On Monday, we were attaching geometric intuition to algebra.

Affine scheme $(X, \mathcal{O}_X)$ $\quad \mathcal{O}_X$: structure sheaf $\quad X$: $\mathrm{Spec}(\mathcal{O})$

if $\mathcal{O}$ order, then say X is singular at point $\wp$ if $\wp$ is not regular

(i.e. if $\wp\mathcal{O}_\wp$ is not principal as ideal in $\mathcal{O}_\wp$)

$\iff$ $\mathfrak{m}/\mathfrak{m}^2$ not generated by single elt. as $\mathcal{O}_\wp/\mathfrak{m}$ - v.s.

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$\mathfrak{m}$: $\wp \cdot \mathcal{O}_\wp$

$\iff$ Jacobian of variety at point $\longleftrightarrow \wp$ is ~~non~~-singular. (not of max. rank.)

Hartshorne I.5 cx. variety (or over alg. closed field)

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return to pages 4, 5 from previous lecture

Function fields : Not $\mathbb{C}[t]$ but $\mathbb{F}_p[t]$ is still closer in structure to rings of integers like $\mathbb{Z} \subseteq \mathbb{Q}$. Preview this briefly...

$\mathbb{F}_p[t]$ is principal ideal domain with primes : monic irreducible of $\mathbb{F}_p[t]$

residue field $\mathbb{F}_p[t] / (g(x))$ $\quad g$: monic irred. of deg d $\quad \simeq \mathbb{F}_q$ with $q = p^d$ (finite field)

$$f(x) \pmod{g(x)} \longmapsto f(\alpha), \quad \alpha : \text{root of } g.$$

which generalizes earlier example $/\mathbb{C}$ where ideals were $g(x) = x - \alpha$

and the map to $\mathbb{C}$ was evaluation at α.

Again, because $\mathbb{F}_p[t]$ is 1-dim'l Noetherian domain, all earlier theory applies.

Two advantages — (1) connections to geometry more immediate
(2) residue fields all have same characteristic.

Often examples considered together — "global fields" either finite ext'n of $\mathbb{Q}$ or finite ext'n of $\mathbb{F}_p(t)$ (fraction field of $\mathbb{F}_p[t]$)
with number fields sometimes referred to as case of "mixed characteristic"

One problem with writing $\mathbb{F}_p(t)$ is that it places undue emphasis on elt. t as generator. $\mathbb{F}_p[t]$ seems natural choice for ring of integers in this context. But if we considered parameter $\frac{1}{t}$ instead, we might use $\mathbb{F}_p[\frac{1}{t}]$ as ring with different prime ideals.

Need coordinate free way to determine primes — valuations.

Suppose K finite ext'n of $\mathbb{F}_p(t)$, with $\mathcal{O}_K$: int. closure of $\mathbb{F}_p[t]$ in K.
Already know $\exists$ valuations on 1-dim'l Noetherian domain corresp. to each non-zero prime $\mathfrak{p} \neq 0$:
$$v_{\mathfrak{p}} : K \longrightarrow \mathbb{Z} \cup \{\infty\}$$
with
- $v_{\mathfrak{p}}(0) = \infty$
- $v_{\mathfrak{p}}(ab) = v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(b)$
- $v_{\mathfrak{p}}(a+b) \geq \min\{v_{\mathfrak{p}}(a), v_{\mathfrak{p}}(b)\}$

and such that
$$(a) = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(a)}$$

Are there any more valuations? Yes. 1 more. (Next chapter all about how to classify valuations on such a domain)

Turns out to be degree: $v_\infty\left(\frac{f}{g}\right) = \deg(g) - \deg(f)$

Alternately define it as the valuation for $\mathfrak{p} = s \cdot \mathbb{F}_p[s]$ with $s = \frac{1}{t}$. (8)

so focus our attention on valuations, not primes per se.

then when we define $Cl(K)$ as in Chow gp earlier,

$$\mathrm{Div}(K) = \left\{ \sum_{v:\mathrm{val}} n_v \cdot v \;\middle|\; n_v \in \mathbb{Z},\ n_v = 0 \text{ for almost all val. } v \right\}$$

Warning: Neukirch and other authors often write $\mathfrak{p}$ for valuation to emphasize similarity. But not same thing.

and similarly map elts

$$K^\times \longrightarrow \mathrm{Div}(K) \qquad \text{with image } P(K) \text{ as before.}$$
$$f \longmapsto \mathrm{div}(f) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(f) \cdot \mathfrak{p}$$

Surprising fact: $Cl(K) = \mathrm{Div}(K) / P(K)$ not finite.

there is a natural subgp., called $Cl^0(K)$, class gp of deg 0 divisors,

which is finite, where $\deg : Cl(K) \longrightarrow \mathbb{Z}$

$$\alpha \longmapsto \deg(\alpha) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \deg(\mathfrak{p})$$

(image of $\mathrm{div}(f) \in P(K)$ is 0 under deg.)

if $\alpha = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p}$

where $\deg(\mathfrak{p})$ = deg. of res. class field over $\mathbb{F}_p$.

Also could have used algebraic geometry,

theory of schemes — paste together affine schemes.

Given $(X, \mathcal{O}_X)$ ask that:

Every point $x \in X$ has open nbhd $\mathcal{U}$ with $\mathcal{U}$ having structure of affine scheme. (Just like affine/proj. varieties) or charts on Riemann surface

(pair is $\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}}$)

Rough idea: detect all prime ideals using open cover of affine schemes.

Example : $K = \mathbb{F}_p(t)$ itself, scheme obtained from pair of

affine schemes $\mathcal{U} = \mathrm{Spec}(\mathbb{F}_p[u])$, $V = \mathrm{Spec}(\mathbb{F}_p[v])$

if we remove the ideal $(u-0)$ from $\mathcal{U}$, then result is

$\mathcal{U} - (u) = \mathrm{Spec}(\mathbb{F}_p[u, u^{-1}])$, $V - (v) = \mathrm{Spec}(\mathbb{F}_p[v, v^{-1}])$

Call $u = \mathfrak{p}_0$, $v = \mathfrak{p}_\infty$ (pre-saging relation to projective space)

Isomorphism of rings $\mathbb{F}_p[u, u^{-1}] \longrightarrow \mathbb{F}_p[v, v^{-1}]$ yields

$$f : u \longmapsto v^{-1}$$

isomorphism of affine schemes $V - (v) \longrightarrow \mathcal{U} - (u)$

$$\mathfrak{p} \longmapsto f^{-1}(\mathfrak{p})$$

Form scheme by identifying $V-(v)$ and $\mathcal{U}-(u)$ in $\mathcal{U} \cup V$.

gives top. space $X = \mathcal{U} \cup V /_{\sim}$ with sheaf of rings $\mathcal{O}_X$ from $\mathcal{O}_{\mathcal{U}}, \mathcal{O}_V$.