Math 8300 - Topics in Algebra: Quantum Groups

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Course Website: `math.umn.edu/~brubaker/8300.html`

Vincent Hall 2, MWF 1:25-2:15 pm

This semester's Topics in Algebra is in the spirit of Peter Webb's first semester course, but the content of that semester is not a prerequisite for this course. We are aiming to study the representation theory of "quantum groups," a variation on Lie groups whose imprint is all over modern mathematics. So the topics and tools presented in Webb's "Representations of Finite-Dimensional Algebras" course[1] will provide welcome context, but will be revisited as needed for our particular examples.

A primary goal for the semester is to explain precisely what is meant by the following paragraph:

The term "quantum group" was used by Drinfeld, in his 1986 ICM lecture at Berkeley, to describe a framework for solutions to certain problems arising in mathematical physics (more precisely, to provide instances of the Yang-Baxter equation in exactly solvable models and the quantum inverse scattering method). In that lecture, Drinfeld formalized these examples as a nice class of Hopf algebras. But in much more down-to-earth terms, quantum groups include certain *deformations* of (universal enveloping algebras of Lie algebras of) Lie groups like $\mathrm{SL}_2(\mathbb{C})$, the set of 2×2 matrices with complex entries and determinant equal to 1. So there are presentations for these algebras involving a parameter q, and setting $q = 1$ gives the presentation for the universal enveloping algebra of the associated Lie algebra. Think of it, roughly, as extending Lie algebra modules from $\mathbb{Z}$ to $\mathbb{Z}[q, q^{-1}]$, where the q now acts as a grading to provide refined information. Moreover, modules for these algebras have a so-called *canonical basis* (Lusztig) or *crystal basis* (Kashiwara) that arises in the limit as $q \to 0$, and these bases appear in many contexts and even help us to understand the structure of modules of the original Lie group in surprising ways. This includes some algorithms for constructing and visualizing highest weight representations, and we may explore methods for computing them in `SageMath`, an open-source computational software, as time permits.

What's remarkable about quantum groups is how ubiquitous they are in modern mathematics, from probability to representation theory, from algebraic combinatorics to knot theory, and on and on. We will spend most of our time discussing their basic structure in simple examples, with connections to representation theory and algebraic combinatorics (since the course is named "Topics in Algebra" after all).

No prior experience with these topics (including Lie theory) will be assumed, but some familiarity with basic subjects in a first graduate course in algebra will be quite helpful. There is no one textbook for the course[2]; we will rely on course notes posted to the course website, though specific textbook references will also be given as we proceed. There will be several optional problem sets assigned over the semester, and an option for a short presentation on a special topic at the end of the semester. These written assignments will be less optional for undergraduates, and all are encouraged to try them.

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[1] See Webb's course notes at:
http://www-users.math.umn.edu/~webb/Teaching/Math8300Notes2019.pdf

[2] A brief word on texts:
Jantzen's "Lectures on Quantum Groups" is very clear, but also provides little insight. Chari and Pressley's "A Guide to Quantum Groups" is encyclopedic and insightful, but perhaps not well-suited to the beginner. Lusztig's "Introduction to Quantum Groups" has useful material, but is very challenging to read. Christian Kassel's "Quantum Groups" (Springer GTM 155) is a good compromise and we will do many of the topics in Parts 1 and 2 of this book.