On Monday, saw how QYBE + train argument \( \rightarrow \) commutativity transfer matrices

(in pictures: \[
\begin{array}{c|c|c|c}
\hline
& & & \\
\hline
\hline
\end{array}
\]
\( = \)
\[
\begin{array}{c|c|c|c}
\hline
& & & \\
\hline
\hline
\end{array}
\]

For our weights, implied that partition function \( Z(x) \) with \( x = (x_1, \ldots, x_r) \)

is symmetric in \( x \) variables.

Let's explore this — which symmetric functions arise as partition functions of

lattice models? Start with 6-vertex models.

End of hour, we were trying to cram in last example —

Given partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \)

\( \rightarrow \lambda + \rho \)

\( \rho = (r-1, \ldots, 0) \)

\( \sim \) made boundary condition

with “up” arrows at

\( \lambda_i + r - i \), down else.

E.g. \( \lambda = (2, 2, 1, 0) \) is partition of \( 8 \),

\( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) with 3 parts.

Then \( \lambda + \rho = (4, 3, 1, 0) \)

\( \rho = (2, 1, 1, 0) \)

with corresponding boundary:

\[
\begin{array}{c|c|c|c}
\hline
\hline
\hline
\end{array}
\]

Let \( Z_\lambda \) := partition function of

[Insert brief interlude on Schur polynomials]

See notes from Lecture #6 (1 + 2)

\[
\begin{align*}
\mathbf{b}_L = & \begin{cases}
\lambda_1 & \text{if } i = 1 \\
\lambda_i & \text{if } i > 1
\end{cases} \\
\lambda_1 & \text{if } i = 1 \\
\lambda_i & \text{if } i > 1
\end{align*}
\]

\( c_i = x_i \), \( c_1 = 1 \), \( c_2 = 1 \).

(\( \Delta(x_i) = 0 \))

Thm: \( Z_\lambda = x^\rho S_\lambda(x) \)
Last time, giving first results on partition functions.

If $M = N$ (square lattice), Boltzmann wts in 6-vertex model, all $= 1$

then $Z_N = \sum \text{Se admissible state} = \# \text{of A3Ms}$

(alternating sign matrices)

\[
\text{bijection:} \quad \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
-1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

EW: 1, NS: -1, all else: 0.

Sketch pf. of following theorem: $Z_N = \frac{1! \cdot 4! \cdot 7! \cdots (3N-2)!}{N! (N+1)! \cdots (2N-1)!}$

(Kuperberg)

(return to earlier notes pages)

Recall Schur polynomial - very special symmetric function

(character values of highest weight representations $V_\lambda$ of $\text{SL}_n(\mathbb{C})$)

Combinatorial definition in terms of semi-standard Young tableaux (SSYT)

Given partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, make diagram of shape $\lambda$

\[
\begin{array}{cccc}
\lambda_1 & \cdots & & \\
& \ddots & \ddots & \\
& & \lambda_r &
\end{array}
\]

SSYT: filling with alphabet $\{1, \ldots, r\}$ s.t. weakly increasing in rows, strictly increasing in columns

\[
\text{wt}(T) = z_1^2 z_2^3 z_3
\]

in our example, $\lambda = (5,3,1)$  

$1 1 2 2 2 3 3$  

for $\lambda = (5,3,1)$  

$\lambda_1 \neq \lambda_2 \neq \lambda_3$
Also skew Schur polys $S_{\lambda / \mu}$ where

we fill skew tableaux:

Skew shape $\lambda / \mu$.

Given partition $\lambda \rightarrow$ partition $\lambda + \rho$ with distinct parts

$\rho = (r-1, r-2, \ldots, 1, 0)$

$\lambda = (2, 2, 0) \rightarrow \lambda + \rho = (4, 3, 0)$

row of arrows: up at columns index

matches parts of down else.

$\lambda + \rho$.

(bijection between partitions and rows of up/down arrows)

Given $\lambda, \mu$, $\mu \leq \lambda$, then make lattice

$\lambda$: r parts

$\mu$: l parts

Thm: (same paper w/ Bostan-Viennot) (i.e. generic columns, prescribed rows)

$2011$ - Commun. in Math Physics, $\mu = \phi$ case:

$Z_{\lambda + \rho} = Z_\rho \cdot S_\lambda \left( \frac{b_2^{(i)}}{a_1^{(i)}}, \ldots, \frac{b_2^{(r)}}{a_1^{(r)}} \right)$

with $Z_\rho = \prod_{i=1}^{r} a_1^{(i)} c_2^{(i)} - \prod_{i<j} (a_1^{(i)} a_2^{(i)} + b_1^{(i)} b_2^{(i)})$

for any vars. with $\Delta^{(i)} = 0 \forall i$

More complicated for $\mu$ gen.
What other symmetric functions are possible?

Know std. symmetric functions $e_r(x)$: monomials with distinct vars

$h_r(x)$: sums of degree $r$ monomials.

$\lambda$ a most important

orthonormal basis w.r.t. symmetric bilinear form $\langle , \rangle$

$q,t$ analogue $\langle , \rangle$ and Macdonald polys. are unique symmetric polys st.:

- $p_{\lambda}(x; q,t)$ are orthogonal w.r.t. $\langle , \rangle$.

- and s.t. $p_{\lambda}(x) = m_{\lambda}(x) + \sum_{\mu < \lambda} \phi_{\lambda \mu} W_{\mu}(x)$

Hierarchy of symmetric polys.

(non-symmetric Macdonald)

$E_\lambda(x; q,t)$ q: composition
sum over permutations of composition.

$p_{\lambda}(x; q,t) + t = q^a$
$q \to 1$

Jack polys.

$p_{\lambda}(x; t)$ (Hall-Littlewood)
$t \to 0$

$s_{\lambda}(x)$ (Schur poly)

April 2019 Porodin-Wheeler "Non-symmetric Macdonald polys via integrable vertex models"

Two facts:
1. If $x$ has $n$ vars, arise naturally from $U_q(\hat{sl}_n)$

std. module: $R$-matrices from $\text{Sym}^n(V)$.

2. Use colored paths to get non-symmetric $V$-refined of symmetric poly.
Why is it interesting that $\Delta_{\lambda}(x)$ (or any other symmetric function) is representable as a partition function?

Answer: Lattice models encode so many nice properties.

- Branching rules.

- More clever rules.

\[
\sum_{\lambda} z^{|\lambda|} \left( \begin{array}{c} n+p \\ r \end{array} \right) \cdot \left( \begin{array}{c} \lambda+p \\ r \end{array} \right)
= \sum_{\lambda} \left( \begin{array}{c} 2r \\ \lambda \end{array} \right) \text{ (glued them together)}
\]

\[\lambda = (5, 4, 1) \quad \lambda = (3, 2, 1) \quad \lambda = (4, 1, 1)\]

\[\lambda' = (3, 2) \quad \lambda' = (2, 2, 1) \quad \lambda' = (4, 3)\]