Definition (Orientation of Vector Space)

V: finite dimensional v.s. / $\mathbb{R}$. $\mathcal{B}_V$: set of bases of V.

then "orientation" of V is a map

$$\underbrace{\Omega}_{\substack{\text{Omega} \\ \text{orientation}}} : \mathcal{B}_V \longrightarrow \{\pm 1\}$$

assignment in which each basis gets positive orientation (+1) or neg. orientation (-1)

with property that assignment respects change of basis:

Given ~~any~~ two bases $\{v\}$, $\{v'\}$, then have change of basis matrix $P_{\{v'\} \to \{v\}}$.

Want $\Omega(\{v'\}) = \operatorname{sgn}(\det(P_{\{v'\} \to \{v\}}))\, \Omega(\{v\})$ (*)

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One way to choose Ω, pick ~~orientation~~ +1 for given basis $\{v\}$.

then all other assignments are determined by (*).

e.g. assign +1 to standard basis. Call result the "standard orientation"

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To define orientation of a manifold, at each point, have to assign orientation to the ~~tangents~~ tangent space.

Of course, want this choice of orientation to be consistent — i.e. vary continuously.

Set up: $\mathcal{B}(M) = \{ (\underline{x}, \underline{v}_1, \ldots, \underline{v}_k) \in (\mathbb{R}^n)^{k+1} \mid \underline{x} \in M \subseteq \mathbb{R}^n$, $\underline{v}_1, \ldots, \underline{v}_k$ give basis of tangent space at $\underline{x}$. $\}$

then define orientation as a continuous map

from $\mathcal{B}(M) \to \{\pm 1\}$,

with $\mathcal{B}_{x_0}(M) = \{ \underbrace{(\underline{x}, v_1, \ldots v_k)}_{\in \mathcal{B}(M)} \mid \underline{x} = x_0 \}$

an orientation on the tangent space $T_{x_0}(M)$ for each $x_0 \in M$.

What does this mean? What is continuous function in this context?

For topological spaces (spaces with declared collection of open sets), then continuous means ~~the~~ the inverse image of every open set is open.

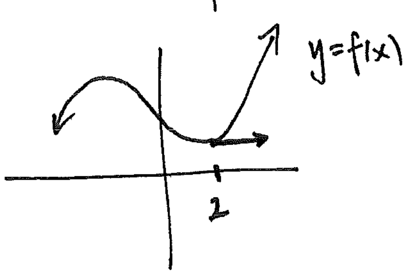
$f: X \to Y$ with $U \subseteq Y$ open, then want $f^{-1}(U) \subseteq X$ open.

(think about ϵ-δ definition for $\mathbb{R}$-valued functions, statement about inverse image of ϵ ball being inside δ-ball.)

On $\{\pm 1\}$, only topology assigns open set to each point $+1, -1$.

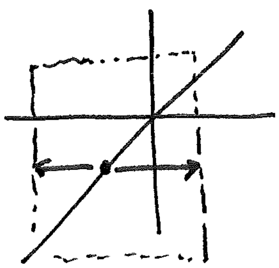
Example: 1-manifold in $\mathbb{R}^2$. $\mathcal{B}(M) \subseteq (\mathbb{R}^2)^2$ ... Hard to draw.

Can draw partial picture of curve $y = f(x)$ in $\mathbb{R}^2$. Draw x coordinate and tangent vector at $(x, f(x))$ in $\mathbb{R}^2$



(this is only in $\mathbb{R}^3$, have some hope ...)

Let (t_1, t_2) be coords of tangent vector.



at $x=2$, suppose tangent line horizontal.

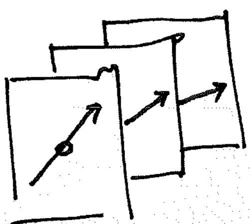
then set of possible tangent vectors is $(t_1, 0)$ with $t_1 \neq 0$.

the point $(0,0)$ is omitted as it is not a basis for tangent space.

Suppose $(1,0)$ gets assigned $+1$. Then $(t_1, 0) \longmapsto \begin{cases} +1 & \text{if } t_1 > 0 \\ -1 & \text{if } t_1 < 0 \end{cases}$

and this defines orientation at $(2, f(2))$ on manifold.

As $x \to \infty$, get lines all with hole at $(0,0)$ in plane. (t_1, t_2)



Often just want to know M has orientation and find one explicitly.

Try: Pick a k-form ϕ and set orientation to be $\text{sign}(\phi)$.

works if $\phi(\underline{x})(v_1, \ldots, v_k) \neq 0 \quad \forall\, x \in M$, bases $v_1, \ldots, v_k$ of $T_x(M)$.

For general curve C in $\mathbb{R}^n$, try to find non-vanishing tangent vector $\underline{t}$ varying continuously in $\underline{x}$, define

$$\Omega^{\underline{t}}_{\underline{x}}(\underline{v}) = \text{sgn}\left(\underline{t}(\underline{x}) \cdot \underline{v}\right)$$

check this: Any two bases $\underline{v}_1, \underline{v}_2$ at $\underline{x}$ for $T_{\underline{x}}(C)$ differ by non-zero constant $\underline{c}$.

Change of basis is 1×1 matrix $[c]$ from one to other.

$$\text{sgn}(\underline{t}(\underline{x}) \cdot c v_1) = \text{sgn}(c) \cdot \text{sgn}(\underline{t}(\underline{x}) \cdot v_1)$$

so our proposed function respects change of basis at each pt.

$\underline{t}(\underline{x})$ chosen continuous, non-vanishing so $\underline{t}(\underline{x}) \cdot \underline{v}$ also continuous non-vanishing for bases $\underline{v}$ $\subseteq (\mathbb{R}^n)^2$

Finally, sgn is continuous on $\mathbb{R} - \{0\}$.

so composition is continuous.

Construction for surfaces: $S \subset \mathbb{R}^3$ smooth 2-manifold.

$\vec{n} : S \longrightarrow \mathbb{R}^3$ vector field varying continuously with respect to $\underline{x} \in S$ and such that $\vec{n}(\underline{x})$ doesn't lie in the tangent space $T_{\underline{x}}(S)$ for any $\underline{x}$ (in particular, is not $\underline{0}$)

then define an orientation

$$\Omega^{\vec{n}}(\underline{v}_1, \underline{v}_2) = \operatorname{sgn}\left(\det\begin{pmatrix} | & | & | \\ \vec{n}(\underline{x}) & \underline{v}_1 & \underline{v}_2 \\ | & | & | \end{pmatrix}\right)$$