 Name: ____________

1. Let $1_D(x,y)$ denote the characteristic function on the unit disc D in $\mathbb{R}^2$. Is the function
$$f = \log_2(x^2+y^2)1_D(x,y)$$
Lebesgue integrable on $\mathbb{R}^2$? Explain your answer.

Yes. Consider the partitioning of unit disc D by annuli with radii $\frac{1}{2^k}, \frac{1}{2^{k+1}}$, $k=0,1,\ldots$

so $A_k = \{ (r,\theta) \mid r \in [\frac{1}{2^k}, \frac{1}{2^{k+1}}] \}$.

Let $f_k = f \cdot 1_{A_k}$. We claim that

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^2} |f_k| \, |d(x,y)| < \infty \quad (*)$$

so since $\sum_k f_k = f$ then f will be L-integrable.

Indeed $\sup_{A_k} f_k = \log_2(\frac{1}{2^{2k}}) = -2k$

so $(*) \lesssim \sum_{k=0}^{\infty} 2k \cdot \underbrace{\text{vol}(A_k)}_{\pi(\frac{1}{2^k})^2 - \pi(\frac{1}{2^{k+1}})^2}$ which converges by favorite series test, e.g. ratio test.

2. a) If f is a Lebesgue integrable function on $\mathbb{R}^n$ and g is a Riemann integrable function on $\mathbb{R}^n$, prove that the product fg is Lebesgue integrable on $\mathbb{R}^n$.

Since f is L-integrable, write it as $f \underset{L}{=} \sum_k f_k$, f_k: R-integrable

Thus $fg \underset{L}{=} \sum_k f_k \cdot g$. We must show $\sum_k \int_{\mathbb{R}^n} |f_k g| \, |d^n \underline{x}| < \infty$.

But g is bounded, since R-integrable, so $\sup |g| < \infty$. And

$$\sum_k \int_{\mathbb{R}^n} |f_k g| \, |d^n \underline{x}| \leq \underbrace{\sup |g|}_{\text{finite}} \cdot \underbrace{\sum_k \int_{\mathbb{R}^n} |f_k| \, |d^n x|}_{\text{finite since } f \text{ is L-int.}} < \infty .$$

b) Give sufficient conditions for which the function

$$F(t) = \int_{\mathbb{R}^n} f(t, \boldsymbol{x}) \, |d^n \boldsymbol{x}|$$

can be differentiated under the integral sign.

(Hint: To bring a limit inside an integral, use the dominated convergence theorem.)

$$\frac{d}{dt} F(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \frac{f(t+h, \underline{x}) - f(t, \underline{x})}{h} \, |d^n x|$$

so to apply dominated convergence theorem, need for some $\epsilon > 0$

if $|h| < \epsilon$ then $\left| \dfrac{f(t+h, \underline{x}) - f(t, \underline{x})}{h} \right| \leq g(\underline{x})$

for some L-integrable function g for all t. Also need $\frac{d}{dt} f(t,x)$ defined a.e.

3. a) Exhibit a relaxed parametrization γ for the surface of revolution S obtained by rotating the curve

$$z^2 = (1+x)^3 \quad \text{with } x \in [-1, 0]$$

about the z-axis in $\mathbb{R}^3$.

(Hint: First parametrize curve $z^2 = (1+x)^3$ in $\mathbb{R}^2$ using a single parameter t. When you search for functions f and g so that $x = f(t)$ and $z = g(t)$, think about choosing simple polynomials so that the degrees of both sides of the curve equation match.)

To parametrize the curve, set $z = t^3$, $x = t^2 - 1$

then surface of revolution is $\begin{pmatrix} t \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} (t^2-1)\cos\theta \\ (t^2-1)\sin\theta \\ t^3 \end{pmatrix}$

2 pts

We could do piecewise parametrization via $z = \pm\sqrt{(1+x)^3}$ with $x = t$, but this is much messier in the end.

b) Prove your answer in (a) gives a relaxed parametrization of S. In particular, find a set $X \subset U$ of points x on which $\gamma : U \longrightarrow S$ either:

- fails to be one-one, C^1, with Lipschitz derivative at x, or
- $[D\gamma(x)]$ is not one-one.

