The case when $q$ is a rt. of 1. So assume that has order $l': q^{l'} = 1$

this implies $[l']_q = 0$ and so $[i']_q = 0$ when $i > l' > 2$.

Proposition: $E^l, F^l, K^l, K^{-l}$ are in center of $U_q$. (check on generators)

If repeated use of $KEK^{-1} = q^{2l}E$, $KF^{-1} = q^{l}F$, shows

$K^l, K^{-l}$ commute with $E, F$, so $K^l, K^{-l}$ are central.

also $KEK^{-1} = q^{2l}E^l$, similarly for $F$. Finally $[E, F^l] = [q]^{l}F^{-l}$

Similarly for $[F, E^l]$. //

if $3|l$, this

in fact, closer look reveals if $l$ even, then with $l = 2l'$

in fact $[l']_q = 0$ since numerator is $q - q^{-l}$, which shows

that $E^{l'}, F^{l'}, K^{l'}, K^{-l'}$ are central.

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From now on, we assume that $l$ is odd, $l > 3$. (if $l$ even, just replace statements with $l'$ s.t. $l = 2l'$)

Return to our universal module $M(\lambda), \lambda \in k^*$

then $K \cdot m_\lambda = \lambda q^{-2l} m_\lambda = \lambda m_\lambda$.

$E \cdot m_\lambda = 0$ since its action involved $[l]_q$.

so any vector of form $m_\lambda - bm_0$, any $b \in k$, is a highest

weight vector

Consider: $Z_b(\lambda) := M(\lambda) / U(m_\lambda - bm_0)$

which is spanned by $F_1^1(m_\lambda - bm_0) = m_0 + bm_0$, $i > 0$,

$U(m_\lambda - bm_0)$ since $K(m_\lambda - bm_0) = \lambda(m_\lambda - bm_0), E(m_\lambda - bm_0) = 0$.
Thus $\mathbb{Z}_b(\lambda)$ is spanned by $m_j \forall j \in \mathbb{N}$ (or rather $[m_j]$, a little silly since $[[F^j]]$)

($l$-dimensional module) with

$$K \cdot m_i = \lambda q^{-2i} m_i, \quad F \cdot m_i = \begin{cases} \sum m_{i+1} & \text{if } i \leq l-1 \\ b \cdot m_0 & \text{if } i = l-1 \end{cases}$$

$$E \cdot m_i = \begin{cases} 0 & \text{if } i = 0 \\ \left[ i \right]_q^b \frac{\lambda q^{i-1} - \lambda^{-1} q^{-i}}{q - q^{-1}} m_{i-1} & (\geq 0) \end{cases}$$

and since $\lambda q^{-2i}, 0 \leq i < l$ are distinct, weight spaces are all 1-dim'l

Note: $F^l \cdot m_0 = b m_0$, so if $b \neq 0$, then

* $F$ doesn't act nilpotently on $\mathbb{Z}_b(\lambda)$ (unlike before, when $q$ not rt of 1, $F^L M = 0 \Rightarrow 0$)
* $F$ need not be of the form $\pm q^a$, $a \in \mathbb{Z}$.

Is this a simple module?

Prop. (2.12 in Jantzen) $q^l$ primitive $l$-th rt of 1, $l$ odd, $\geq 3$.

- If $b \neq 0$ or if $\lambda^2 q^l \neq 1$ then $\mathbb{Z}_b(\lambda)$ is a simple $U_q$-module.
- If $b = 0$ and $\lambda = \pm q^n$ ($0 \leq n < l$) then $\mathbb{Z}_b(\lambda)$ is:
  - Simple if $n = l - 1$.
  - If $n < l - 1$, then $\langle v_{i_1} \ldots v_{i_n} \rangle$ span a submodule of $\mathbb{Z}_b(\lambda)$
    and this is unique non-trivial submodule.

If: Let $M$ be non-zero submodule. $\mathbb{Z}_b(\lambda)$ is a direct sum of its one dim'l
weight spaces and $M$ is closed under action of $K$, so $M$ is direct sum of
weight spaces $M \cap \mathbb{Z}_b(\lambda)_{m}$. Weight vectors are those $\mathbb{N}$ in $M$.

our named basis above.
Choose \( j > 0 \) minimal with \( mj \in M \). \( M \) is closed under \( F \), so
\[
M_i = F^{i-j} Mj \quad \forall j \leq i \leq l.
\]
As in our earlier pf, if \( j = 0 \)
\[
\text{then } M \cong Z_b(\lambda), \text{ i.e. not proper.}
\]
So we may assume \( j > 0 \).

Now the weird property of \( F \) acting by \( F \cdot M_{j-1} = bmn \), says this only happens if \( b = 0 \) (necessary condition for proper submodule). With \( b = 0 \):
\[
M \text{ spanned by } m_i, \quad i \geq j, \text{ and } E \cdot m_i \text{ is a multiple of } m_{j-1},
\]
so by minimality of \( j \), and \( M \) closed under \( E \), must be \( E \cdot m_j = 0 \).

but then knowing \( E \) acts in range \( Z_0(\lambda) \) by
\[
[j]_\lambda \rho^{1-\delta - \lambda^{-1}e} \quad \delta = q^{-j-1}
\]
on \( m_j \), then numerator is 0. Can't be \( [j]_\lambda \rho^{1-\delta - \lambda^{-1}e} \) giving 0, \( \delta = q^{-j-1} \)

since \( 0 < j < l \), the order of \( b \). so \( \lambda q^{1-j} - \lambda^{-1}q^{-1} = 0 \Rightarrow \lambda = q^{2(j-l)} \), (another necessary cond. for proper submodule.)

Now check that where \( b = 0 \), \( \lambda = q^n \), \( 0 \leq n < l-1 \), then the \( m_i \) with \( i > n \) are a submodule. (straightforward. in particular \( E \cdot m_{n+1} = 0 ) \)

In particular we learn if \( 0 \leq n < l \), recover same structure of

Simple \( U \)-modules \( L(n, \pm) \) in \( Z_0(\pm q^n) \) (compare two actions)

Also learn that \( Z_0(\pm q^n) \) with \( 0 \leq n < l-1 \) can't be semisimple, because the complement of the \( L(n, \pm) \) wouldn't have form described above.
In fact, there are no simple finite-dim'l $U_g$-modules of dimension $> l$.

(Kassel, Prop. VI.5.2) \[ k = \mathbb{C}/\text{alg. closed field}. \]

pf sketch: if $V$ finite-dim'l, dim $> l$, then do cases:

(a) there is a lowest weight vector: $Kv = \lambda v$, $Fv = 0$.

then $\langle v, E^1v, \ldots, E^{l-1}v \rangle$ is a submodule \( \text{(of dim } \leq l) \)

\[ \text{reason: } E \cdot E^{l-1}v = E^l v \text{ and } E^l \text{ central so acts by scalar on } V \]

(b) No lowest wt. vector. Take $v$ s.t. $Kv = \lambda v$, consider

$\langle v, Fv, \ldots, F^{l-1}v \rangle$: similar idea as before.

\[ \text{Need to use commutation reln's of } E, F \text{ to check this.} \]