Math 8300 – Quantum Groups – Problem Set 2
Due: Friday, February 21

1. A Hopf algebra H is a bialgebra with additional structure. That is, it has an antipode map $S : H \longrightarrow H$ such that

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon,$$

where m and Δ are multiplication and comultiplication, η and ε are unit and counit in H.

a) Draw commutative diagrams representing the above displayed axioms.

b) Explain, using diagrams, why every finite dimensional Hopf algebra H has a dual Hopf algebra structure on the underlying dual vector space.

c) Show that S is an antialgebra map. That is, $S(m(a,b)) = m(S(b), S(a))$ and $S(1) = 1$. Again, m is multiplication here. (Note: We do not assume that S as a linear map has an inverse.)

2. Show that the tensor algebra $T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ can be given a Hopf algebra structure. (Lectures 8 and 9 included definitions of the necessary comultiplication, counit, and antipode maps.)

3. Show that the Hopf algebra structure on kG, the group algebra of a finite group G, has dual Hopf algebra $k(G)$, the algebra of functions on G valued in k, with multiplication defined pointwise: $(f \cdot g)(x) := f(x)g(x)$. That is, write down the five pairing axioms from Lecture 10, p. 3, and show that they are satisfied for an appropriate choice of non-degenerate bilinear pairing.

4. In lecture 12, we weakened cocommutativity in a bialgebra, asking for a family of isomorphisms $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ for all modules A, B (as opposed to requiring it to be the simple "flip" map $\tau : a \otimes b \mapsto b \otimes a$). These must be compatible with associativity (whose diagrams are a pair of a commutative hexagons - see p. 2A in the lecture 12 notes). The existence of the map $c_{A,B}$ is equivalent to the existence of an element R in $H \otimes H$ with certain properties. Show that the compatibility of $c_{A,B}$ with associativity is equivalent to the element R satisfying the pair of properties:

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \qquad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$$

5. Let E, F, K, K^{-1} denote the generators of $U_q(\mathfrak{sl}_2)$ with relations as given in Lecture 16. Show the following relations for any integer $m > 0$:

$$[E, F^m] = [m]_q F^{m-1} \frac{q^{-(m-1)}K - q^{(m-1)}K^{-1}}{q - q^{-1}},$$

$$[E^m, F] = [m]_q E^{m-1} \frac{q^{(m-1)}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}$$

where

$$[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$$