3 pts

Explain why the set X has volume 0.

These are smooth functions in parametrization, but they may fail to be one-one where curve touches z-axis in x-z plane. Then all θ give same point. This occurs when $t = 1, -1$ (places where x-coord $= 0$).

$$[D\gamma\begin{pmatrix} t \\ \theta \end{pmatrix}] = \begin{bmatrix} 2t\cos\theta & -(t^2-1)\sin\theta \\ 2t\sin\theta & (t^2-1)\cos\theta \\ 3t^2 & 0 \end{bmatrix}$$

which has non-trivial kernel when $t = 0$, θ arbitrary

Circle of radius 1, which has volume 0.

Finally, fails to be one-one when $\theta = 2\pi$ and $\theta = 0$ give same curves, again of vol. 0.

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c) Set up an iterated integral to compute the 2-volume of this surface of revolution.

(Your answer should have integrand in the form $\sqrt{p(t,\theta)}$ for some polynomial p in t and $\sin\theta$ and $\cos\theta$. But definitely don't try to integrate this!)

$$\begin{bmatrix} 2t\cos\theta & 2t\sin\theta & 3t^2 \\ -(t^2-1)\sin\theta & (t^2-1)\cos\theta & 0 \end{bmatrix} \begin{bmatrix} 2t\cos\theta & -(t^2-1)\sin\theta \\ 2t\sin\theta & (t^2-1)\cos\theta \\ 3t^2 & 0 \end{bmatrix}$$

2 pts

$$= \begin{bmatrix} 4t^2\overbrace{(\cos^2\theta+\sin^2\theta)}^{1} + 9t^4 & 0 \\ 0 & (t^2-1)^2\cdot\underbrace{(\sin^2\theta+\cos^2\theta)}_{1} \end{bmatrix}$$

$$\det = (4t^2+9t^4)\cdot(t^2-1)^2$$

Compute
$$\int_0^{2\pi}\int_{-1}^{1} \sqrt{(4t^2+9t^4)(t^2-1)^2}\,dt\,d\theta.$$

4. a) State Gauss' Theorema Egregium on the area of discs $D_r(p)$ at a point p on a 2-manifold S.

For r sufficiently small

$$D_r(p) = \pi r^2 - \frac{\pi}{12} K(p) r^4 + o(r^4)$$

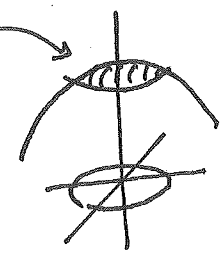
b) Show that the theorem holds at the north pole $(x, y, z) = (0, 0, 1)$ of the unit sphere $x^2 + y^2 + z^2 = 1$ by computing both sides appearing in Gauss' Theorem. You may use the fact that the disc $D_r(0, 0, 1)$ on the sphere projects to a disk of radius $\sin r$ in the xy-plane.

For computing $K(p)$ at north pole, we have

$$z = f(x,y) = \sqrt{1-x^2-y^2} = 1 - \frac{1}{2}(x^2+y^2) + o(x^2, y^2)$$

so $a_{2,0} = a_{0,2} = -1$ and its determinant is $K(p) = 1$.

To find surface area of $D_r(p)$:



can either use spherical coords, or

write surface as

$$\begin{pmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{pmatrix} = \begin{pmatrix} x \\ y \\ -\frac{1}{2}(x^2+y^2) + o(x^2, y^2) \end{pmatrix}$$

then, using above fact, in ρ, θ coords in x-y plane

$$\begin{pmatrix} \rho \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ -\frac{1}{2}\rho^2 \end{pmatrix} \quad \text{with } U = \{(\rho,\theta) \mid \rho < \sin r\}$$

$$\mathcal{U} \longrightarrow D_r(0,0,1)$$

So compute

$$\int_0^{2\pi}\int_0^{\sin r} \sqrt{\det (D\gamma^T)(D\gamma)}\, d\rho\, d\theta = \pi r^2 + \frac{\pi}{12} r^4 + o(r^4)$$

requires simplifying using $\sqrt{1+t} = 1 + \frac{t}{2} + o(t)$.