Defined $\partial_M X$ for X : compact subset of k-manifold M.

and smooth boundary $\partial_M^S X$ (determined by add'l smooth function $g : \mathcal{U} \subseteq \mathbb{R}^k \to \mathbb{R}$ for each point $\underline{x} \in \partial_M X$)

So sets X we wish to consider have : "piece with boundary"

$$\mathrm{vol}_{k-1}(\partial_M^S X) < \infty$$

$$\mathrm{vol}_{k-1}(\partial_M^{NS} X) = 0.$$

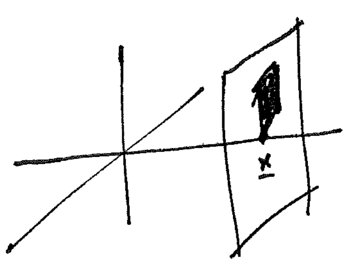
$\partial_M^{NS} X$: non-smooth boundary pts. $= \partial_M X \setminus \partial_M^S X$.

FACTS : ① $\mathrm{vol}_k(X) < \infty$ if X : piece with boundary.

② If X is piece with boundary, then $g(X)$ is piece with boundary if $g : A\underline{x} + \underline{c}$ is linear map with A : invertible $n \times n$ matrix.

Running example in section : k parallelogram anchored at $\underline{x} \in \mathbb{R}^n$, P, spanned by $v_1, \dots, v_k$. Think of it as sitting in k-hyperplane spanned by $v_1, \dots, v_k$ anchored at $\underline{x}$.

this is our M.



To find $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ defining M, just pick A with $\ker(A) = \langle v_1, \dots, v_k \rangle$

define $F(\underline{y}) = A(\underline{y}) - A(\underline{x})$

fixed rooted point of k-parallelogram.

Boundary of P are $(k-1)$-parallelograms spanned by $k-1$ of the k vectors $v_1, \dots, v_k$.

Our function g, locally defining boundary, is the linear function

$$\alpha_i : \mathbb{R}^n \to \mathbb{R} \quad \text{such that} \quad \ker \begin{bmatrix} A \\ \alpha_i \end{bmatrix} = \text{span}\,(k-1 \text{ vectors spanning boundary } k-1 \text{ parallelogram})$$

then additional inequality is

$$\alpha_i(\underline{x}) \le \alpha_i(\underline{y}) \le \alpha_i(\underline{x}+\underline{y})$$

So $g(y) := \alpha_i(y) - \alpha_i(\underline{x})$ or $\alpha_i(y) - \alpha_i(\underline{x}+\underline{y})$.

non-smooth part is intersection of two $k-1$ planes, a $k-2$ plane which has $k-1$-volume equal to 0.

How do we orient boundary? (Need this to do oriented integration on $\partial_M X$ required in Stokes' theorem)

Answer: Not so bad. Just use orientation on any open set $\mathcal{U} \subseteq M$ that contains our compact set X.

(so even manifolds w/o global orientation are ok as long as we find orientation on $\mathcal{U} \supseteq X$).

Idea: Have orientation on $X \subset M$ k-manifold with $T_{\underline{x}} M$ having basis spanned by k vectors in $\mathbb{R}^n$.

$T_{\underline{x}}\, \partial_M^s X$ will have $k-1$ vectors. Complete this to a basis of $T_{\underline{x}} M$ using one more vector.

$$(v_1, \dots, v_{k-1}) \longmapsto \Omega^M_{\underline{x}}(v_{\text{special}}, v_1, \dots, v_{k-1}).$$

What is v_{special}? Use our extra condition $g(x) = 0$ to pick consistent choice of v_{special}.

if, given $\underline{v} \in T_{\underline{x}} M \setminus T_{\underline{x}} \partial_M^s X$, we have

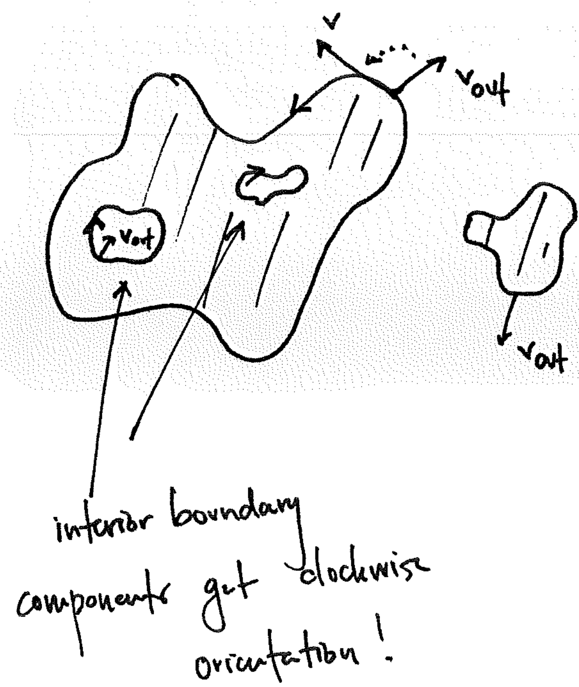
$[Dg(\underline{x})] \cdot \underline{v} > 0$, this means $\underline{v}$ points into domain X

$[Dg(\underline{x})] \cdot \underline{v} < 0$, then $\underline{v}$ points outward from domain X.

choose $\underline{v}_{out}$ to be $v_{special}$ — that is, for each x, pick v_{out} in $T_{\underline{x}} M$.

this is our orientation.

Example: $M = \mathbb{R}^2 \supseteq X$: compact



$$\Omega_{\underline{x}}^{\partial X}(\underline{v}) := \text{sgn} \det(v_{out}, \underline{v})$$

with $\underline{v} \in T_{\underline{x}} \partial_M^s X$

v_{out} first means move counterclockwise from v_{out} to v in pos. orientation.

Example 2: Surface $M \subseteq \mathbb{R}^3$ with orientation given by normal vector $n(\underline{x})$.

with $\Omega_{\underline{x}}^M(\underline{v}_1, \underline{v}_2) = \text{sgn} \det[n(\underline{x}), \underline{v}_1, \underline{v}_2]$

then set $v_1 = v_{out}$ get $\Omega_{\underline{x}}^{\partial X}(\underline{v}) = \text{sgn} \det[n(\underline{x}), v_{out}, \underline{v}]$

Can do 3-manifolds as open chunks of $\mathbb{R}^3$ bounded by surfaces as well.

Or 1-manifolds whose boundary is points, to recover FTC.