Cornerstone of algebraic geometry : Hilbert's Nullstellensatz. In one form,

$$\left\{ \begin{matrix} \text{max. ideals} \\ \text{in } \mathbb{C}[\underline{x}] \end{matrix} \right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{ \begin{matrix} \text{pts. in} \\ \mathbb{C}^n \end{matrix} \right\}. \qquad \underline{x} = (x_1, \dots, x_n)$$

For $n=1$, this is just statement :

$$(x-a) \longleftrightarrow a \quad \text{and} \quad \mathbb{C}[x]/_{(x-a)} \simeq \mathbb{C}$$

$$f \pmod{x-a} \longmapsto f(a)$$

which in turn implies if we define

V : variety given as zero locus of $f_1(x), \dots, f_r(x)$

then I : ideal gen. by $\langle f_1, \dots, f_r \rangle$. There exists a bijection

$$\left\{ \begin{matrix} \text{max. ideals} \\ \text{of } \mathbb{C}[x]/_I \end{matrix} \right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{ \begin{matrix} \text{pts. of} \\ V \end{matrix} \right\}$$

Let $\mathbb{C}(C)$: function field of smooth proj. variety of dim 1, C,

view C as affine variety $\quad \mathbb{C}(C) = \text{Frac}\left(\mathbb{C}[\underline{x}]/I(C) \right)$ e.g. C : zeros of $y^2 = x^3 + x$ in $\mathbb{C}[x,y]$.

Given $f \in \mathbb{C}(C)^\times$, then

$$\text{div}(f) = \sum_{p \in C} \text{ord}_p(f) \cdot (P) \quad \text{formal sum.} \quad \text{"principal divisors"}$$

Arbitrary divisor $\quad D = \sum_{p \in C} n_p \cdot (P)$ with $n_p \in \mathbb{Z}$, $n_p = 0$ for almost all p.

$$\text{Pic}(C) = \text{Div}(C) / P(C) \quad \longleftarrow \quad D_1 \sim D_2 \text{ if } D_1 - D_2 = \text{div}(f) \text{ for some } f \in \mathbb{C}(C)^\times.$$

Mention use of $\text{Div}(C)$ in Riemann Roch theorem.

We want to play the same game with 1-dim'l Noetherian domain.

Nullstellensatz $\Rightarrow$ pts on $\mathcal{O}$ should be maximal ideals, and

thus $\mathrm{Div}(\mathcal{O}) = \bigoplus_{\mathfrak{p}:\text{prime}} \mathbb{Z}\cdot\mathfrak{p}$ (For $\mathcal{O}$ Dedekind, $\mathrm{Div}(\mathcal{O}) = J(\mathcal{O})$, the set of all fractional ideals)

Want to define principal divisors assoc. to elements $f \in \mathrm{Frac}(\mathcal{O})^\times = K^\times$

Idea: $f \longmapsto (f) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(f)}$ (*) viewed as elt of $\mathrm{Div}(\mathcal{O})$. (mult. vs. add. notation)

and $\mathrm{ord}_{\mathfrak{p}}(f)$ was valuation in DVR $\mathcal{O}_{\mathfrak{p}}$.

But in general, if $\mathcal{O}$ is 1-dim'l Noetherian domain, $\mathcal{O}_{\mathfrak{p}}$ is <u>not</u> a DVR. (not nec. int. closed.)

Still can define valuation: Given $f = \frac{a}{b} \in K^\times$,

set $\mathrm{ord}_{\mathfrak{p}}(f) = l_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{p}}/a\mathcal{O}_{\mathfrak{p}}) - l_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{p}}/b\mathcal{O}_{\mathfrak{p}})$

where $l_{\mathcal{O}_{\mathfrak{p}}}(M)$ is length as $\mathcal{O}_{\mathfrak{p}}$-module: size of maximal chain of submodules $M \supsetneq M_1 \supsetneq \cdots \supsetneq 0$.

This generalizes valuation in DVR since all ideals are a power of the maximal ideal

$(a) = a\cdot\mathcal{O}_{\mathfrak{p}} = \mathfrak{p}^l$ then max. chain is $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^l \supsetneq \mathfrak{p}\cdot\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^l \cdots$ so has length l.

check that $l_{\mathcal{O}_{\mathfrak{p}}}$ is multiplicative on short exact sequences so it indeed defines a valuation.

Now (*) makes sense for $\mathcal{O}$ 1-dim'l Noeth. domain, additively $f \mapsto \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(f)\cdot\mathfrak{p}$

Then $\mathrm{Div}(\mathcal{O}) / P(\mathcal{O}) =:$ Chow gp. $CH^1(\mathcal{O})$

If $\mathcal{O}$ ~~then~~ Dedekind, then we've shown $J(\mathcal{O}) / P(\mathcal{O}) = \mathrm{Pic}(\mathcal{O}) \simeq CH^1(\mathcal{O})$

But in general, we have a map

$$\mathrm{Pic}(\mathcal{O}) \longrightarrow CH^1(\mathcal{O})$$

induced by $\mathfrak{a} \longmapsto \mathrm{div}(\mathfrak{a}) = \sum_{\mathfrak{p}} -\mathrm{ord}_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) \cdot \mathfrak{p}$ where $\mathfrak{a}$ invertible so principal in localizations, hence $= \alpha_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$ at each $\mathfrak{p}$.

a homomorphism on $J(\mathcal{O})$ to $\mathrm{Div}(\mathcal{O})$ that respects their quotients.

