Stokes' theorem: $\displaystyle\int_{\partial_M X} \varphi = \int_X d\varphi$. Need to understand $d\varphi$ – "exterior derivative" of $\varphi$.

$X \subseteq M$ "good"

$\varphi$: $(k-1)$ form (field) on $X$

Should generalize FTC: In our language, $X = [a,b] \subseteq \mathbb{R}$.

with $\partial_M X = \{a,b\}$ (orientation on $b$ is $+$, on $a$ is $-$)

and $\varphi$ is 0-form, aka function, call it $F$.

then $\displaystyle\int_{\partial_M X} F = F(b) - F(a)$. $\displaystyle\int_{[a,b]} dF$ indicates $dF = \frac{d}{dx} F$

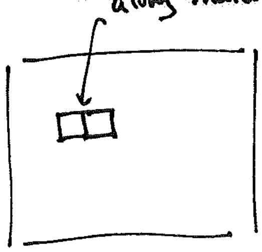
(exterior deriv. on $F$ is derivative)

$= \displaystyle\lim_{h \to 0} \frac{1}{h}\left(F(x+h) - F(x)\right)$

Also want Stokes' theorem to be true,

So how to define $d\varphi$ so that it holds?

along shared boundary have opposite orientation.



define $\displaystyle d\varphi(\underline{x})(\underline{v}_1, \dots, \underline{v}_k) = \lim_{h \to 0} \frac{1}{h^k} \int_{\partial P_{\underline{x}}(h\underline{v}_1, \dots, h\underline{v}_k)} \varphi$

boundary of a $k$-//-ogram is an almost everywhere $(k-1)$-manifold.

Concoct this in a way makes Stokes' theorem believable.

Trouble is: can we compute $d\varphi$ for $\varphi$ a $k$-form with $k > 0$.

Amazingly, YES!

Big theorem on computing $d\varphi$, $\varphi$ a $k$-form.

① the limit exists if $\varphi$ is nice:

$$\varphi = \sum_{i_1,\dots,i_k} \underbrace{a_{i_1,\dots,i_k}(\underline{x})}_{\substack{C^2\text{- functions} \\ \text{on nbhd } U}} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

② linearity: $\varphi, \psi$ $k$-forms, $a, b$ constants in $\mathbb{R}$,

then $d(a\varphi + b\psi) = a\,d\varphi + b\,d\psi$.

③ constants :

$$\varphi = \sum_{i_1,\dots,i_k} \underbrace{c_{i_1,\dots,i_k}}_{\text{constant}} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

"constant form" since coeffs. don't depend on

then $d\varphi = 0$

(special case of ⑤ so skippable )

④ $\underline{df}$, $f$ : function (aka 0-form) :

$$df = [Df] = \sum_{i=1}^{n} (D_i f)\, dx_i$$

⑤ Given $f$, $d(f\, dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

---

All you need :

$$d(e^x \cdot y \, dx \wedge dz + 2x \, dy \wedge dz)$$

$$\overset{②}{=} d(e^x y \, dx \wedge dz) + d(2x \, dy \wedge dz)$$

$$\overset{⑤}{=} \underbrace{d(e^x y)}_{= \dots} \wedge dx \wedge dz + \underbrace{d(2x)}_{2\,dx} \wedge dy \wedge dz$$

$$d(e^x y) = \underbrace{D_x(e^x y)}_{e^x y\,dx} dx + \underbrace{D_y(e^x y)}_{e^x dy} dy + \underbrace{D_z(e^x y)}_{=0} dz$$

$$e^x y\,dx + e^x dy$$

so $d(e^x y) \wedge dx \wedge dz = (e^x y\,dx + e^x dy) \wedge dx \wedge dz$

$$= e^x y\, dx \wedge \underbrace{dx}_{=0} \wedge dz + e^x dy \wedge \underbrace{dx}_{} \wedge dz$$

$=0$ (earlier property of wedge product)

$-e^x\, dx \wedge dy \wedge dz$.

$-e^x \det[\ ]$

---

Two useful corollaries (both of which follow from big theorem by computation)

Thm 6.7.7 in H-H: $\varphi$ nice, then $d(d\varphi) = 0$.

Thm. 6.7.8 in H-H: $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$.

($\varphi, \psi$ nice, $\varphi$ a $k$-form, $\psi$ an $l$-form)

---

HW: 6.7.2, 5, 6, 8, 10, 11      6.8: 3, 6, 7, 10

0-form field $f$ so $df$ is 1-form.

$$df(x)(v) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{1}{h} \int_{\partial P_x(h\underline{v})} f = \lim_{h\to 0} \frac{f(\underline{x}+h\underline{v}) - f(\underline{x})}{h}$$

$$= [Df(x)]\, v$$

---

i.e. $= (D_1 f)\, v_1 + \cdots + (D_n f)\, v_n$

<u>Harder</u>:

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_n}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}.$$

Translate form to origin. to find value at any particular pt.

Use Taylor expansion for $f$. $\quad f = \underbrace{f(\underline{0}) + D_1 f(\underline{0})\, x_1 + \cdots + D_n f(\underline{0})\, x_n}_{[Df(\underline{0})]\, \underline{x}}$

show that linear term is what contributes to limit.

$+$ remainder. $\leq C \cdot |\underline{x}|^2$ for some $C$

Integrate over faces of parallelograms $2\binom{k+1}{}$ of them.

parametrize them and integrate:

pairs with one vector fixed at $0 \cdot v_i$ or $h \cdot v_i$

other vectors free to roam: $(t_1, \ldots, t_k) \longmapsto 0 \cdot v_i + t_1 v_1 + \cdots + t_k v_{k+1}$

consider orientation.

pairs have cancelling constant terms. Work out linear term. $\quad t_j \in [0,1]$

linear terms of opposing faces: ( need to sum over all indices $\overset{i}{\cancel{\phantom{xx}}}$, let's just pick one, later sum up... )

omitted indices

$$\int_{[0,1]^k} \underbrace{p_1(f)\,(\gamma_{1,i}(\underline{t}) - \gamma_{0,i}(\underline{t}))}\; dx_{i_1}\wedge\cdots\wedge dx_{i_k}\,(\underline{v}_1,\ldots,\hat{\underline{v}}_i,\ldots,\underline{v}_{k+1})\,|d^k\underline{t}|$$

params of opposing faces

$$\underbrace{[Df(\underline{0})]\,(\underbrace{hv_i + \gamma_{0,i}(\underline{t})}_{\gamma_{1,i}(\underline{t})}) - [Df(\underline{0})]\,\gamma_{0,i}(\underline{t})}_{\text{constant indep. of } t\,!}$$

$$= h\cdot[Df(\underline{0})]\,v_i$$

So summing over all:

$$= \sum_{i=1}^{k+1} (-1)^{i-1}\,\frac{h^{k+1}}{h^{k+1}}\,[Df(\underline{0})]\,v_i\,\left(dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)(v_1,\ldots,\hat{v}_i,\ldots,v_{k+1})$$

$$\underset{\text{wedge prod formula}}{=} df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}\,(\underline{0})\,(\underline{v}_1,\ldots,\underline{v}_{k+1}) \quad ✓$$

<u>Easier about this</u>: normally $D\gamma_{i,1}$, $D\gamma_{i,0}$ inserted into form change depending on $\underline{t}$. Not here since //-ogram is linear.