4 proofs from H-H :

Theorem 4.3.10 (volume 0 discontinuities)

Theorem 4.11.7 (Lebesgue integrals are well-defined)

Theorem 5.4.1 (Gauss' theorem on curvature)

Proposition 6.9.7 (Easy case of Stokes' theorem)

Mini-exam : ① Compute $\displaystyle\int_C x^2 y^2 \, |dx\,dy|$ where $C : \begin{array}{l} 1 \le xy \le a \\ x \le y \le bx \end{array} \subseteq \mathbb{R}^2$

Hint : $u = xy$, $v = y/x$.

② Show that $\dfrac{1}{1+|\underline{x}|^3}$ is L-integrable in $\mathbb{R}^2$

Hint : Break up plane into annuli of outer/inner radius $2^i, 2^{i-1}$.

③ Find the area of the surface in $\mathbb{R}^4$ given by $\begin{array}{l} x_1^2 + x_2^2 = a^2 \\ x_3^2 + x_4^2 = b^2 \end{array}$

Hint : use pairs of polar coordinates.

④ Find an orientation for M : $x_1^2 + x_2^2 + x_1 x_4^2 = 1$.
(that is prove it defines oriented manifold, so check that it is smooth manifold too)

Also Problem 6.22 in Review for chapter 6.

Theorem 4.3.10 in H-H: A bounded function $f: \mathbb{R}^n \to \mathbb{R}$ with bounded support is R-integrable if it is continuous except on a set of volume 0.

Proof sketch: We use the "iff" criterion for integrability:

f bounded, bounded support $\iff$ f is R-integrable.

and for all $\epsilon > 0$, $\exists N$ st.

$$\sum_{\substack{C \in \mathcal{D}_N \\ \mathrm{osc}_C(f) \geq \epsilon}} \mathrm{vol}_n C < \epsilon \qquad (\text{just need } \Rightarrow)$$

$(\Rightarrow)$ is proved by considering $\lim_{N \to \infty} U_N(f) - L_N(f)$.

Express $U_N(f) - L_N(f)$ in terms of sums over cubes with $\underbrace{\mathrm{osc}_C(f) > \epsilon}_{\text{these are small by assumption}}$

and $\underbrace{\mathrm{osc}_C(f) \leq \epsilon}_{\text{these are small by compact supp. of } f}$.

To apply the criterion, show that discontinuities of f are covered by cubes (finitely many) with arbitrarily small volume. These cubes are then covered by nearest neighbors. On these we can't control $\mathrm{osc}_C(f)$.

Remaining points of $\mathrm{Supp}(f)$ have f continuous and are bounded away from discontinuities, so have $\mathrm{osc}_C(f) \leq \epsilon$ for $N \gg 0$.