If we have a solution R'' to YBE, how does it prove commutativity of transfer matrices?

Pictorial proof. Start with $Z \left( \begin{array}{c}
\longrightarrow \\
1 \\
\longrightarrow
\end{array} \right)$

Step 1.

Claim 1.

$Z \left( \begin{array}{c}
1 \\
% \\
2 \\
\vdots \\
\longrightarrow
\end{array} \right) = wt \left( \begin{array}{c}
\rightarrow \\
1 \\
\leftarrow
\end{array} \right) \cdot Z \left( \begin{array}{c}
\rightarrow \\
1 \\
\leftarrow
\end{array} \right)$

Step 2.

Use YBE to conclude...

$Z \left( \begin{array}{c}
\longrightarrow \\
2 \\
\longrightarrow \\
1 \\
\vdots
\end{array} \right) \overset{YBE}{=} \overset{YBE}{=} \overset{\text{again}}{=} \overset{\text{again}}{=}$

so these transfer matrices commute!

$Z \left( \begin{array}{c}
\rightarrow \\
2 \\
\leftarrow
\end{array} \right) \cdot Z \left( \begin{array}{c}
\rightarrow \\
1 \\
\leftarrow
\end{array} \right) \cdot wt \left( \begin{array}{c}
\rightarrow \\
1 \\
\leftarrow
\end{array} \right)$

Algebraic Interpretation

$V_1 \quad V_{(2)}$

$R \in \text{End}(V \otimes V)$

vertical strand - one copy of $V$

horizontal strand - other copy of $V$.

In $3$ strands.

$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$

as endoms.

In our case: $R'' R(\lambda_1)_{12} R(\lambda_2)_{23}$

pictorial ft is given algebraically in 7.5.3 of Chari-Pressley.
Does there exist a solution $R''$ to QYBE in our 6-vertex model example?

Yes, in fact given $R(\lambda_1), R(\lambda_2)$, then $R''(\lambda_3) = R(\lambda_3)$ with

$\lambda_3 = \lambda_1 + \lambda_2 + \frac{\lambda_1 \lambda_2 \lambda_3}{(q + q^{-1}) \lambda_3 \lambda_2}$.

Natural to ask for general conditions on 6-vertex weights so that a QYBE exists. (next page)

Definition: A vertex model is "solvable" (a.k.a. "integrable") if it's Boltzmann weights satisfy a QYBE.

Definition: QYBE $\iff$ commuting transfer matrices $\iff$ closed form solution to partition function of a lattice model

Add to this:

Quantum group modules.

Technical Sentence: Quantum groups are examples of "quasitriangular Hopf algebras" $(H, R \in H \otimes H)$ with nice properties, including abstract

on $H \otimes H \otimes H$, $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$ (QYBE). And given $\varphi: \rho \otimes \nu \rightarrow \rho \otimes \nu$, then $(\rho \otimes \nu)(R)$ is matrix valued solution of QYBE. $\varepsilon \in \text{End}(\rho \otimes V)$.
When does a solution to Yang-Baxter equation exist? (6-vertex model)

Given $S, T \in \text{End}(V \otimes V)$, want $R \in \text{End}(V \otimes V)$ s.t.

$$Z(R \otimes S) = Z(T \otimes S)$$

$S$ has 6 wts. Let

- $w_S(SW) = a_1$
- $w_s(NW) = b_1$
- $w_s(EW) = c_1$
- $w_S(NE) = a_2$
- $w_S(SE) = b_2$
- $w_S(NS) = c_2$

Define invariants

$$\Delta_1(S) = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_1b_1}$$

$$\Delta_2(S) = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2a_2b_2}$$

Then (Baxter, Brubaker-Bump-Friedberg)

- General case
- Field-free case

$$\Delta_1(S) = \Delta_1(T) \iff \exists R \text{ s.t. QYBE is solved}.$$  

$$\Delta_2(S) = \Delta_2(T)$$

(In fact, we give exact description for weights in solution depending on $S, T$)

Conclude this week with an example: In ice, energy of every oxygen site is same. (There are other interesting molecules with hydrogen bonds where this is not the case)

So might as well take all wts = 1.

Then we're computing $Z$ as the sum of admissible states with fixed boundary condition.

Very special case: $M = N$ (our rectangle is a square)

and boundary is always: in in “domain wall boundary conditions”

\[ \text{DWBC} \]
Replace state of race with a matrix, entries at each vertex

\[ N = 3 : \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

is an ASM.

The only one not a permutation matrix is an ASM.

To find \( Z \) in this case, exactly one 1 in each row, each column.

How many ASMs are there?

1, 2, 7, ... ?

Conjecture (Mills, Robbins, Ramsey)

\[ \# \text{ of } N \times N \text{ ASMs} = \frac{1! \cdot 4! \cdot 7! \cdots \cdot (3N-2)!}{N! \cdot (N+1)! \cdots \cdot (2N-1)!} \]

(first proved by Zeilberger - difficult proof establishing bijection with totally symmm., self-complem., plane partitions)

Solution (Kuperberg):

B-Hz. weights = 1 won't help us in our techniques.

Use clever weights, with non-trivial YBE.

Let \( [x]_q = \frac{x^q - x^{-q}}{q - q^{-1}} \). Wts are:

\[
\begin{align*}
\text{E}\text{w} & : [x^{-1}]_q \\
\text{N}\text{s} & : [x^{-1}]_q \\
\text{S}\text{w} & : [x-1]_q \\
\text{N}\text{e} & : [x]_q \\
\text{S}\text{e} & : [x]_q \\
\text{N}\text{i}w & : [x]_q \\
\text{R}(x) & : (*) \cdot A(n; x)
\end{align*}
\]

At each vertex \( x : = d_{i} - B_{i} \) in pos. \( i,j \).

Solve for \( Z(n; \alpha, B) \) - \( Z \) symmetric in \( \alpha, B \)'s.

\( \alpha_{0} \)

\( \alpha_{n-1} \)

\( p_{0} \)

\( p_{n-1} \)

\( \cdots \)