:

distributive: $\phi \wedge (\theta_1 + \theta_2) = \phi \wedge \theta_1 + \phi \wedge \theta_2$

associative: $\theta_1 \wedge (\theta_2 \wedge \theta_3) = (\theta_1 \wedge \theta_2) \wedge \theta_3$.

skew-commutativity: $\phi$ : $k$-form, $\theta$ : $l$-form then

$$\phi \wedge \theta = (-1)^{k \cdot l} \, \theta \wedge \phi .$$

Finally, objects to be integrated are not $k$-forms so to speak. They are assignments of $k$-forms to points in open set $U \subseteq \mathbb{R}^n$

$$\varphi : U \longrightarrow \underbrace{A_c^k(\mathbb{R}^n)}_{k\text{-forms on } \mathbb{R}^n} .$$

Think of it as follows.
map $\varphi$ from set to collection of functions on $k$-vectors.

given $u \in U$, $k$-vectors in $\mathbb{R}^n$ give $k$-parallelogram based at $u$.

$\varphi$ assigns to this $k$-parallelogram a number.

Really $\varphi(u) \in$ $k$-form. so $\varphi(u)(v_1, \dots, v_k)$ is a number.

Think of it (from volume point of view)

as $\varphi(P_{\underline{u}}(v_1, \dots, v_k))$

function of $k$-parallelograms based at $u$.

Example: On $\mathbb{R}^3$, consider

$$\varphi : \mathbb{R}^3 \longrightarrow dx_1 \wedge dx_3 \qquad \text{(same at all points)}$$

or $$\varphi : \mathbb{R}^3 \longrightarrow \underbrace{e^{x_1} \cos x_2}_{f(x_1, x_2, d_3) \,\cdot\, k\text{-form.}} \, dx_1 \wedge dx_3$$

Punchline: Given a parametrization of a manifold $\gamma : U \rightarrow V$

$$\int_V \varphi = \int_U \varphi\left( P_{\gamma(u)} ( D_1\gamma(u), \dots, D_k\gamma(u) ) \right) \, |d^k \underline{u}| .$$