On Monday, ended with "working definition" of quantum groups as non-commutative, non-cocommutative Hopf algebras - Hopf: bialgebra with antipode.

### Bialgebra:

Given an $H$-modules $M_1, M_2$ (mean: algebra modules) or even more precisely - left $H$-module structure

then $M_1 \otimes M_2$ has left $H$-module structure via

$$\Delta(h) \cdot (m_1 \otimes m_2) = \sum h_{(1)} \cdot m_1 \otimes h_{(2)} \cdot m_2$$

### Antipode:

$S : H \to H$ playing role of inverse (anti-algebra, anti-coalgebra map) though not necessarily $S^2 = \text{id}$.

Recall that if $M$ is an $H$-module, then we can define left $H$-module structure in $M^* = \text{Hom}_k(M, k) \cong \text{linear functionals}$ on $M$

via

$$(h \cdot f)(m) \overset{\text{def}}{=} f(S(h)m)$$

[For generic algebra $A$, with left $A$-module $M$ then $M^*$ comes with structure of a right $A$-module]

Also, if $S$ invertible (not guaranteed in defin of Hopf alg.) then can also define another

left $h$-module:

$$(h \cdot f)(m) \overset{\text{def}}{=} f(S^{-1}(h)m)$$

(if $S^2 = \text{id}$ then $S = S^{-1}$, so they are the same. But might be different in general.)

### Bialgebra:

if $H$ is finite dim'l, then antipode is bijective (so invertible)

[existence of (both) duals: "rigid monoidal category"]
Proposition:

Example: \( U_q(b_+): \) algebra generated by \( 1, X, g, g^{-1} \) with relations

\[
\begin{align*}
g g^{-1} &= 1 = g^{-1} g, \\
g X &= g X g 
\end{align*}
\]

\( 1, X, H, H^{-1} \) with \( H H^{-1} = 1 = H^{-1} H, \) \( H X = g X H \).

Then the maps

\[
\begin{align*}
\Delta X &= X \otimes 1 + H \otimes X \\
\Delta H &= H \otimes H \\
\Delta H^{-1} &= H^{-1} \otimes H^{-1}
\end{align*}
\]

not Commutative!

\( S(x) = -H^{-1}X \)

\( S(H) = H^{-1} \)

\( S(H^{-1}) = H \) (may be extended to)

Infinite dimensional non-commutative Hopf algebra with

\( S \) invertible, but

\( S^2 \circ (x) = g^{-1}X. \)

Proof: Extend defn of \( \Delta, \varepsilon \) on generators multiplicatively, check that they are algebra maps (consistent with relations in algebra)

For example, want to show extending \( \Delta \) mult. gives \( \Delta(HX) = \Delta(g X H) \):

\[
\begin{align*}
\Delta(HX) &= \Delta(H) \Delta(X) \\
&= (H \otimes H) (X \otimes 1 + H \otimes X) \\
&= HX \otimes H + H^2 \otimes HX
\end{align*}
\]

while

\[
\begin{align*}
\Delta(g X H) &= g \Delta(X) \Delta(H) \\
&= g (X \otimes 1 + H \otimes X) (H \otimes H) \\
&= g (XH \otimes H) + g : (H^2 \otimes XH) \\
&= H X \otimes H + H^2 \otimes H X 
\end{align*}
\]

etc. As before, want to extend antipode \( S \) as antialgebra map. Then check antipode axioms on generators.
check it is not cocommutative: immediate on generator $X$.

$$\Delta(X) = X \otimes 1 + X \otimes H \quad \Downarrow \tau$$

$$1 \otimes 1 X + X \otimes H \neq \Delta(X).$$

Also, what is dimension of $U_q(b_t)$? Not finite dim'el since $\{ H^a X^b \}$ are linearly independent. a.e $z, b \in \mathbb{Z}, b \neq 0$.

