In our last lecture, we saw that weakening cocommutativity from condition
\[ A \otimes B \rightarrow B \otimes A \]
to any isomorphisms \( e_{A,B} \) compatible ("braided monoidal category")
with associativity, module action
\[ \leftrightarrow \text{ giving invertible elt. } R \in H \otimes H \]
satisfying certain properties.

Restate def'n and lemma and prove lemma.

Reminder about \( R \) - "universal R matrix"

given \( \varphi : \text{rep of } H \), then \( (\varphi \otimes \varphi)(R) \) defines elt. in \( V \otimes V \)
giving solution to QYBE on \( V \otimes V \).

more generally, we can choose
three reps \( \varphi_1, \varphi_2, \varphi_3 \) with spaces \( (U, V, W) \).
get QYBE on \( U \otimes V \otimes W \).
A quasitriangular Hopf algebra is a pair \((H, R)\) where

\[ T \circ \Delta H = R (\Delta h) R^{-1} \quad \forall h \in H. \]

So \( R = id \) would be "cocommutative" condition and s.t.

\[ R \in H \otimes H \text{ on both sides.} \]

Reference: Lecture 5 of Majid, QG Primer

In \( H_{(1)} \otimes H_{(2)} \otimes H_{(3)} \) we have the following identities:

\((\Delta \otimes id) R = R_{13} R_{23} \quad (id \otimes \Delta) R = R_{13} R_{12}\)

(where, as before \( R_{13} \) means \( (\otimes) (\otimes) (\otimes) \))

\( R = R^{(a)} \otimes R^{(b)} \), then \( R_{13} \) means \( R^{(a)} \otimes 1 \otimes R^{(b)} \)

Lemma: If \((H, R)\) is quasitriangular, then

1. \((\varepsilon \otimes id) R = (id \otimes \varepsilon) R = 1\)
2. \((id \otimes S) R^{-1} = R\)
3. \(R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}\) in \(H \otimes H \otimes H\). (abstract QYBE)

If, for \((\Delta \otimes id) R = R_{13} R_{23}\) from axioms,

\((\Delta \otimes id) (R^{(a)} \otimes R^{(b)}) = \Delta (R^{(a)}) \otimes R^{(b)}\)

if

\( R = R^{(a)} \otimes R^{(b)} \)

\( R^{(a)}, R^{(b)} \in H\) (in general, might be linear combination of \(H\) elements)

Apply \( \varepsilon \otimes id \otimes id \),

\( \varepsilon (R^{(1)}) (R^{(2)}) \otimes R^{(3)} \)

Coalgebra axioms →

\( R^{(a)} \)
For \( 1 \), \( (\Delta \otimes \text{id}) R = R_{13} R_{23} \) from axioms,

and \( (\varepsilon \otimes \text{id}) \Delta = \text{id} \) (coalgebra axiom) \( \varepsilon \otimes \text{id} \)

So \( (\varepsilon \otimes \text{id}) (\Delta \otimes \text{id}) R \)

\[= R \]

and on the other hand \( = (\varepsilon \otimes \text{id} \otimes \text{id}) R_{13} R_{23} \)

\[= (\varepsilon \otimes \text{id}) R \varepsilon(1) R \]

\( E \text{ is alg. map.} \)

since \( R \) invertible, then comparing two sides, \( (\varepsilon \otimes \text{id}) R = 1 \) as desired.

use other coalgebra axiom for \( \text{id} \otimes E \) to prove \( (\text{id} \otimes E) R = 1 \) in same fashion.

\( \text{(2)} \) is straightforward since \( E \) is simple map, so checking axioms easy.

\( R_{12} R_{13} R_{23} = R_{12} (\Delta \otimes \text{id}) R = (\tau \otimes \Delta \otimes \text{id})(R) R_{12}^{\text{apr}} \) (4)

so \( R_{12} \Delta (x) R_{12}^{-1} = \tau \Delta (x) \forall x \in H \)

i.e. \( R_{12} \Delta (R^{(e)}) = \tau \Delta (R^{(e)}) \cdot R_{12}^{\text{apr}} \)

\( (4) = (\tau \otimes \text{id} \otimes \text{id}) (R) R_{12} \)

\[= (\tau \otimes \text{id}) (R_{13} R_{23}) R_{12} \]

\[= R_{23} R_{13} R_{12} \]

\[\checkmark \]

NOTE THE PARENTHESES!