Last time, outlined 3 part plan:

1. try to find an orientation
2. try to find a parametrization (relaxed)
3. determine if parametrization is orientation preserving.

Big theorem (later today): If we define

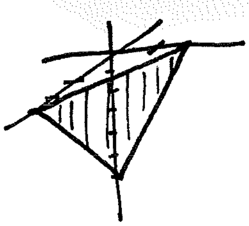
$$\int_M \varphi \overset{\text{def}}{=} \int_{\mathcal{U}} \varphi(\gamma(u))(D\gamma(u))\, |d^k u|$$

for any relaxed $\gamma : \mathcal{U} \to M$ (then this is well-defined definition)

(Better to write out first as equality of two integrals w.r.t. relaxed param.)

Suppose true for now and do some examples.

$M =$ plane in $\mathbb{R}^3$. e.g. $2x + 3y - z = 6$ with $x, y \geq 0$, $z \leq 0$.



Find orientation — Pick nowhere vanishing vector at each point not in tangent plane.

Obvious choice: constant normal vector $(2, 3, -1)$.

$$\Omega : (\underline{x}, \vec{v}_1, \vec{v}_2) \longmapsto \operatorname{sgn}\left(\det\begin{pmatrix} 2 & | & | \\ 3 & \vec{v}_1 & \vec{v}_2 \\ -1 & | & | \end{pmatrix}\right)$$

Find parametrization: $z = 2x + 3y - 6$ so

$$\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ 2x+3y-6 \end{pmatrix}.$$

$$D\gamma(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$$ for all x, y.

$$\Omega : (\underline{x}, \vec{v}_1, \vec{v}_2) = \operatorname{sgn}\left(\det\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}\right)$$

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix} = 2\cdot(-2) - 1\cdot 6 = -10. \quad \text{get } -1$$

at all points!

orientation reversing! Could change vector to $(-2,-3,1)$ defining orientation.

Could change parametrization.

compose with map like $A : \mathbb{R}^k \to \mathbb{R}^k$
$\underline{x} \mapsto (-x_1, x_2, \dots, x_k)$

$$\det A = \left|\begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}\right| = -1.$$

map $A^{-1}(U) \to M$ instead of $U \to M$.

or just take final integral to be negative of orientation-reversing one.

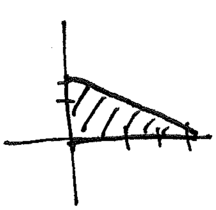
Now can integrate any two-form field. $\varphi = x^2\, dy \wedge dz$.

$$\int_P \varphi = \iint_{\substack{\text{Triangle} \\ \text{in } xy\text{-plane}}} x^2\, dy \wedge dz.$$

if we used $\varphi' = z\, dx \wedge dz$
then substitute
$z = 2x + 3y - 6$.

$$= \iint_{\text{Triangle}} \underbrace{x^2\, dy \wedge dz \begin{pmatrix} x \\ y \\ 2x+3y-6 \end{pmatrix} (D\gamma(x,y))}_{x^2 \cdot \det \text{ of submatrix}} \, |d(x,y)|$$

Triangle:



$$\begin{pmatrix} 1 & 0 \\ \boxed{\begin{matrix} 0 & 1 \\ 2 & 3 \end{matrix}} \end{pmatrix} = -2.$$

$$= \iint -2x^2 \cdot |d(x,y)|$$

No harder for arbitrary graph $z = f(x,y)$. Remember how to define normal vector.