Go further - put topology on space of all prime ideals of $\mathcal{O}$, $\mathrm{Spec}(\mathcal{O})$, called Zariski topology. Defined by setting closed sets to be $\{ \mathfrak{p} \mid \mathfrak{p} \supseteq \mathfrak{a} \}$ where $\mathfrak{a}$ varies over ideals of $\mathcal{O}$.

not-necessarily-Hausdorff topological space.

Just as in $\mathbb{C}[x] / x-a \simeq \mathbb{C}$

$f \pmod{x-a} \longrightarrow f(a)$

then functions on X are elts of $\mathcal{O}$, f, whose "evaluation at point" is just $f \bmod \mathfrak{p}$.

Weird part: In complex case, for any a,

$\mathbb{C}[x] / x-a \simeq \mathbb{C}$ so functions viewed as taking values in same space $\mathbb{C}$.

Here we just have values in $\mathcal{O}/\mathfrak{p}$ for each $\mathfrak{p}$.

We are including in $\mathrm{Spec}(\mathcal{O})$ the 0-ideal.

The residue field at 0 is $\mathrm{Frac}\left(\mathcal{O}/_{(0)}\right) = \mathrm{Frac}(\mathcal{O}) =: K$.

so value of $f \in \mathcal{O}$ at (0)-ideal is just f considered as elt. of K.

The "pt" corresponding to the 0-ideal is often referred to as "generic pt"

Open sets are complements of closed sets.

Note (0) is not a closed pt. since can't find $\mathfrak{a}$ s.t. $\{\mathfrak{p} \mid \mathfrak{p} \supseteq \mathfrak{a}\} = (0)$.

But the points $\mathfrak{p}$ are closed, and $\mathrm{Spec}(\mathcal{O}) = X$ is closed

finite sets of primes are closed,

so Open sets are $\mathrm{Spec}(\mathcal{O}) \setminus \{\text{finite \# of primes}\}$, and we have a topology on X.

Good: weakest topology for which pts closed and polynomial maps are continuous

Bad: Too weak to distinguish arithmetic. - All fields have $\mathrm{Spec}(K) = \{\text{pt.}\}$ corresponding to 0-ideal.

So geometry all the same.

Richer structure, an affine scheme, by considering $\underset{\mathrm{Spec}(\mathcal{O})}{\overset{X}{\shortparallel}}$ together with its structure sheaf $\mathcal{O}_X$.

Remember sheaf of rings is just assignment of ring for every open set $\mathcal{U} \mapsto \mathcal{F}(\mathcal{U})$ in topology, with good compatibility properties

presheaf: if for $V \subseteq \mathcal{U}$ have given homom. $\rho_{\mathcal{U},V} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$ "restriction"

(typo in Neukirch here)

+ simple axioms $\mathcal{F}(\emptyset) = 0$

$\rho_{\mathcal{U}\mathcal{U}}$ identity map.

$\rho_{V,W} \circ \rho_{\mathcal{U},V} = \rho_{\mathcal{U},W}$

Add additional conditions on elts of $\mathcal{F}(\mathcal{U})$ to obtain a bona fide sheaf

For us, $\mathcal{O}_X$ is sheaf of rings for Zariski topology: (Assume $\mathcal{O}$ 1-dim'l Noetherian domain)

$$\mathcal{U} \longmapsto \mathcal{O}(\mathcal{U}) =: \mathcal{O}S^{-1} \quad \text{where } S = \left(X \setminus \bigcup_{\wp \in \mathcal{U}} \wp\right)$$

(non-empty)

$$= \left\{ \frac{f}{g} \;\middle|\; \underbrace{g \not\equiv 0 \bmod \wp}_{g \text{ "evaluated at the point } \wp\text{"}} \ \forall \wp \in \mathcal{U} \right\}$$

with natural inclusions

$$\mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(V) \quad \text{if} \quad V \subseteq \mathcal{U} \quad \text{so} \quad S_V \supseteq S_{\mathcal{U}}$$

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presheaf $\to$ sheaf: For every section $s \in \mathcal{F}(\mathcal{U})$, any open cover $\{\mathcal{U}_i\}$ of $\mathcal{U}$

(i) if $s|_{\mathcal{U}_i} = s'|_{\mathcal{U}_i}$ $\forall i$, then $s = s'$

(ii) if $s_i \in \mathcal{F}(\mathcal{U}_i)$ $\forall i$ with $s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ $\forall i, j$.

then $\exists$ s with $s|_{\mathcal{U}_i} = s_i$ $\forall i$.

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check this is satisfied by structure sheaf $\mathcal{O}_X$.

Easy examples: (a) K: field, then $(X, \mathcal{O}_X)$ is $X = \{\text{pt.}\}$ (generic pt. (0))

whose structure sheaf is just the association $X \mapsto K$

(b) $\mathcal{O}$ is DVR with maximal ideal $\wp$ then $X = \{0, \wp\}$

with sheaf
$$\wp \longmapsto \mathcal{O}_\wp = \mathcal{O} \quad \text{since } \mathcal{O}_\wp = \left\{ \frac{a}{s} \;\middle|\; a \in \mathcal{O},\ s \in \underbrace{\mathcal{O} \setminus \wp}_{\text{units in DVR}} \right\}$$
$$0 \longmapsto \mathrm{Frac}(\mathcal{O}) =: K$$