$$S^2(X) = S(S(X)) = S(-H^{-1}X) = -S(X) S(H^{-1}) = + H^{-1}X \
\text{more importantly, } S^2(u) = H^{-1}uH \quad \forall u \in U_q(b_t) \text{ since} \quad S^2 \text{ is homom. of } U_q(b_t) \text{, and this shows } S^2 \text{ bijective, so } S \text{ bijective, hence invertible.}$$

Dual Hopf algebras. In finite-dim'el case, dual vector space $H^*$ has same dimension as $H$ and $H^{**} \cong H$.

so we tend to think of $H, H^*$ symmetrically, write $\langle \phi, v \rangle := \phi(v)$ for "evaluation map" to reflect this symmetry.

then we can give formulas for $H^*$ as Hopf algebra using Hopf alg. structure on $H$:

$$\langle \phi \cdot f, a \rangle := \langle \phi \otimes f, \Delta a \rangle$$
$$\langle 1, a \rangle := \varepsilon(a)$$
$$\langle \Delta \phi, a \otimes b \rangle := \langle \phi, ab \rangle$$
$$\varepsilon(\phi) := \langle \phi, 1 \rangle$$
$$\langle S \phi, a \rangle := \langle \phi, Sa \rangle$$

as axioms are symmetric
upon reversing arrows
+ interchanging roles of $(m, \eta)$ and $(\Delta, \varepsilon)$.

Here, works because $(H \otimes H)^* \cong H^* \otimes H^*$ are equalities of v.s. in finite case.
More generally, say $H, H'$ are "dually paired" if $\exists <, > : H' \otimes H \to k$

satisfying previous 5 axioms.

$(H: \text{finite dim} \Rightarrow \text{then evaluation map is unique such } <, > : \text{Infinite dim} \text{ is more subtle. (In fact, there can be more than one choice for } H').)$

Examples: $k[G], \mathcal{U}(sl) \text{ or } T(V), \text{ now } \mathcal{U}_q(b_+). \text{ Find dually paired Hopf algebras for them.}$

if $G \text{ finite, then we know.}$

$(kG)^* : \text{algebra of functions on } G \Rightarrow \text{call it } k(G)$

with values in $k$, pointwise products: $(f \cdot g)(x) = f(x)g(x)$

$\forall x \in G, f, g \in k(G).$

so coproduct

$(\Delta f)(x,y) = f(xy) \text{ identify } k(G) \otimes k(G) = k(G \times G)$

Count $E(f) := f(e) \text{ where } e \text{ id. in } G$

antipode $S f(x) = f(x^{-1}) \text{ using earlier } <, >$

properties.

\[ 2 \text{ } \mathcal{U}(gl)'s \text{ dually paired Hopf algebra } \Rightarrow \text{Recall that } \mathcal{U}g : \text{finite dimensional ex semisimple Lie algebra}$

\[ \curvearrowright G \subset \text{Mat}_n(\mathbb{C}), \text{ a complex Lie group. In fact, } G \text{ is complex alg. variety,}$

so $G \text{ can be cut out of } \text{Mat}_n(\mathbb{C}) \text{ by polynomial equations } \exists p(x) \neq 0 \text{ for some } n$

Write $\mathcal{C}[G] \text{ for coordinate algebra of this algebraic variety.}$

\[ = \mathcal{C}[x_{ij}]_{i,j=1}^n \text{ / } \exists p(x) = 0 \text{ matrix coordinates.} \]
Example 2 cont.: Matrix mult. suggests natural defn of comult. / counit on \( \mathbb{C}[G] \).

\[ \Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj} \quad \varepsilon(x_{ij}) = \delta_{ij} \]

The Kronecker 3

- \( = 0 \) if \( i \neq j \)
- \( = 1 \) if \( i = j \)

also can define over \( \mathbb{k}[G] \) since structure constants are integers.

So then \( \mathbb{Z}[G] \) is well-defined and then we can tensor with \( \mathbb{k} \).

Antipode is more complicated in terms of cofactors in \( \sum x_{ij} \).

Removing one row and one column.

so \( \mathbb{k}[S_{L_2}] \) has generators \( a, b, c, d \) then \( S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \)

Slight abuse of notation...